Nice Hamel bases under the Covering Property Axiom

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Abstract

In the paper we prove that axiom $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$, which follows from the Covering Property Axiom CPA and holds in the iterated perfect set model, implies that there exists a Hamel basis which is a union of less than continuum many pairwise disjoint perfect sets. We will also give two consequences of this last fact.

1 The result and its consequences

In this paper we will use standard set theoretic terminology as in [2]. We will consider the real line $\mathbb{R}$ as a linear space over the rationals $\mathbb{Q}$. Any linear base of this space will be referred to as a *Hamel base*. For $A \subset \mathbb{R}$ we will write $\mathrm{LIN}(A)$ to denote the linear subspace of $\mathbb{R}$ spanned by A.

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Axiom $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ was introduced by the authors in [5], where it is shown that it holds in the iterated perfect set model. Also, $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ is a version of the axiom CPA which is described in a monograph [6].

It is known that $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ captures, to a big extend, the essence of the iterated perfect set model. This follows from a resent result of J. Zapletal [13] who proved that for a "nice" cardinal invariant κ if $\kappa < \mathfrak{c}$ holds in any forcing extension than $\kappa < \mathfrak{c}$ follows already from $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$.

For the reader convenience, we will restate $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$, along with necessary definitions, in the next section.

The main result of this paper is the following theorem.

Theorem 1.1 $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ *implies that there exists a family $\mathcal{H}$ of ω_1 pairwise disjoint perfect subsets of $\mathbb{R}$ such that $H = \bigcup \mathcal{H}$ is a Hamel basis.*

This theorem will be proved in the following sections. For the rest of this section we will discuss its two consequences.

Let $\mathcal{I}$ be a translation invariant ideal on $\mathbb{R}$. We say that a subset X of $\mathbb{R}$ is *$\mathcal{I}$-rigid* provided $X, \mathbb{R} \setminus X \notin \mathcal{I}$ but $X \triangle (r + X) \in \mathcal{I}$ for every $r \in \mathbb{R}$. An easy inductive construction gives a non-measurable subset X of $\mathbb{R}$ without the Baire property which is $[\mathbb{R}]^{<\mathfrak{c}}$-rigid. (First such a construction, under CH, comes from Sierpiński [12]. Compare also [8].) Thus, under CH or MA there are $\mathcal{N} \cap \mathcal{M}$-rigid sets, where $\mathcal{N}$ and $\mathcal{M}$ stand for the ideals of measure zero and of the ideal meager subsets of $\mathbb{R}$, respectively. Recently these sets have been studied by Laczkovich [11] and Cichoń, Jasiński, Kamburelis, and Szczepaniak [1]. In particular, Laczkovich [11, Theorem 2] implies that there is no $\mathcal{N} \cap \mathcal{M}$-rigid set in the random and Cohen models. The next corollary shows that the existence of such sets follows from $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$.

Corollary 1.2 $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ *implies there exists an $\mathcal{N} \cap \mathcal{M}$-rigid set X which is neither measurable nor has it the Baire property.*

Proof. Let $\mathcal{H} = \{Q_\xi : \xi < \omega_1\}$ be from Theorem 1.1 and for every $\xi < \omega_1$ let $L_\xi = \mathrm{LIN}\left(\bigcup_{\eta<\xi} Q_\eta\right)$. Then $\mathbb{R}$ is an increasing union of L_ξ's and each L_ξ belongs to $\mathcal{N} \cap \mathcal{M}$, since it is a proper Borel subgroup of $\mathbb{R}$.

Since, under $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$, the cofinalities of the ideals $\mathcal{N}$ and $\mathcal{M}$ is equal to ω_1 (see [4] or [6]), there is a family $\{C_\xi : \xi < \omega_1\}$ such that every $S \in \mathcal{M} \cup \mathcal{N}$ is a subset of some C_ξ. By induction choose $X_0 = \{x_\xi : \xi < \omega_1\} \subset \mathbb{R}$ such that

$$x_\xi \notin C_\xi \cup \mathrm{LIN}(L_\xi \cup \{x_\zeta : \zeta < \xi\}).$$

Then X_0 intersects the complement of every set from $\mathcal{M}\cup\mathcal{N}$. Define

$$X=\bigcup_{\xi<\omega_1}(x_\xi+L_\xi)$$

and notice that $X_0\subset X$ and $2\,X_0\subset\mathbb{R}\setminus X$. Thus, both X and $\mathbb{R}\setminus X$ intersects the complement of every set from $\mathcal{M}\cup\mathcal{N}$. In particular, $X,\mathbb{R}\setminus X\notin\mathcal{M}\cup\mathcal{N}$.

Next notice that for every $r\in L_\zeta$

$$X\triangle(r+X)\subset\bigcup_{\xi<\zeta}[(x_\xi+L_\xi)\cup(r+x_\xi+L_\xi)]\in\mathcal{N}\cap\mathcal{M}.$$

Thus, X is $\mathcal{N}\cap\mathcal{M}$-rigid, but also $\mathcal{N}$-rigid and $\mathcal{M}$-rigid. These last two facts imply that X is neither measurable nor does it have the Baire property. ■

Our second application of Theorem 1.1 is the following result.

Corollary 1.3 $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ *implies there exists a function* $f\colon\mathbb{R}\to\mathbb{R}$ *such that for every* $h\in\mathbb{R}$ *the difference function* $\Delta_h(x)=f(x+h)-f(x)$ *is Borel; however, for every* $\alpha<\omega_1$ *there is an* $h\in\mathbb{R}$ *such that* Δ_h *is not of Borel class* α*.*

Note that answering a question of Laczkovich [10] Filipów and Recław [7] gave an example of such an f under CH. Recław also asked (private communication) whether such a function can be constructed in absence of CH. Corollary 1.3 gives an affirmative answer to this question. It is an open question whether such a function exists in ZFC.

