Uncountable γ-sets under axiom $\mathrm{CPA}_{\mathrm{cube}}^{\mathrm{game}}$

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Abstract

In the paper we formulate a Covering Property Axiom $\mathrm{CPA}_{\mathrm{cube}}^{\mathrm{game}}$, which holds in the iterated perfect set model, and show that it implies the existence of uncountable strong γ-sets in $\mathbb{R}$ (which are strongly meager) as well as uncountable γ-sets in $\mathbb{R}$ which are not strongly meager. These sets must be of cardinality $\omega_1 < \mathfrak{c}$, since every γ-set is universally null, while $\mathrm{CPA}_{\mathrm{cube}}^{\mathrm{game}}$ implies that every universally null has cardinality less than $\mathfrak{c} = \omega_2$.

We will also show that $\mathrm{CPA}_{\mathrm{cube}}^{\mathrm{game}}$ implies the existence of a partition of $\mathbb{R}$ into ω_1 null compact sets.

1 Axiom $\mathrm{CPA}_{\mathrm{cube}}^{\mathrm{game}}$ and other preliminaries

Our set theoretic terminology is standard and follows that of [3]. In particular, $|X|$ stands for the cardinality of a set X and $\mathfrak{c} = |\mathbb{R}|$. The Cantor set 2^ω will be denoted by a symbol $\mathfrak{C}$. We use term *Polish space* for a complete separable metric space `without isolated points`. For a Polish space X, the symbol

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$\mathrm{Perf}(X)$ will denote the collection of all subsets of X homeomorphic to $\mathfrak{C}$. We will consider $\mathrm{Perf}(X)$ as ordered by inclusion.

Axiom CPA$^{\rm game}_{\rm cube}$ was first formulated by Ciesielski and Pawlikowski in [4]. (See also [6].) It is a simpler version of a Covering Property Axiom CPA which holds in the iterated perfect set model. (See [4] or [6].) In order to formulate CPA$^{\rm game}_{\rm cube}$ we need the following terminology and notation. A subset C of a product $\mathfrak{C}^\omega$ of the Cantor set is said to be a *perfect cube* if $C = \prod_{n\in\omega} C_n$, where $C_n \in \mathrm{Perf}(\mathfrak{C})$ for each n. For a fixed Polish space X let $\mathcal{F}_{\rm cube}$ stand for the family of all continuous injections from a perfect cube $C \subset \mathfrak{C}^\omega$ onto a set P from $\mathrm{Perf}(X)$. We consider each function $f \in \mathcal{F}_{\rm cube}$ from C onto P as a coordinate system imposed on P. We say that $P \in \mathrm{Perf}(X)$ is a *cube* if we consider it with (implicitly given) witness function $f \in \mathcal{F}_{\rm cube}$ onto P, and Q is a *subcube of a cube* $P \in \mathrm{Perf}(X)$ provided $Q = f[C]$, where $f \in \mathcal{F}_{\rm cube}$ is the witness function for P and $C \subset \mathrm{dom}(f) \subset \mathfrak{C}^\omega$ is a perfect cube. Here and in what follows symbol $\mathrm{dom}(f)$ stands for the domain of f.

We say that a family $\mathcal{E} \subset \mathrm{Perf}(X)$ is *cube dense* in $\mathrm{Perf}(X)$ provided every cube $P \in \mathrm{Perf}(X)$ contains a subcube $Q \in \mathcal{E}$. More formally, $\mathcal{E} \subset \mathrm{Perf}(X)$ is cube dense provided

$$\forall f \in \mathcal{F}_{\rm cube}\ \exists g \in \mathcal{F}_{\rm cube}\ (g \subset f \ \&\ \mathrm{range}(g) \in \mathcal{E}). \tag{1}$$

It is easy to see that the notion of cube density is a generalization of a notion of density with respect to $\langle \mathrm{Perf}(X), \subseteq\rangle$, that is, if $\mathcal{E}$ is cube dense in $\mathrm{Perf}(X)$ then $\mathcal{E}$ is dense in $\mathrm{Perf}(X)$. On the other hand, the converse implication is not true, as shown by the following simple example.

Example 1.1 ([5, 6]) *Let $X = \mathfrak{C}\times\mathfrak{C}$ and let $\mathcal{E}$ be the family of all $P \in \mathrm{Perf}(X)$ such that either all vertical sections of P are countable, or else all horizontal sections of P are countable. Then $\mathcal{E}$ is dense in $\mathrm{Perf}(X)$, but it is not cube dense in $\mathrm{Perf}(X)$.*

It is also worth to notice that in order to check that $\mathcal{E}$ is cube dense it is enough to consider in condition (1) only functions f defined on the entire space $\mathfrak{C}^\omega$, that is

Fact 1.2 ([4, 5, 6]) *$\mathcal{E} \subset \mathrm{Perf}(X)$ is cube dense if and only if*

$$\forall f \in \mathcal{F}_{\rm cube},\ \mathrm{dom}(f) = \mathfrak{C}^\omega,\ \exists g \in \mathcal{F}_{\rm cube}\ (g \subset f \ \&\ \mathrm{range}(g) \in \mathcal{E}). \tag{2}$$

Let $\mathrm{Perf}^*(X)$ stand for the family of all sets P such that either $P \in \mathrm{Perf}(X)$ or P is a singleton in X. In what follows we will consider singletons as *constant cubes*, that is, with the constant coordinate function from $\mathfrak{C}^\omega$ onto the singleton. In particular, a subcube of a constant cube is the same singleton.

Consider the following game $\mathrm{GAME}_{\rm cube}(X)$ of length ω_1. The game has two players, Player I and Player II. At each stage $\xi < \omega_1$ of the game Player I can play an arbitrary cube $P_\xi \in \mathrm{Perf}^*(X)$ and Player II must respond with a subcube Q_ξ of P_ξ. The game $\langle\langle P_\xi, Q_\xi\rangle \colon \xi < \omega_1\rangle$ is won by Player I provided

$$\bigcup_{\xi<\omega_1} Q_\xi = X;$$

otherwise the game is won by Player II.