PROOF. The proof is quite similar to that for Corollary 1.2.

Let $\mathcal{H}=\{Q_\xi\colon\xi<\omega_1\}$ be from Theorem 1.1. For every $\xi<\omega_1$ define $L_\xi=\mathrm{LIN}\left(\bigcup_{\eta<\xi}Q_\eta\right)$ and choose a Borel subset B_ξ of Q_ξ of Borel class greater than ξ. Define

$$X=\bigcup_{\xi<\omega_1}(B_\xi+L_\xi)$$

and let f be the characteristic function χ_X of X.

To see that f is as required note that

$$\Delta_{-h}(x)=\left[\chi_{(h+X)\setminus X}-\chi_{X\setminus(h+X)}\right](x).$$

So, it is enough to show that each of the sets $(h+X)\setminus X$ and $X\setminus(h+X)$ is Borel, though they can be of arbitrary high class. For this, notice that for every $h\in L_{\alpha+1}\setminus L_\alpha$ we have

$$h+X=h+\bigcup_{\xi<\omega_1}(B_\xi+L_\xi)=\bigcup_{\xi\le\alpha}(h+B_\xi+L_\xi)\cup\bigcup_{\alpha<\xi<\omega_1}(B_\xi+L_\xi)$$

and that the sets $\bigcup_{\xi\le\alpha}(h+B_\xi+L_\xi)\subset L_{\alpha+1}$ and $\bigcup_{\alpha<\xi<\omega_1}(B_\xi+L_\xi)$ are disjoint. So

$$(h+X)\setminus X=\bigcup_{\xi\le\alpha}(h+B_\xi+L_\xi)\setminus X=\bigcup_{\xi\le\alpha}(h+B_\xi+L_\xi)\setminus\bigcup_{\xi\le\alpha}(B_\xi+L_\xi)$$

is Borel, since each set $B_\xi+L_\xi$ is Borel. (It is a subset of $Q_\xi+L_\xi$, which is homeomorphic to $Q_\xi\times L_\xi$ via addition function.) Similarly, set $X\setminus(h+X)$ is Borel.

Finally notice that for $h\in Q_\alpha\setminus B_\alpha$ the set

$$(h+X)\setminus X=\bigcup_{\xi\le\alpha}(h+B_\xi+L_\xi)$$

is of Borel class greater than α, since so is $(h+Q_\alpha)\cap[(h+X)\setminus X]=h+B_\alpha$. Thus, $\Delta_h(x)$ can be of an arbitrarily high Borel class. ∎

2 $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ and how it implies the theorem

In what follows the Cantor set 2^ω will be denoted by a symbol $\mathfrak{C}$. For a Polish space X (i.e., a complete separable metric space) $\mathrm{Perf}(X)$ will stand for a collection of all subsets of X homeomorphic to the Cantor set $\mathfrak{C}$.

The main notion behind a formulation of a $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ is that of a prism in a Polish space X and of its subprism. A *prism* in X is a perfect set $P\in\mathrm{Perf}(X)$ which comes with (implicitly given) coordinate system, that is, a homeomorphism from $\mathfrak{C}^\alpha$, $0<\alpha<\omega_1$, onto P. If P is a prism with a coordinate function $f\colon\mathfrak{C}^\alpha\to P$ then its *subprism* is any set of the form $f[E]$, where E is an *iterated perfect set*, that is, it belongs to the family $\mathbb{P}_\alpha$ to be defined latter.

In addition, we consider every singleton as a (trivial) prism, whose only subprism is itself. We also define $\mathrm{Perf}^*(X)$ as a family of all sets P such that either $P\in\mathrm{Perf}(X)$ or P is a singleton.

$\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ is expressed in terms of the following game $\mathrm{GAME}_{\mathrm{prism}}(X)$ of length ω_1. The game has two players, Player I and Player II. At each stage $\xi < \omega_1$ of the game Player I plays a prism $P_\xi \in \mathrm{Perf}^*(X)$ and Player II must respond with a subprism Q_ξ of P_ξ. The game $\langle\langle P_\xi, Q_\xi\rangle : \xi < \omega_1\rangle$ is won by Player I provided

$$\bigcup_{\xi<\omega_1} Q_\xi = X;$$

otherwise the game is won by Player II.

By a strategy for Player II we will understand any function S such that $S(\langle\langle P_\eta, Q_\eta\rangle : \eta < \xi\rangle, P_\xi)$ is a subprism of P_ξ, where $\langle\langle P_\eta, Q_\eta\rangle : \eta < \xi\rangle$ is any partial game. (We abuse here slightly the notation, since function S depends also on the implicitly given coordinate functions making each P_η a prism.) A game $\langle\langle P_\xi, Q_\xi\rangle : \xi < \omega_1\rangle$ is played according to a strategy S for Player II provided $Q_\xi = S(\langle\langle P_\eta, Q_\eta\rangle : \eta < \xi\rangle, P_\xi)$ for every $\xi < \omega_1$. A strategy S for Player II is a *winning strategy* for Player II provided Player II wins any game played according to the strategy S.

Now, we can formulate the axiom.

$\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$: $\mathfrak{c} = \omega_2$ and for any Polish space X Player II has no winning strategy in the game $\mathrm{GAME}_{\mathrm{prism}}(X)$.

Now, Theorem 1.1 follows quite easily form the axiom and the following lemma, which proof will take the reminder of this paper.

Lemma 2.1 *Let $M \subset \mathbb{R}$ be a sigma-compact and linearly independent. Then for every prism P in $\mathbb{R}$ there exist a subprism Q of P and a compact subset R of $P \setminus M$ such that $M \cup R$ is a maximal linearly independent subset of $M \cup Q$.*

Proof of Theorem 1.1. For a linearly independent sigma-compact set $M \subset \mathbb{R}$ and a prism P in $\mathbb{R}$ let $Q(M,P) = Q$ and $R(M,P) = R \subset P \setminus M$ be as in Lemma 2.1. Consider Player II strategy S given by

$$S(\langle\langle P_\eta, Q_\eta\rangle : \eta < \xi\rangle, P_\xi) = Q\left(\bigcup\{R_\eta : \eta < \xi\}, P_\xi\right),$$

where R_η's are defined inductively by $R_\eta = R(\bigcup\{R_\zeta : \zeta < \eta\}, P_\eta)$.

By $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ strategy S is not a winning strategy for Player II. So there exists a game $\langle\langle P_\xi, Q_\xi\rangle : \xi < \omega_1\rangle$ played according to S in which Player II loses, that is, $\mathbb{R} = \bigcup_{\xi<\omega_1} Q_\xi$.

Let $\mathcal{H} = \{R_\xi : \xi < \omega_1\}$ and notice that $\bigcup \mathcal{H}$ is a Hamel basis. Indeed, clearly $\bigcup \mathcal{H}$ is linearly independent. To see that it spans $\mathbb{R}$ it is enough to notice that $\mathrm{LIN}(\bigcup_{\eta<\xi} R_\eta) = \mathrm{LIN}(\bigcup_{\eta<\xi} Q_\eta)$ for every $\xi < \omega_1$.

Although sets in $\mathcal{H}$ need not to be perfect, they are clearly pairwise disjoint and compact. Thus, the theorem follows immediately from the following remark. ■

Remark 2.2 If there exists a family $\mathcal{H}$ of ω_1 pairwise disjoint compact subsets of $\mathbb{R}$ such that $\bigcup \mathcal{H}$ is a Hamel basis then there exists such an $\mathcal{H}$ with $\mathcal{H} \subset \mathrm{Perf}(\mathbb{R})$.

Proof. Let $\mathcal{H}_0$ be a family of ω_1 pairwise disjoint compact subsets of $\mathbb{R}$ such that $\bigcup \mathcal{H}_0$ is a Hamel basis. Partitioning each $H \in \mathcal{H}_0$ into its perfect part and singletons from scattered part we can assume that $\mathcal{H}_0$ contains only perfect sets and singletons. To get $\mathcal{H}$ as required fix a perfect set $P_0 \in \mathcal{H}_0$ and an $x \in P_0$ and notice that if we replace each $P \in \mathcal{H}_0 \setminus \{P_0\}$ with $px + qP$ for some $p, q \in \mathbb{Q} \setminus \{0\}$ then the resulting family will still be pairwise disjoint with union being a Hamel basis. Thus, without loss of generality, we can assume that every open interval in $\mathbb{R}$ contains ω_1 perfect sets from $\mathcal{H}_0$. Now, for every singleton $\{x\}$ in $\mathcal{H}_0$ we can choose a sequence $P_1^x > P_2^x > P_3^x > \cdots$ from $\mathcal{H}_0$ converging to x, and replace a family $\{x\} \cup \{P_n^x : n < \omega\}$ with its union. (We assume that we choose different sets P_n^x for different singletons.) If $\mathcal{H}$ is such a modification of $\mathcal{H}_0$ then $\mathcal{H}$ is as desired. ■

3 Iterated perfect sets and fusion lemmas for prisms

Let $0 < \alpha < \omega_1$. To define $\mathbb{P}_\alpha$ we need to consider the family $\Phi_{\mathrm{prism}}(\alpha)$ of all continuous injections $f : \mathfrak{C}^\alpha \to \mathfrak{C}^\alpha$ with the property that

$$f(x) \restriction \beta = f(y) \restriction \beta \iff x \restriction \beta = y \restriction \beta \qquad \text{for all } \beta < \alpha \text{ and } x, y \in \mathfrak{C}^\alpha \quad (1)$$

or, equivalently, such that for every $\beta < \alpha$

$$f \restriction\restriction \beta \stackrel{\text{def}}{=} \{\langle x \restriction \beta, y \restriction \beta\rangle : \langle x, y\rangle \in f\}$$

is a one-to-one function from $\mathfrak{C}^\beta$ into $\mathfrak{C}^\beta$. For example, if $\alpha = 3$ then $f \in \Phi_{\mathrm{prism}}(\alpha)$ provided there exist continuous functions $f_0 : \mathfrak{C} \to \mathfrak{C}$, $f_1 : \mathfrak{C}^2 \to \mathfrak{C}$, and $f_2\colon \mathfrak{C}^3 \to \mathfrak{C}$ such that $f(x_0,x_1,x_2) = \langle f_0(x_0), f_1(x_0,x_1), f_2(x_0,x_1,x_2)\rangle$ for all $x_0,x_1,x_2 \in \mathfrak{C}$ and maps f_0, $\langle f_0, f_1\rangle$, and f are one-to-one. Functions f from $\Phi_{\rm prism}(\alpha)$ were first introduced, in more general setting, in [9] where they are called *projection-keeping homeomorphisms*. Note that

$$\Phi_{\rm prism}(\alpha) \text{ is closed under the compositions} \tag{2}$$

and that for every $0<\beta<\alpha$

$$\text{if } f\in\Phi_{\rm prism}(\alpha) \text{ then } f\restriction\restriction\beta\in\Phi_{\rm prism}(\beta). \tag{3}$$

We define $\mathbb{P}_\alpha$ as

$$\mathbb{P}_\alpha = \{\text{range}(f)\colon f\in\Phi_{\rm prism}(\alpha)\}.$$

The simplest possible elements of $\mathbb{P}_\alpha$ are the *perfect cubes*, that is, the sets of the form $\prod_{\beta<\alpha} C_\beta$, where $C_\beta\in\mathfrak{C}$ for every $\beta<\alpha$. (If f_β is a continuous injection from $\mathfrak{C}$ onto P_β and $f\colon\mathfrak{C}^\alpha\to\mathfrak{C}^\alpha$ is given by $f(x)(\beta)=f_\beta(x_\beta)$ then $f\in\Phi_{\rm prism}(\alpha)$ and $\text{range}(f)=\prod_{\beta<\alpha}C_\beta$.)