By a strategy for Player II we will understand any function S such that $S(\langle\langle P_\eta,Q_\eta\rangle\colon\eta<\xi\rangle,P_\xi)$ is a subcube of P_ξ, where $\langle\langle P_\eta,Q_\eta\rangle\colon\eta<\xi\rangle$ is any partial game. (We abuse here slightly the notation, since function S depends also on the implicitly given coordinate functions $f_\eta\colon\mathfrak{C}^\omega\to P_\eta$ making each P_η a cube.) A game $\langle\langle P_\xi,Q_\xi\rangle\colon\xi<\omega_1\rangle$ is played according to a strategy S for Player II provided $Q_\xi=S(\langle\langle P_\eta,Q_\eta\rangle\colon\eta<\xi\rangle,P_\xi)$ for every $\xi<\omega_1$. A strategy S for Player II is a *winning strategy* for Player II provided Player II wins any game played according to the strategy S.

Here is the axiom.

CPA$_{\rm cube}^{\rm game}$**:** $\mathfrak{c}=\omega_2$ and for any Polish space X Player II has no winning strategy in the game GAME$_{\rm cube}(X)$.

Notice that

Proposition 1.3 ([4, 6]) *Axiom* CPA$_{\rm cube}^{\rm game}$ *implies*

CPA$_{\rm cube}$**:** $\mathfrak{c}=\omega_2$ *and for every Polish space X and every cube dense family $\mathcal{E}\subset{\rm Perf}(X)$ there is an $\mathcal{E}_0\subset\mathcal{E}$ such that $|\mathcal{E}_0|\le\omega_1$ and $|X\setminus\bigcup\mathcal{E}_0|\le\omega_1$.*

In [4] (see also [6]) it was proved that CPA$_{\rm cube}$ (so, also CPA$_{\rm cube}^{\rm game}$) implies that ${\rm cof}(\mathcal{N})=\omega_1$ and that all perfectly meager sets and all universally null sets have cardinality at most ω_1.

In what follows we will also use the following simple fact. Its proof can be found in [5] and [6].

Claim 1.4 *Consider $\mathfrak{C}^\omega$ with standard topology and standard product measure. If G is a Borel subset of $\mathfrak{C}^\omega$ which is either of second category or of positive measure then G contains a perfect cube $\prod_{i<\omega}P_i$.*

2 Disjoint coverings by ω_1 null compacts

Theorem 2.1 *Assume that* CPA$_{\rm cube}^{\rm game}$ *holds and let X be a Polish space. If $\mathcal{D}\subset{\rm Perf}(X)$ is $\mathcal{F}_{\rm cube}$-dense and it is closed under perfect subsets then there exists a partition of X into ω_1 disjoint sets from $\mathcal{D}\cup\{\{x\}\colon x\in X\}$.*

In the proof we will use the following easy lemma.

Lemma 2.2 *Let X be a Polish space and let $\mathcal{P}=\{P_i\colon i<\omega\}\subset{\rm Perf}^*(X)$. For every cube $P\in{\rm Perf}(X)$ there exists a subcube Q of P such that either $Q\cap\bigcup_{i<\omega}P_i=\emptyset$ or $Q\subset P_i$ for some $i<\omega$.*

PROOF. Let $f\in\mathcal{F}_{\rm cube}$ be such that $f[\mathfrak{C}^\omega]=P$.

If $P\cap\bigcup_{i<\omega}P_i$ is meager in P then, by Claim 1.4, we can find a subcube Q of P such that $Q\subset P\setminus\bigcup_{i<\omega}P_i$.

If $P\cap\bigcup_{i<\omega}P_i$ is not meager in P then there exists an $i<\omega$ such that $P\cap P_i$ has a non-empty interior in P. Thus, there exists a basic clopen set C in $\mathfrak{C}^\omega$, which is a perfect cube, such that $f[C] \subset P_i$. So, $Q = f[C]$ is a desired subcube of P. ■

Proof of Theorem 2.1. For a cube $P \in \mathrm{Perf}(X)$ and a countable family $\mathcal{P} \subset \mathrm{Perf}^*(X)$ let $D(P) \in \mathcal{D}$ be a subcube of P and $Q(\mathcal{P}, P) \in \mathcal{D}$ be as in Lemma 2.2 used with $D(P)$ in place of P. For a singleton $P \in \mathrm{Perf}^*(X)$ we just put $Q(\mathcal{P}, P) = P$.

Consider the following strategy S for Player II:

$$S(\langle\langle P_\eta, Q_\eta\rangle : \eta < \xi\rangle, P_\xi) = Q(\{Q_\eta : \eta < \xi\}, P_\xi).$$

By CPA$_{\rm cube}^{\rm game}$ strategy S is not a winning strategy for Player II. So there exists a game $\langle\langle P_\xi, Q_\xi\rangle : \xi < \omega_1\rangle$ played according to S in which Player II looses, that is, $X = \bigcup_{\xi<\omega_1} Q_\xi$.

Notice that for every $\xi < \omega_1$ either $Q_\xi \cap \bigcup_{\eta<\xi} Q_\eta = \emptyset$ or there is an $\eta < \omega_1$ such that $Q_\xi \subset Q_\eta$. Let

$$\mathcal{F} = \left\{ Q_\xi : \xi < \omega_1 \ \& \ Q_\xi \cap \bigcup_{\eta<\xi} Q_\eta = \emptyset \right\}.$$

Then $\mathcal{F}$ is as desired. ■

Since a family of all measure zero perfect subsets of $\mathbb{R}^n$ is $\mathcal{F}_{\rm cube}$-dense we get the following corollary.