Note also that

$$\text{if } f\in\Phi_{\rm prism}(\alpha) \text{ and } P\in\mathbb{P}_\alpha \text{ then } f[P]\in\mathbb{P}_\alpha. \tag{4}$$

Indeed, if $P=g[\mathfrak{C}^\alpha]$ for some $g\in\Phi_{\rm prism}(\alpha)$ then, by condition (2), we have $f[P]=f[g[\mathfrak{C}^\alpha]]=(f\circ g)[\mathfrak{C}^\alpha]\in\mathbb{P}_\alpha$.

In what follows for a fixed $0<\alpha<\omega_1$ and $0<\beta\le\alpha$ the symbol π_β will stand for the projection from $\mathfrak{C}^\alpha$ onto $\mathfrak{C}^\beta$. We will always consider $\mathfrak{C}^\alpha$ with the following standard metric ρ: fix an enumeration $\{\langle\beta_k,n_k\rangle\colon k<\omega\}$ of $\alpha\times\omega$ and for distinct $x,y\in\mathfrak{C}^\alpha$ define

$$\rho(x,y)=2^{-\min\{k<\omega: x(\beta_k)(n_k)\neq y(\beta_k)(n_k)\}}. \tag{5}$$

The open ball in $\mathfrak{C}^\alpha$ with a center at $z\in\mathfrak{C}^\alpha$ and radius $\varepsilon>0$ will be denoted by $B_\alpha(z,\varepsilon)$. Notice that in this metric any two open balls are either disjoint or one is a subset of another. Also for every $\gamma<\alpha$ and $\varepsilon>0$

$$\pi_\gamma[B_\alpha(x,\varepsilon)]=\pi_\gamma[B_\alpha(y,\varepsilon)] \quad \text{for every } x,y\in\mathfrak{C}^\alpha \text{ with } x\restriction\gamma=y\restriction\gamma. \tag{6}$$

It is also easy to see that any $B_\alpha(z,\varepsilon)$ is a clopen set and, in fact, it is a perfect cube in $\mathfrak{C}^\alpha$, so it belongs to $\mathbb{P}_\alpha$.

For a fixed $0 < \alpha < \omega_1$ let $\{\langle \beta_k, n_k\rangle : k < \omega\}$ be an enumeration of $\alpha \times \omega$ used in the definition (5) of the metric ρ and let

$$A_k = \{\langle \beta_i, n_i\rangle : i < k\} \quad \text{for every } k < \omega. \tag{7}$$

In what follows we will need the following simple fusion lemma, which can be found in [5]. For reader convenience we include here its short proof.

Lemma 3.1 *Let $0 < \alpha < \omega_1$ and for $k < \omega$ let $\mathcal{E}_k = \left\{E_s \in \mathbb{P}_\alpha : s \in 2^{A_k}\right\}$. Assume that for every $k < \omega$, $s, t \in 2^{A_k}$, and $\beta < \alpha$ we have:*

(i) *the diameter of E_s is less than or equal to 2^{-k},*

(ii) *if $i < k$ then $E_s \subset E_{s\restriction i}$,*

(ag) *(agreement) if $s \restriction (\beta \times \omega) = t \restriction (\beta \times \omega)$ then $\pi_\beta[E_s] = \pi_\beta[E_t]$,*

(sp) *(split) if $s \restriction (\beta \times \omega) \neq t \restriction (\beta \times \omega)$ then $\pi_\beta[E_s] \cap \pi_\beta[E_t] = \emptyset$.*

Then $Q = \bigcap_{k<\omega} \bigcup \mathcal{E}_k$ belongs to $\mathbb{P}_\alpha$.

Proof. For $x \in \mathfrak{C}^\alpha$ let $\bar{x} \in 2^{\alpha\times\omega}$ be defined by $\bar{x}(\beta, n) = x(\beta)(n)$.

First note that, by conditions (i) and (sp), for every $k < \omega$ the sets in $\mathcal{E}_k$ are pairwise disjoint and each of the diameter at most 2^{-k}. Thus, taking into account (ii), the function $h\colon \mathfrak{C}^\alpha \to \mathfrak{C}^\alpha$ defined by

$$h(x) = r \iff \{r\} = \bigcap_{k<\omega} E_{\bar{x}\restriction A_k}$$

is well defined and is one-to-one. It is also easy to see that h is continuous and that $Q = h[\mathfrak{C}^\alpha]$. Thus, we need to prove only that $h \in \Phi_{\rm prism}(\alpha)$, that is, that h is projection-keeping.

To show this fix $\beta < \alpha$, put $S = \bigcup_{i<\omega} 2^{A_i}$, and notice that, by (i) and (ag), for every $x \in \mathfrak{C}^\alpha$ we have

$$\begin{aligned}
\{h(x) \restriction \beta\} &= \pi_\beta\left[\bigcap\{E_{\bar{x}\restriction A_k} : k < \omega\}\right] \\
&= \bigcap\{\pi_\beta[E_{\bar{x}\restriction A_k}] : k < \omega\} \\
&= \bigcap\{\pi_\beta[E_s] : s \in S \ \&\ s \subset \bar{x}\} \\
&= \bigcap\{\pi_\beta[E_s] : s \in S \ \&\ s \restriction (\beta \times \omega) \subset \bar{x}\}.
\end{aligned}$$

Now, if $x \restriction \beta = y \restriction \beta$ then for every $s \in S$

$$s \restriction (\beta \times \omega) \subset \bar{x} \quad \Leftrightarrow \quad s \restriction (\beta \times \omega) \subset \bar{y}$$

so $h(x) \restriction \beta = h(y) \restriction \beta$.

On the other hand, if $x \restriction \beta \neq y \restriction \beta$ then there exists a $k < \omega$ big enough such that for $s = \bar{x} \restriction A_k$ and $t = \bar{y} \restriction A_k$ we have $s \restriction (\beta \times \omega) \neq t \restriction (\beta \times \omega)$. But then $\{h(x) \restriction \beta\}$ and $\{h(y) \restriction \beta\}$ are subsets of $\pi_\beta[E_s]$ and $\pi_\beta[E_t]$, respectively, which, by (sp), are disjoint. So, $h(x) \restriction \beta \neq h(y) \restriction \beta$. ∎

In what follows we will also need the following simple fact, which follows from the fact that every dense G_δ subset of a Polish space $X \times X$ contains a product $G \times P$, where G is dense G_δ in X and $P \in \mathrm{Perf}(X)$. For the proof see e.g. [4] or [6].

Claim 3.2 *Let $0 < \alpha < \omega_1$. If G is a second category Borel subset of $\mathfrak{C}^\alpha$ then G contains a perfect cube $\prod_{\beta<\alpha} P_\beta$.*

We will also use the following variant of Kuratowski-Ulam theorem, which can be deduced from the classical Kuratowski-Ulam theorem via a simple closure argument. Its proof can be found in [3] or [6].

Lemma 3.3 *Let $0 < \alpha < \omega_1$. For every comeager set $H \subset \mathfrak{C}^\alpha$ there exists a comeager set $G \subset H$ such that for every $x \in G$ and $\beta < \alpha$ the set*

$$G_{x \restriction \beta} = \left\{ y \in \mathfrak{C}^{\alpha \setminus \beta} \colon (x \restriction \beta) \cup y \in G \right\}$$

is comeager in $\mathfrak{C}^{\alpha \setminus \beta}$.

4 Proof of Lemma 2.1

Let X be a Polish space, $0 < n < \omega$, and $F \subset X^n$ be an n-ary relation. We say that a set $S \subset X$ is F-*independent* provided $F(x(0), \ldots, x(n-1))$ does not hold for any one-to-one $x \colon n \to S$. For a family $\mathcal{F}$ of finitary relations on X (i.e., relations $F \subset X^n$ where $0 < n < \omega$) we say that $S \subset X$ is $\mathcal{F}$-independent provided S is F-independent for every $F \in \mathcal{F}$. We will use the term *unary relation* for any 1-ary relation.

Proposition 4.1 *Let $0 < \alpha < \omega_1$ and $\mathcal{F}$ be a countable family of closed finitary relations on $\mathfrak{C}^\alpha$. Assume that every unary relation in $\mathcal{F}$ is nowhere dense in $\mathfrak{C}^\alpha$ and that for every $F \in \mathcal{F}$ there exists a comeager subset G_F of $\mathfrak{C}^\alpha$ such that*

(ex) *for every F-independent finite set $S \subset G_F$, $x \in S$, and $\beta < \alpha$ the set*

$$\left\{z \in \mathfrak{C}^{\alpha\setminus\beta}\colon\ S \cup \{z \cup x \restriction \beta\} \subset G_F \text{ is } F\text{-independent}\right\}$$

is dense in $\mathfrak{C}^{\alpha\setminus\beta}$.

Then there is an $E \in \mathbb{P}_\alpha$ which is $\mathcal{F}$-independent.

Note that without the assumption that the unary relations in $\mathcal{F}$ are nowhere dense the proposition is false: the unary relation $F = \mathfrak{C}^\alpha$ satisfies the condition (ex) (with $G_F = \mathfrak{C}^\alpha$) and no non-empty set is F-independent. On the other hand, for any n-ary relation $F \in \mathcal{F}$ with $n > 1$ condition (ex) implies that F is nowhere dense in $(\mathfrak{C}^\alpha)^n$. However, not every nowhere dense binary relation satisfies (ex). For example $F = \{\langle x, y\rangle\colon x(0) = y(0)\}$ is nowhere dense and it does not satisfy (ex) if $\alpha > 1$.

Proof. First notice that applying Lemma 3.3, if necessary, we can assume that for every $F \in \mathcal{F}$, $x \in G_F$, and $\beta < \alpha$ the set $(G_F)_{x\restriction\beta}$ is comeager in $\mathfrak{C}^{\alpha\setminus\beta}$. But this implies that each set from the condition (ex) is comeager in $\mathfrak{C}^{\alpha\setminus\beta}$ since it is an intersection of $(G_F)_{x\restriction\beta}$ and an open set $\left\{z \in \mathfrak{C}^{\alpha\setminus\beta}\colon\ S \cup \{z \cup x \restriction \beta\} \text{ is } F\text{-independent}\right\}$. In particular, if we put $G = \bigcap_{F\in\mathcal{F}} G_F$ then G is comeager in $\mathfrak{C}^\alpha$ and it is easy to see that it satisfies the following condition.

(EX) For every $\mathcal{F}$-independent finite set $S \subset G$, $x \in S$, and $\beta < \alpha$ the set

$$\left\{z \in \mathfrak{C}^{\alpha\setminus\beta}\colon\ S \cup \{z \cup x \restriction \beta\} \subset G \text{ is } \mathcal{F}\text{-independent}\right\}$$

is dense in $\mathfrak{C}^{\alpha\setminus\beta}$.