Corollary 2.3 CPA$_{\rm cube}^{\rm game}$ *implies that there exists a partition of* $\mathbb{R}^n$ *into* ω_1 *disjoint closed nowhere dense measure zero sets.*

Note that the conclusion of Corollary 2.3 does not follow from the fact that $\mathbb{R}^n$ can be covered by ω_1 perfect measure zero subsets. (See [10, thm. 6].)

3 Uncountable γ-sets

In this subsection we will prove that CPA$_{\rm cube}^{\rm game}$ implies the existence of an uncountable γ-set. Recall that a subset T of a Polish space X is a *γ-set* provided for every open ω-cover $\mathcal{U}$ of T there is a sequence $\langle U_n \in \mathcal{U} : n < \omega\rangle$ such that $T \subset \bigcup_{n<\omega} \bigcap_{i>n} U_i$, where $\mathcal{U}$ is an *ω-cover* of T if for every finite set $A \subset T$ there a $U \in \mathcal{U}$ with $A \subset U$.

γ-sets were introduced by Gerlits and Nagy [8]. They were studied by Galvin and Miller [7], Recław [12], Bartoszyński, Recław [2], and others. It is known that under the Martin's axiom there are γ-sets of cardinality continuum [7]. On the other hand, every γ-set is strong measure zero [8], so it is consistent with ZFC that every γ-set is countable. Moreover, CPA$_{\rm cube}^{\rm game}$ implies that every γ-set has cardinality at most $\omega_1 < \mathfrak{c}$, since every strong measure zero is universally null and under CPA$_{\rm cube}^{\rm game}$ every universally null has cardinality $\leq \omega_1$.

In what follows we will use the characterization of γ-sets due to Recław [12]. To formulate it we need to fix some terminology. Thus, in what follows we will consider $\mathcal{P}(\omega)$ as a Polish space by identifying it with 2^ω via characteristic functions. For $A, B \subset \omega$ we will write $A \subseteq^* B$ when $|A \setminus B| < \omega$. We say that a family $\mathcal{A} \subset \mathcal{P}(\omega)$ is *centered* provided $\bigcap \mathcal{A}_0$ is infinite for every finite $\mathcal{A}_0 \subset \mathcal{A}$; and $\mathcal{A}$ *has a pseudointersection* provided there exists a $B \in [\omega]^\omega$ such that $B \subseteq^* A$ for every $A \in \mathcal{A}$. In addition for the rest of this section $\mathcal{K}$ will stand for the family of all continuous functions from $\mathcal{P}(\omega)$ to $\mathcal{P}(\omega)$ and for $A \in \mathcal{P}(\omega)$ we put $A^* = \{B \in \mathcal{P}(\omega) \colon B \subseteq^* A\}$.

Proposition 3.1 (Recław [12]) *For $T \subset \mathcal{P}(\omega)$ the following conditions are equivalent.*

(i) *T is a γ-set.*

(ii) *For every $f \in \mathcal{K}$ if $f[T]$ is centered than $f[T]$ has a pseudointersection.*

In the proof that follows we will apply axiom CPA$^{\rm game}_{\rm cube}$ to the cubes from the space $\mathcal{K}$. The fact that the subcubes given by the axiom cover $\mathcal{K}$ will allow us to use the above characterization to conclude that the constructed set is indeed a γ-set. It is also possible to construct an uncountable γ-set by applying axiom CPA$^{\rm game}_{\rm cube}$ to the space $\mathcal{Y}$ of all ω-covers of $\mathcal{P}(\omega)$,[1] similarly as in Section 5. However, we believe that greater diversification of spaces to which we apply CPA$^{\rm game}_{\rm cube}$ makes the paper more interesting.

In what follows we will need the following two lemmas.

Lemma 3.2 *For every countable set $Y \subset \mathcal{P}(\omega)$ the set*

$$\mathcal{K}_Y = \{f \in \mathcal{K} \colon f[Y] \text{ is centered}\}$$

is Borel in $\mathcal{K}$.

Proof. Let $Y = \{y_i \colon i < \omega\}$ and note that

$$\mathcal{K}_Y = \bigcap_{n,k<\omega} \bigcup_{m \geq k} \bigcap_{i<n} \{f \in \mathcal{K} \colon m \in f(y_i)\}.$$

So, $\mathcal{K}_Y$ is a G_δ set, since each set $\{f \in \mathcal{K} \colon m \in f(y_i)\}$ is open in $\mathcal{K}$. ■

Lemma 3.3 *Let $Y \subset \mathcal{P}(\omega)$ be countable and such that $[\omega]^{<\omega} \subset Y$. For every $W \in [\omega]^\omega$ and a compact set $Q \subset \mathcal{K}_Y$ there exist $V \in [W]^\omega$ and a continuous function $\varphi \colon Q \to [\omega]^\omega$ such that $\varphi(f)$ is a pseudointersection of $f[Y] \cup f[V^*]$ for every $f \in Q$.*

Moreover, if $\mathcal{J}$ is an infinite family of non-empty pairwise disjoint finite subsets of W then we can choose V such that it contains infinitely many J's from $\mathcal{J}$.

[1] More precisely, if $\mathcal{B}_0$ is a countable base for $\mathcal{P}(\omega)$ and $\mathcal{B}$ is the collection of all finite unions of elements from $\mathcal{B}_0$ then we can define $\mathcal{Y}$ as $\mathcal{B}^\omega$ considered with the product topology, where $\mathcal{B}$ is taken with discrete topology.

PROOF. First notice that there exists a continuous $\psi\colon Q \to [\omega]^\omega$ such that $\psi(f)$ is a pseudointersection of $f[Y]$ for every $f \in Q$.

Indeed, let $Y = \{y_i\colon i < \omega\}$ and for every $f \in Q$ let $\psi(f) = \left\{n_i^f\colon i<\omega\right\}$, where $n_0^f = \min f(y_0)$ and $n_{i+1}^f = \min\left\{n \in \bigcap_{j\le i} f(y_j)\colon n > n_i^f\right\}$. The set in the definition of n_{i+1}^f is non-empty, since $f[Y]$ is centered, as $f \in Q \subset \mathcal{K}_Y$. It is easy to see that ψ is continuous and that $\psi(f)$ is as desired.