Let $\{F_k\colon k < \omega\}$ be an enumeration of $\mathcal{F}$ with infinite repetitions. Also, for $k < \omega$ let $A_k = \{\langle \beta_i, n_i\rangle\colon i < k\}$ be as in the condition (7). By induction on $k < \omega$ we will construct two sequences: $\langle \varepsilon_k > 0\colon k < \omega\rangle$ converging to 0 and $\left\langle \left\{x_s \in G\colon s \in 2^{A_k}\right\} : k < \omega\right\rangle$ of $\mathcal{F}$-independent sets such that for every $\beta < \alpha$, $k < \omega$, and $s, t \in 2^{A_k}$

(a) $x_s \restriction \beta = x_t \restriction \beta$ if and only if $s \restriction \beta \times \omega = t \restriction \beta \times \omega$;

(b) if $E_s = B_\alpha(x_s, \varepsilon_k)$ and $\mathcal{E}_k = \{E_s : s \in 2^{A_k}\}$ then $\mathcal{E}_k$'s satisfy (ii), (ag), and (sp) from Lemma 3.1;

(c) if F_k is an n-ary relation then $F_k(z_0, \ldots, z_{n-1})$ does not hold provided each z_i is chosen from a different ball from $\mathcal{E}_k$.

Before we construct such sequences, let us first note that $E = \bigcap_{k<\omega} \bigcup \mathcal{E}_k$ is as desired. Indeed, $E \in \mathbb{P}_\alpha$ by Lemma 3.1. To see that E is $\mathcal{F}$-independent pick an n-ary relation $F \in \mathcal{F}$, $\{z_0, \ldots, z_{n-1}\} \in [E]^n$, and find a $k < \omega$ with $F_k = F$ which is big enough so that ε_k is smaller than the distance between z_i and z_j for all $i < j < n$. Then z_i's must belong to distinct elements of $\mathcal{E}_k$ so, by (c), $F(z_0, \ldots, z_{n-1})$ does not hold.

For $k = 0$ we pick an arbitrary $\mathcal{F}$-independent $x_\emptyset \in G$ by choosing an arbitrary element of G which does not belong to any nowhere dense unary relation from $\mathcal{F}$. Also, we choose an $\varepsilon_0 \in (0, 1]$ ensuring (c), which can be done since F_0 is closed. (This is a non-trivial requirement only when F_0 is an unary relation.) Clearly (a)-(c) are satisfied.

Assume that for some $k < \omega$ the construction is done up to the level k. For $s \in 2^{A_k}$ and $j < 2$ let $s\hat{\ }j = s \cup \{\langle\langle \beta_k, n_k\rangle, j\rangle\} \in 2^{A_{k+1}}$ and define $x_{s\hat{\ }0} = x_s$. Let $\{s_i : i < 2^k\}$ be an enumeration of 2^{A_k} and put $S = \{x_{s\hat{\ }0} : s \in 2^{A_k}\}$. Points $x_{s_i\hat{\ }1} \in G \cap E_{s_i}$ will be chosen by induction on $i \leq 2^k$ such that the set $S_i = S \cup \{x_{s_j\hat{\ }1} : j < i\}$ is $\mathcal{F}$-independent and the condition (a) is satisfied for the elements of S_i. Clearly, by the inductive assumption (a) is satisfied for the elements of $S_0 = S$. So, assume that for some $i \leq 2^k$ the set S_i is already constructed. We need to find an appropriate $x_{s_i\hat{\ }1} \in G \cap E_{s_i}$. Let $\beta < \alpha$ be maximal such that there is an $s \in \{s\hat{\ }0 : s \in 2^{A_k}\} \cup \{s_j\hat{\ }1 : j < i\}$ with $s \restriction \beta \times \omega = (s_i\hat{\ }1) \restriction \beta \times \omega$ and let $x = x_s \restriction \beta$. We will choose $x_{s_i\hat{\ }1}$ extending x and such that $x_{s_i\hat{\ }1}(\beta) \neq x_t(\beta)$ for all $x_t \in S_i$. Notice that this will ensure that the condition (a) is satisfied for the elements of S_{i+1}. Surprisingly, more difficult condition to insure will be that $x_{s_i\hat{\ }1} \in E_{s_i} = B_\alpha(x_{s_i\hat{\ }0}, \varepsilon_k)$, since at the first glance it is not even obvious that

$$B_\alpha(x_{s_i\hat{\ }0}, \varepsilon_k) \text{ contains an extension of } x. \tag{8}$$

To argue for this first notice that maximality of β insures that $\beta \geq \beta_k$, since $s_i\hat{\ }0 \in S_i$ and $(s_i\hat{\ }0) \restriction \beta_k \times \omega = (s_i\hat{\ }1) \restriction \beta_k \times \omega$. If $\beta = \beta_k$ we have $x = x_{s_i\hat{\ }0} \restriction \beta$ and (8) is obvious. So, assume that $\beta > \beta_k$. Then there is a

$j < i$ such that $s = s_j\hat{\ }1$. We also have $s_j \restriction \beta \times \omega = s_i \restriction \beta \times \omega$ so, by the inductive assumption, $x_{s_j} \restriction \beta = x_{s_i} \restriction \beta$.

Now, let $n < \omega$ be the smallest such that $2^{-n} < \varepsilon_k$. Then, by the definition of the metric on $\mathfrak{C}^\alpha$, the fact that $x_s = x_{s_j\hat{\ }1} \in E_{s_j} = B_\alpha(x_{s_j}, \varepsilon_k)$ means that $x_s(\gamma)(m) = x_{s_j}(\gamma)(m)$ for every $\langle \gamma, m\rangle \in A_n$. Therefore, we have $x(\gamma)(m) = x_s(\gamma)(m) = x_{s_j}(\gamma)(m) = x_{s_i}(\gamma)(m)$ for every $\langle \gamma, m\rangle \in A_n$ with $\gamma < \beta$. Thus, we can extend x to an element $y \in \mathfrak{C}^\alpha$ for which $y(\gamma)(m) = x_{s_i}(\gamma)(m)$ for every $\langle \gamma, m\rangle \in A_n$. But this y witnesses (8).

To finish the construction of $x_{s_i\hat{\ }1}$ notice that by (8) we can find an open ball B in $\mathfrak{C}^{\alpha\setminus\beta}$ such that $\{x\} \times B \subset B_\alpha(x_{s_i\hat{\ }0}, \varepsilon_k)$. Decreasing B, if necessary, we can also insure that $y(\beta) \neq x_t(\beta)$ for every $t \in S_i$ and $y \in \{x\} \times B$. By condition (EX) we can find a $z \in B$ such that $S_i \cup \{x \cup z\} \subset G$ is $\mathcal{F}$-independent. We put $x_{s_i\hat{\ }1} = x \cup z$.

Thus, we constructed an $\mathcal{F}$-independent set $\{x_{s\hat{\ }j} \colon s \in 2^{A_k}\ \&\ j < 2\} \subset G$ satisfying (a) and such that $x_{s\hat{\ }0}, x_{s\hat{\ }1} \in E_s$ for every $s \in 2^{A_k}$. To finish the construction insuring (a)-(c) we need to choose an $\varepsilon_{k+1} \leq 2^{-(k+1)}$ small enough to guarantee the following properties.

- $E_{s\hat{\ }j} = B_\alpha(x_{s\hat{\ }0}, \varepsilon_{k_1}) \subset E_s$ for every $s \in 2^{A_k}$ and $j < 2$. This will ensure condition (ii).
- Condition (sp) holds. This can be done, since (a) is satisfied.
- Condition (c) is satisfied. This can be done since $\left\{x_s \colon s \in 2^{A_{k+1}}\right\}$ is $\mathcal{F}$-independent and F_{k+1} is a closed relation.

Note that (ag) is guaranteed by (a) and our definition of E_s's. This finishes the proof of Proposition 4.1. ■

We say that an n-ary relation F on a Polish space X is *symmetric* provided for any sequence $\langle x_i \in X \colon i < n\rangle$ and any permutation π of n

$$F\left(x_0, \ldots, x_{n-1}\right) \text{ holds if and only if } F\left(x_{\pi(0)}, \ldots, x_{\pi(n-1)}\right) \text{ holds.}$$

For such an F and $A \subset X$ we put

$$F * A = A \cup \{x \in X \colon (\exists a_1, \ldots, a_{n-1} \in A)\ \ F(x, a_1, \ldots, a_{n-1})\}.$$

If F is unary relation we interpret the above as $F * A = A \cup F$. If $\mathcal{F}$ is a family of symmetric finitary relations on X then we put $\mathcal{F} * A = \bigcup_{F \in \mathcal{F}} F * A$. Also, an *$\mathcal{F}$-closure of* A, denoted by $\mathrm{cl}_{\mathcal{F}}(A)$, is the least $B \subset X$ containing A such that $\mathcal{F} * B = B$. Note that $\mathrm{cl}_{\mathcal{F}}(A) = \bigcup_{n<\omega} \mathcal{F}^n * A$, where $\mathcal{F}^0 * A = A$ and $\mathcal{F}^{n+1} * A = \mathcal{F} * (\mathcal{F}^n * A)$. Thus, if $\mathcal{F}$ is a countable family of closed symmetric finitary relations then $\mathrm{cl}_{\mathcal{F}}(A)$ is F_σ in X for a sigma-compact $A \subset X$ since $F * K$ is closed for every $F \in \mathcal{F}$ and compact $K \subset X$.

We are the most interested in these notions when we are concerned with either linear independence (over $\mathbb{Q}$) or algebraic independence in $\mathbb{R}$. In the first case $\mathcal{F} = \mathcal{F}_{\mathrm{lin}}$ is defined as the family of all relations F_ℓ of all $\langle x_0, \ldots, x_{n-1} \rangle$ for which

$$\ell(x_{\pi(0)}, \ldots, x_{\pi(n-1)}) = 0 \text{ for some permutation } \pi \text{ of } n, \tag{9}$$

where ℓ is a non-zero linear function with rational coefficients. In this case $\mathcal{F}$-independence stands for linear independence (over $\mathbb{Q}$) and $\mathrm{cl}_{\mathcal{F}}(A)$ is the linear span of A. When $\mathcal{F}$ is the family of all relations F_ℓ, where ℓ spans over all non-zero polynomials with rational coefficients, then $\mathcal{F}$-independence stands for algebraic independence, while $\mathrm{cl}_{\mathcal{F}}(A)$ is the algebraic closure of $\mathbb{Q}(A)$.