We will define a sequence $\langle J_i \in \mathcal{J}\colon i<\omega\rangle$ such that $\max J_i < \min J_{i+1}$ for every $i<\omega$. We are aiming for $V = \bigcup_{i<\omega} J_i$.

A set $J_0 \in \mathcal{J}$ is chosen arbitrarily. Now, if J_i is already defined for some $i<\omega$ we define J_{i+1} as follows. Let $w_i = 1+\max J_i$. Thus $J_i \subset w_i$. For every $f \in Q$ define

$$m_i^f = \min\left(\psi(f) \cap \bigcap f[\mathcal{P}(w_i)]\right).$$

The set $\psi(f)\cap\bigcap f[\mathcal{P}(w_i)]$ is infinite, since $\psi(f)$ is a pseudointersection of $f[Y]$ while $\mathcal{P}(w_i) \subset Y$. Let $k_i^f = \min K_i^f$, where

$$K_i^f = \left\{k \ge w_i\colon m_i^f \in f(a) \text{ for all } a \subset \omega \text{ with } a\cap k \subset w_i\right\}.$$

The fact that $K_i^f \ne \emptyset$ follows from the continuity of f since $m_i^f \in f(a)$ for all $a \subset w_i$. Notice that, by the continuity of ψ and the assignment of k_i^f, for every $p<\omega$ the set $U_p = \{f \in Q\colon k_i^f < p\}$ is open in Q. Since sets $\{U_p\colon p<\omega\}$ form an increasing cover of Q, compactness of Q implies the existence of $p_i<\omega$ such that $Q \subset U_{p_i}$. Thus, $w_i \le k_i^f < p_i$ for every $f \in Q$. We define J_{i+1} as an arbitrary element of $\mathcal{J}$ disjoint with p_i and notice that

$$m_i^f \in f(a) \text{ for every } f \in Q \text{ and } a \subset \omega \text{ with } a \cap \min J_{i+1} \subset w_i.$$

This finishes the inductive construction.

Let $V = \bigcup_{i<\omega} J_i \subset W$ and $\varphi(f) = \{m_i^f\colon i<\omega\}$. It is easy to see that φ is continuous (though, we will not use this fact). To finish the proof it is enough to show that $\varphi(f)$ is a pseudointersection of $f[Y]\cup f[V^*]$ for every $f \in Q$.

So, fix an $f \in Q$. Clearly $\varphi(f) \subset \psi(f)$ is a pseudointersection of $f[Y]$ since so was $\psi(f)$. To see that $\varphi(f)$ is a pseudointersection of $f[V^*]$ take an $a \subseteq^* V$. Then for almost all $i<\omega$ we have $a\cap\min J_{i+1} \subset w_i$, so that $m_i^f \in f(a)$. Thus $\varphi(f) \subseteq^* f(a)$. ■

Theorem 3.4 CPA$^{\rm game}_{\rm cube}$ *implies that there exists an uncountable γ-set in $\mathcal{P}(\omega)$.*

PROOF. For $\alpha<\omega_1$ and an $\subseteq^*$-decreasing sequence $\mathcal{V} = \{V_\xi \in [\omega]^\omega\colon \xi<\alpha\}$ let $W(\mathcal{V}) \in [\omega]^\omega$ be such that $W(\mathcal{V}) \subsetneq^* V_\xi$ for all $\xi<\alpha$. Moreover, if $P \in \mathrm{Perf}^*(\mathcal{K})$ is a cube then we define a subcube $Q = Q(\mathcal{V},P)$ of P and an infinite subset $V = V(\mathcal{V},P)$ of $W = W(\mathcal{V})$ as follows. Let $Y = \mathcal{V}\cup[\omega]^{<\omega}$ and choose a subcube Q of P such that either $Q\cap\mathcal{K}_Y = \emptyset$ or $Q \subset \mathcal{K}_Y$. This can be done by Claim 1.4 since $\mathcal{K}_Y$ is Borel. If $Q\cap\mathcal{K}_Y=\emptyset$ we put $V=W$. Otherwise we apply Lemma 3.3 to find V.

Consider the following strategy S for Player II:

$$S(\langle\langle P_\eta, Q_\eta\rangle : \eta < \xi\rangle, P_\xi) = Q(\{V_\eta : \eta < \xi\}, P_\xi),$$

where sets V_η are defined inductively by $V_\eta = V(\{V_\zeta : \zeta < \eta\}, P_\eta)$. In other words, Player II remembers (recovers) sets V_η associated with the cubes P_η played so far, and he uses them (and Lemma 3.3) to get the next answer $Q_\xi = Q(\{V_\eta : \eta < \xi\}, P_\xi)$, while remembering (or recovering each time) the set $V_\xi = V(\{V_\eta : \eta < \xi\}, P_\xi)$.

By CPA$^{\rm game}_{\rm cube}$ strategy S is not a winning strategy for Player II. So there exists a game $\langle\langle P_\xi, Q_\xi\rangle : \xi < \omega_1\rangle$ played according to S in which and Player II loses, that is, $\mathcal{K} = \bigcup_{\xi<\omega_1} Q_\xi$. Let $\mathcal{V} = \{V_\xi : \xi < \omega_1\}$ be a sequence associated with this game, which is strictly $\subseteq^*$-decreasing, and let $T = \mathcal{V} \cup [\omega]^{<\omega}$. We claim that T is a γ-set.

In the proof we use Lemma 3.2. So, let $f \in \mathcal{K}$ be such that $f[T]$ is centered. There exists an $\alpha < \omega_1$ such that $f \in Q_\alpha$. Since $f[\{V_\xi : \xi < \alpha\} \cup [\omega]^{<\omega}] \subset f[T]$ we must have applied Lemma 3.3 in the choice of Q_α and V_α. Therefore, the family $f[\{V_\xi : \xi < \alpha\} \cup [\omega]^{<\omega} \cup V^*_\alpha]$ has a pseudointersection. So, $f[T]$ has a pseudointersection too, since $T \subset \{V_\xi : \xi < \alpha\} \cup [\omega]^{<\omega} \cup V^*_\alpha$. ■

Since $\mathcal{P}(\omega)$ embeds into any Polish space, we conclude that, under CPA$^{\rm game}_{\rm cube}$, any Polish space contains an uncountable γ-set. In particular, there exists an uncountable γ-set $T \subset \mathbb{R}$.

4 γ-sets in $\mathbb{R}$ which are not strongly meager

Recall (see e.g. [1, p. 437]) that a subset X of $\mathbb{R}$ is *strongly meager* provided $X + G \neq \mathbb{R}$ for every measure zero subset G of $\mathbb{R}$. This is a notion which is dual to a strong measure zero subset of $\mathbb{R}$, since Galvin, Mycielski, and Solovay proved (see e.g. [1, p. 405]) that: $X \subset \mathbb{R}$ is strong measure zero if and only if $X + M \neq \mathbb{R}$ for every meager subset M of $\mathbb{R}$.

Now, although every γ-set is strong measure zero, under the Martin's axiom Bartoszyński and Recław [2] constructed a γ-set T in $\mathbb{R}$ which is not strongly meager. In what follows we will show that the existence of such a set follows also from CPA$^{\rm game}_{\rm cube}$. The construction is a generalization of that used in the proof of Theorem 3.4.

In the proof we will use the following notation. For $A, B \subset \omega$ we define $A + B$ as the symmetric difference between A and B. Upon identification of a set $A \subset \omega$ with its characteristic function $\chi_A \in 2^\omega$ this definition is motivated by the fact that $\chi_{A+B}(n) = \chi_A(n) +_2 \chi_B(n)$, where $+_2$ is the addition modulo 2. Also, let $\bar{\mathcal{J}} = \{J_n \in [\omega]^{2^n} : n < \omega\}$ be a family of pairwise disjoint sets and let $\tilde{G}$ be the family of all $W \subset \omega$ which are disjoint with infinitely many $J \in \bar{\mathcal{J}}$. Notice that $\tilde{G}$ has measure zero with respect to the standard measure on $\mathcal{P}(\omega)$ induced by the product measure on 2^ω.

Lemma 4.1 *If $\mathcal{J} \in [\bar{\mathcal{J}}]^\omega$ and P is a cube in $\mathcal{P}(\omega)$ then there exists a subcube Q of P and a set $V \subset \bigcup \mathcal{J}$ containing infinitely many $J \in \mathcal{J}$ such that $V + Q \subset \tilde{G}$.*

Proof. Let $D=\bigcup\mathcal{J}$ and

$$\begin{aligned} H &= \{\langle U,W\rangle\in\mathcal{P}(D)\times\mathcal{P}(\omega)\colon (U+W)\cap J=\emptyset \text{ for infinitely many } J\in\mathcal{J}\}\\ &\subseteq \{\langle U,W\rangle\in\mathcal{P}(D)\times\mathcal{P}(\omega)\colon U+W\in\tilde{G}\}. \end{aligned}$$

Note that H is a G_δ subset of $\mathcal{P}(D)\times\mathcal{P}(\omega)$ since $H_J=\{\langle U,W\rangle\colon (U+W)\cap J=\emptyset\}$ is open for every $J\in\mathcal{J}$. Moreover horizontal sections of H are dense in $\mathcal{P}(D)$. So, $\bar{H}=H\cap(\mathcal{P}(D)\times P)$ is a dense G_δ subset of $\mathcal{P}(D)\times P$, as all its horizontal sections are dense. Thus, by Kuratowski-Ulam theorem, there is a dense G_δ subset $\mathcal{K}_0$ of $\mathcal{P}(D)$ such that for every $U\in\mathcal{K}_0$ the vertical section $\bar{H}_U$ of $\bar{H}$ is dense in P. Now, since

$$\mathcal{K}_1=\{U\in\mathcal{P}(D)\colon J\subset U \text{ for infinitely many } J\in\mathcal{J}\}$$

is a dense G_δ there is a $V\in\mathcal{K}_0\cap\mathcal{K}_1$. In particular, V contains infinitely many $J\in\mathcal{J}$ and $\bar{H}_V$ is a dense G_δ subset of P. So, by Claim 1.4, there exists a subcube Q of P contained in $\bar{H}_V$. Thus, $Q\subset\bar{H}_V\subset\{W\in P\colon V+W\in\tilde{G}\}$ and so $V+Q\subset\tilde{G}$. ■

Theorem 4.2 CPA$^{\rm game}_{\rm cube}$ *implies that there exists a γ-set $T\subset\mathcal{P}(\omega)$ such that $T+\tilde{G}=\mathcal{P}(\omega)$.*

Proof. We will use CPA$^{\rm game}_{\rm cube}$ for the space $X=\mathcal{K}\cup\mathcal{P}(\omega)$, a direct sum of $\mathcal{K}$ and $\mathcal{P}(\omega)$.

For $\alpha<\omega_1$ and an $\subseteq^*$-decreasing sequence $\mathcal{V}=\{V_\xi\in[\omega]^\omega\colon\xi<\alpha\}$ such that each V_ξ contains infinitely many $J\in\bar{\mathcal{J}}$ let $W(\mathcal{V})\in[\omega]^\omega$ be such that $\mathcal{J}=\{J\in\bar{\mathcal{J}}\colon J\subset W(\mathcal{V})\}$ is infinite and $W(\mathcal{V})\subsetneq^* V_\xi$ for all $\xi<\alpha$. For a cube $P\in\mathrm{Perf}^*(\mathcal{K})$ we define a subcube $Q=Q(\mathcal{V},P)$ of P and an infinite subset $V=V(\mathcal{V},P)$ of $W=W(\mathcal{V})$ as follows. By Claim 1.4 we can find subcube P' of P such that either $P'\subset\mathcal{K}$ or $P'\subset\mathcal{P}(\omega)$.

If $P'\subset\mathcal{K}$ we proceed as in the proof of Theorem 3.4. We put $Y=\mathcal{V}\cup[\omega]^{<\omega}$ and we use Claim 1.4 to find a subcube Q of P' such that either $Q\cap\mathcal{K}_Y=\emptyset$ or $Q\subset\mathcal{K}_Y$. If $Q\cap\mathcal{K}_Y=\emptyset$ we put $V=W$. Otherwise we apply Lemma 3.3 to find V. If $P'\subset\mathcal{P}(\omega)$ we use Lemma 4.1 to find Q and V.

Consider the following strategy S for Player II:

$$S(\langle\langle P_\eta,Q_\eta\rangle\colon\eta<\xi\rangle,P_\xi)=Q(\{V_\eta\colon\eta<\xi\},P_\xi),$$

where sets V_η are defined inductively by $V_\eta=V(\{V_\zeta\colon\zeta<\eta\},P_\eta)$. By CPA$^{\rm game}_{\rm cube}$ strategy S is not a winning strategy for Player II. So there exists a game $\langle\langle P_\xi,Q_\xi\rangle\colon\xi<\omega_1\rangle$ played according to S in which and Player II loses, that is, $X=\bigcup_{\xi<\omega_1}Q_\xi$. Let $\mathcal{V}=\{V_\xi\colon\xi<\omega_1\}$ be a sequence associated with this game, which is strictly $\subseteq^*$-decreasing, and let $T=\mathcal{V}\cup[\omega]^{<\omega}$. We claim that T is as desired.

The argument that T is a γ-set is the same as in the proof of Theorem 3.4. To see that $\mathcal{P}(\omega)\subset T+\tilde{G}$ notice that for every $A\in\mathcal{P}(\omega)$ there is an $\alpha<\omega_1$ such that $A\in Q_\alpha$. But then at step α we used Lemma 4.1 to find Q_α and V_α. In particular, $V_\alpha+Q_\alpha\subset\tilde{G}$. So, $A\in Q_\alpha\subset V_\alpha+\tilde{G}\subset T+\tilde{G}$. ■

Corollary 4.3 CPA$^{\rm game}_{\rm cube}$ *implies that there exists a γ-set $X \subset \mathbb{R}$ which is not strongly meager.*

Proof. This is the argument from [2]. Let T be as in Theorem 4.2 and let $f\colon \mathcal{P}(\omega) \to [0,1]$, $f(A) = \sum_{i<\omega} 2^{-(i+1)}\chi_A(i)$. Then f is continuous, so $X = f[T]$ is a γ-set. Let $H = \bigcap_{m<\omega} \bigcup_{n>m} f[J_n]$. Then H has measure zero and it is easy to see that $[0,1] = f[\mathcal{P}(\omega)] \subset f[T] + H = X + H$. Then $\bar{G} = H + \mathbb{Q}$ has measure zero and $X + \bar{G} = \mathbb{R}$. ■

5 Uncountable strongly meager γ-sets in $\mathbb{R}$

Let X be a Polish space with topology τ. We say that $\mathcal{U} \subset \tau$ is a cover of $Z \subset [X]^{<\omega}$ provided for every $A \in Z$ there is a $U \in \mathcal{U}$ with $A \subset U$. Following [7] we say that a subset S of X is a *strong γ-set* provided there exists an increasing sequence $\langle k_n < \omega \colon n < \omega \rangle$ such that for every sequence $\langle J_n \subset \tau \colon n < \omega \rangle$, where each J_n is a cover of $[X]^{k_n}$, there exists a sequence $\langle D_n \in J_n \colon n < \omega \rangle$ with $X \subset \bigcup_{n<\omega} \bigcap_{m>n} D_m$. It is proved in [7] that every strong γ-set $X \subset \mathbb{R}$ is strongly meager. The goal of this section is to construct, under CPA$^{\rm game}_{\rm cube}$, an uncountable strong γ-set in $\mathcal{P}(\omega)$. So, after identifying $\mathcal{P}(\omega)$ with its homeomorphic copy in $\mathbb{R}$, this will become an uncountable γ-set in $\mathbb{R}$ which is strongly meager. Under Martin's axiom a strong γ-set in $\mathcal{P}(\omega)$ of cardinality continuum exists, see [7].

Let $\mathcal{B}_0$ be a countable basis for the topology of $\mathcal{P}(\omega)$ and let $\mathcal{B}$ be the collection of all finite unions of elements from $\mathcal{B}_0$. Since every open cover of $[\mathcal{P}(\omega)]^k$, $k < \omega$, contains a refinement from $\mathcal{B}$, in the definition of strong γ-set it is enough to consider only sequences $\langle J_n \colon n < \omega \rangle$ with $J_n \subset \mathcal{B}$.

Now, consider $\mathcal{B}$ with the discrete topology. Since $\mathcal{B}$ is countable, the space $\mathcal{B}^\omega$, considered with the product topology, is a Polish space and so is $\mathcal{X} = (\mathcal{B}^\omega)^\omega$. For $J \in \mathcal{X}$ we will write J_n in place of $J(n)$. It is easy to see that a subbasis for the topology of $\mathcal{X}$ is given for the clopen sets

$$\{J \in \mathcal{X} \colon J_n(m) = B\},$$

where $n, m < \omega$ and $B \in \mathcal{B}$.

For the reminder of this section fix an increasing sequence $\langle k_n < \omega \colon n < \omega \rangle$ such that $k_n \geq n\,2^n + n$ for every $n < \omega$. Then we have the following lemma.

Lemma 5.1 *Let $X \in [\omega]^\omega$ and let F be a countable subset of $\mathcal{P}(\omega)$ such that $[\omega]^{<\omega} \subset F$. Assume that P is a compact subset of $\mathcal{X}$ such that for every $J \in P$ and $n < \omega$ the family $J_n[\omega] = \{J_n(m) \colon m < \omega\}$ covers $[F]^{k_n}$. Then there exists a set $Y \in [X]^\omega$ and for each $J \in P$ a sequence $\langle D^J_n \in J_n \colon n < \omega \rangle$ such that $F \cup Y^* \subset \bigcup_{n<\omega} \bigcap_{m>n} D^J_m$.*

Proof. Let $\{F_n \colon n < \omega\}$ be an enumeration of $[\omega]^{<\omega}$ such that $F_n \subset n$ for all $n < \omega$ and let $F = \{f_n \colon n < \omega\}$. We will construct inductively the sequences $\langle s_n \in X \colon n < \omega \rangle$ and $\langle \{D^J_n \in J_n[\omega] \colon J \in P\} \colon n < \omega \rangle$ such that for every $n < \omega$, $J \in P$, and $A \subset \omega$ we have

(i) $\{f_i\colon i<n\}\subset D^J_n$ and $s_n<s_{n+1}$;

(ii) if $i<j\leq n+1$ and $(A\cap s_{n+1})\setminus\{s_0,\ldots,s_n\}=F_i$ then $A\in D^J_j$.

We chose $s_0\in X$ and $\{D^J_n\in J_n[\omega]\colon J\in P\}$ arbitrarily. Then conditions (i) and (ii) are trivially satisfied. Next, assume that the sequence $\{s_i\colon i\leq n\}$ is already constructed. We will construct s_{n+1} and sets D^J_{n+1} as follows.

Let
$$Q=\{q\in[\omega]^{<\omega}\colon q\setminus\{s_0,\ldots,s_n\}=F_i \ \text{ for some } \ i\leq n\}.$$
Then $|Q|\leq(n+1)\,2^{n+1}$ and $|Q\cup\{f_0,\ldots,f_n\}|\leq k_{n+1}$.

Fix $J\in P$. Since $J_{n+1}[\omega]$ covers $[F]^{\leq k_{n+1}}$, there exists a $\bar D^J_{n+1}\in J_{n+1}[\omega]$ containing $Q\cup\{f_0,\ldots,f_n\}$. Since $\bar D^J_{n+1}$ is open and covers finite set Q, there is an $s^J_{n+1}>s_n$ in X such that for every $q\in Q$
$$\{x\subset\omega\colon x\cap s^J_{n+1}=q\cap s^J_{n+1}\}\subset\bar D^J_{n+1}.$$
Notice that

($*$) for every $A\subset\omega$ and $\bar s_{n+1}\geq s^J_{n+1}$ condition (ii) holds.

Indeed, assume that $(A\cap\bar s_{n+1})\setminus\{s_0,\ldots,s_n\}=F_i$ for some $i<j\leq n+1$. If $j\leq n$ then $n\geq 1$ and since $F_i\subset i\subset s_{n-1}$ we have
$$(A\cap s_n)\setminus\{s_0,\ldots,s_{n-1}\}=(A\cap\bar s_{n+1})\setminus\{s_0,\ldots,s_n\}=F_i.$$
So, by the inductive assumption, $A\in D^J_j$. If $j=n+1$ then $q=A\cap\bar s_{n+1}\in Q$. So $A\in\{x\subset\omega\colon x\cap\bar s_{n+1}=q\cap\bar s_{n+1}\}\subset\{x\subset\omega\colon x\cap s^J_{n+1}=q\cap s^J_{n+1}\}\subset\bar D^J_{n+1}$, finishing the proof of ($*$).

For each $J\in P$ let $m^J<\omega$ be such that $J_{n+1}(m^J)=\bar D^J_{n+1}$ and define $U_J=\{K\in\mathcal{X}\colon K_{n+1}(m^J)=\bar D^J_{n+1}\}$. Then U_J is an open neighborhood of J. In particular, $\{U_J\colon J\in P\}$ is an open cover of a compact set P, so there exists a finite $P_0\subset P$ such that $P\subset\bigcup\{U_{\bar J}\colon\bar J\in P_0\}$. Choose $s_{n+1}\in X$ such that $s_{n+1}\geq\max\{s^{\bar J}_{n+1}\colon\bar J\in P_0\}$. Moreover, for every $J\in P$ choose $\bar J\in P_0$ such that $J\in U_{\bar J}$ and define $D^J_{n+1}=\bar D^{\bar J}_{n+1}$. It is easy to see that, by ($*$), conditions (i) and (ii) are preserved. This completes the inductive construction.

Put $Y=\{s_n\colon n<\omega\}$. To see that it satisfies the lemma pick an arbitrary $J\in P$. We will show that $F\cup Y^*\subset\bigcup_{n<\omega}\bigcap_{m>n}D^J_m$.

Clearly $F\subset\bigcup_{n<\omega}\bigcap_{m>n}D^J_m$ since, by (i), $f_n\in D^J_m$ for every $m>n$. So, fix an $A\in Y^*$. Then $A\setminus Y=F_i$ for some $i<\omega$. Let $n<\omega$ be such that $i<n$ and $s_n>\max F_i$. Then for every $m>n$ we have $i<m\leq m+1$ and $(A\cap s_{m+1})\setminus\{s_0,\ldots,s_m\}=F_i$. So, by (ii), we have $A\in D^J_m$ for every $m>n$. Thus, $A\in\bigcap_{m>n}D^J_m$. ■

Lemma 5.2 *If $F\subset\mathcal{P}(\omega)$ is countable then the set*
$$\mathcal{X}_F=\{J\in\mathcal{X}\colon J_n[\omega] \text{ covers } [F]^{k_n} \text{ for every } n<\omega\}$$
is Borel in $\mathcal{X}$.

PROOF. This follows from the fact that

$$\mathcal{X}_F=\bigcap_{n<\omega}\ \bigcap_{A\in[F]^{k_n}}\ \bigcup_{m<\omega}\ \bigcup_{A\subset B\in\mathcal{B}}\{J\in\mathcal{X}\colon J_n(m)=B\}$$

since each set $\{J\in\mathcal{X}\colon J_n(m)=B\}$ is clopen in $\mathcal{X}$. Thus, $\mathcal{X}_F$ is a G_δ-set. ■

Theorem 5.3 CPA$^{\rm game}_{\rm cube}$ *implies that there exists an uncountable strong γ-set in $\mathcal{P}(\omega)$.*

PROOF. For $\alpha<\omega_1$ and an $\subseteq^*$-decreasing sequence $\mathcal{V}=\{V_\xi\in[\omega]^\omega\colon\xi<\alpha\}$ let $W(\mathcal{V})\in[\omega]^\omega$ be such that $W(\mathcal{V})\subsetneq^* V_\xi$ for all $\xi<\alpha$. Moreover, if $P\in\mathrm{Perf}^*(\mathcal{X})$ is a cube then we define a subcube $Q=Q(\mathcal{V},P)$ of P and an infinite subset $Y=V(\mathcal{V},P)$ of $X=W(\mathcal{V})$ as follows. Let $F=\mathcal{V}\cup[\omega]^{<\omega}$ and choose a subcube Q of P such that either $Q\cap\mathcal{X}_F=\emptyset$ or $Q\subset\mathcal{X}_F$. This can be done by Claim 1.4 since $\mathcal{X}_F$ is Borel. If $Q\cap\mathcal{X}_F=\emptyset$ we put $Y=X$. Otherwise we apply Lemma 5.1 to find Y.

Consider the following strategy S for Player II:

$$S(\langle\langle P_\eta,Q_\eta\rangle\colon\eta<\xi\rangle,P_\xi)=Q(\{V_\eta\colon\eta<\xi\},P_\xi),$$

where sets V_η are defined inductively by $V_\eta=V(\{V_\zeta\colon\zeta<\eta\},P_\eta)$. By CPA$^{\rm game}_{\rm cube}$ strategy S is not a winning strategy for Player II. So there exists a game $\langle\langle P_\xi,Q_\xi\rangle\colon\xi<\omega_1\rangle$ played according to S in which and Player II loses, that is, $\mathcal{X}=\bigcup_{\xi<\omega_1}Q_\xi$. Let $\mathcal{V}=\{V_\xi\colon\xi<\omega_1\}$ be a sequence associated with this game, which is strictly $\subseteq^*$-decreasing, and let $T=\mathcal{V}\cup[\omega]^{<\omega}$. We claim that T is a strong γ-set.

Indeed, let $\langle\mathcal{U}_n\subset\mathcal{B}\colon n<\omega\rangle$ be such that $\mathcal{U}_n$ covers $[T]^{k_n}$ for every $n<\omega$. Then there is a $J\in\mathcal{X}$ such that $J_n[\omega]=\mathcal{U}_n$ for every $n<\omega$. Let $\alpha<\omega_1$ be such that $J\in Q_\alpha$. Then $J\in\mathcal{X}_{\{V_\eta\colon\eta<\alpha\}\cup[\omega]^{<\omega}}$, so we must have used Lemma 5.1 to get Q_α. In particular, there is a sequence $\langle D^J_n\in J_n[\omega]=\mathcal{U}_n\colon n<\omega\rangle$ such that $([\omega]^{<\omega}\cup\{V_\eta\colon\eta<\alpha\})\cup(V_\alpha)^*\subset\bigcup_{n<\omega}\bigcap_{m>n}D^J_m$. So, $T\subset\bigcup_{n<\omega}\bigcap_{m>n}D^J_m$, as $\{V_\eta\colon\alpha\le\eta<\omega_1\}\subset(V_\alpha)^*$. ■

Since every homeomorphic image of a strong γ-set is evidently a strong γ-set, we obtain immediately the following conclusion.

Corollary 5.4 CPA$^{\rm game}_{\rm cube}$ *implies that there exists an uncountable γ-set in $\mathbb{R}$ which is strongly meager.*

It is worth to mention that a construction of an uncountable strong γ-set in $\mathcal{P}(\omega)$ under CPA$^{\rm game}_{\rm cube}$ can be also done in a formalism similar to that used in Section 3. In order to do it, we need the following definitions and facts. For a fixed sequence $\bar{k}=\langle k_n<\omega\colon n<\omega\rangle$ we say that $\mathcal{A}\subset(\mathcal{P}(\omega))^\omega$ is *$\bar{k}$-centered* provided for every $n<\omega$ any k_n-many sets from $\{A(n)\colon A\in\mathcal{A}\}$ have a common point; $B\in\omega^\omega$ is a *quasi-intersection* of $\mathcal{A}\subset(\mathcal{P}(\omega))^\omega$ provided for every $A\in\mathcal{A}$ there is infinitely many $n<\omega$ with $B(n)\in A(n)$. Now, if $\mathcal{K}^*$ is a family of all continuous functions from $\mathcal{P}(\omega)$ to $(\mathcal{P}(\omega))^\omega$ then the following is true:

A set $X \subset \mathcal{P}(\omega)$ is a strong γ-set if and only if there exists an increasing sequence $\bar{k} = \langle k_n < \omega \colon n < \omega \rangle$ such that for every $f \in \mathcal{K}^*$ if $f[X]$ is $\bar{k}$-centered then $f[X]$ has a quasi-intersection.

With this characterization in hand we can construct an uncountable strong γ-set in $\mathcal{P}(\omega)$ by applying CPA$_{\text{cube}}^{\text{game}}$ to the space $\mathcal{K}^*$.

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