We will need also one more notion. For a family $\mathcal{F}$ of closed symmetric finitary relations on X and an $M \subset X$ we define $\mathcal{F}_M$ as the collection of all possible projections of the relations from $\mathcal{F}$ along M. In other words, $\mathcal{F}_M$ is the collection of all (symmetric) relations

$$\{\langle x_0, \ldots, x_{k-1} \rangle \colon (\exists a_k, \ldots, a_{n-1} \in M)\ \ F(x_0, \ldots, x_{k-1}, a_k, \ldots, a_{n-1})\}, \tag{10}$$

where $F \in \mathcal{F}$ is an n-ary relation and $0 < k \leq n$. Note that if M is compact then each relation in $\mathcal{F}_M$ is still closed and for every $A \subset X$ we have

$$\mathrm{cl}_{\mathcal{F}}(M \cup A) = \mathrm{cl}_{\mathcal{F}_M}(A). \tag{11}$$

Also, if M is $\mathcal{F}$-independent then

$$A \cup M \text{ is } \mathcal{F}\text{-independent provided } A \text{ is } \mathcal{F}_M\text{-independent.} \tag{12}$$

Lemma 4.2 *Let $\mathcal{F}$ be an arbitrary family of closed symmetric finitary relations in a Polish space X. Then for every prism P in X there exists a subprism Q of P and a compact $\mathcal{F}$-independent set $R \subset P$ such that $Q \subset \mathrm{cl}_{\mathcal{F}}(R)$.*

PROOF. For $0 < \alpha < \omega_1$ let I_α be the statement:

I_α: the lemma holds for any prism P with witness function $f \colon \mathfrak{C}^\alpha \to P$.

We will prove I_α by induction on α.

First notice that I_α implies the following:

I^*_α: for every $k<\omega$ and continuous functions $g_0,\ldots,g_k\colon \mathfrak{C}^\alpha\to X$ there exist an $E\in\mathbb{P}_\alpha$ and a compact $\mathcal{F}$-independent set $R\subset\bigcup_{i\le k} g_i[\mathfrak{C}^\alpha]$ such that $\bigcup_{i\le k} g_i[E]\subset \mathrm{cl}_{\mathcal{F}}(R)$.

To see that I^*_α holds true for $k=0$, for every n-ary relation $F\in\mathcal{F}$ define $F^0=\{\langle x_0,\ldots,x_{n-1}\rangle\in(\mathfrak{C}^\alpha)^n\colon F(g_0(x_0),\ldots,g_0(x_{n-1}))\}$. By I_α applied to $\mathcal{F}_0=\{F^0\colon F\in\mathcal{F}\}$ we can find an $\mathcal{F}_0$-independent set $R_0\subset\mathfrak{C}^\alpha$ and an $E\in\mathbb{P}_\alpha$ such that $E\subset\mathrm{cl}_{\mathcal{F}_0}(R)$. But then $R=g_0[R_0]$ is compact, $\mathcal{F}$-independent, and $g_0[E]\subset\mathrm{cl}_{\mathcal{F}}(g_0[R_0])=\mathrm{cl}_{\mathcal{F}}(R)$.

To make an inductive step assume that I^*_α holds for some $k<\omega$ and take continuous functions $g_0,\ldots,g_{k+1}\colon\mathfrak{C}^\alpha\to X$. By the inductive assumption we can find an $E_0\in\mathbb{P}_\alpha$ and a compact $\mathcal{F}$-independent set $R_0\subset\bigcup_{i\le k} g_i[\mathfrak{C}^\alpha]$ such that $\bigcup_{i\le k} g_i[E_0]\subset\mathrm{cl}_{\mathcal{F}}(R_0)$. Let $h\in\Phi_{\mathrm{prism}}(\alpha)$ be a mapping from $\mathfrak{C}^\alpha$ onto E_0. Using the case $k=0$ to the function $g_{k+1}\circ h$ and the family $\mathcal{F}_{R_0}$ we can find an $E_1\in\mathbb{P}_\alpha$ and a compact $\mathcal{F}_{R_0}$-independent set $R_1\subset(g_{k+1}\circ h)[\mathfrak{C}^\alpha]$ such that $(g_{k+1}\circ h)[E_1]\subset\mathrm{cl}_{\mathcal{F}_{R_0}}(R_1)$. Then, by (12), we conclude that $R=R_0\cup R_1$ is $\mathcal{F}$-independent. Put $E=h[E_1]\in\mathbb{P}_\alpha$. Then, by (11), we have $g_{k+1}[E]\subset\mathrm{cl}_{\mathcal{F}_{R_0}}(R_1)=\mathrm{cl}_{\mathcal{F}}(R_0\cup R_1)=\mathrm{cl}_{\mathcal{F}}(R)$, while clearly $\bigcup_{i\le k} g_i[E]\subset\bigcup_{i\le k} g_i[E_0]\subset\mathrm{cl}_{\mathcal{F}}(R_0)\subset\mathrm{cl}_{\mathcal{F}}(R)$. Thus, E and R satisfy I^*_α.

Now, we are ready to prove I_α. So, fix $0<\alpha<\omega_1$ and assume that I_γ is true for all $0<\gamma<\alpha$. Let P be a prism in X with witness function $f\colon\mathfrak{C}^\alpha\to P$. We need to find appropriate Q and R.

Let W be the set of all $\beta\le\alpha$ for which there exists an $E\in\mathbb{P}_\alpha$ and an $F\in\mathcal{F}$ such that for every $z\in\pi_\beta[E]$ there is a finite set $R_z\subset P$ for which

$$f[\{x\in E\colon z\subset x\}]\subset F*R_z. \tag{13}$$

Notice that W is non-empty since $\alpha\in W$. So $\beta=\min W$ is well defined. Let $E\in\mathbb{P}_\alpha$ be such that (13) holds for β. Replacing f with its composition with an appropriate function from $\Phi_{\mathrm{prism}}(\alpha)$ (compare (4)), if necessary, we can assume that $E=\mathfrak{C}^\alpha$.

If $\beta=0$ then $f[\mathfrak{C}^\alpha]\subset\mathrm{cl}_{\mathcal{F}}(R_0)$ for some finite set $R_0\subset P$, and we can find an $\mathcal{F}$-independent finite $R\subset R_0$ with $f[\mathfrak{C}^\alpha]\subset\mathrm{cl}_{\mathcal{F}}(R)$. (Note that if T is $\mathcal{F}$-independent and $x\in X\setminus\mathrm{cl}_{\mathcal{F}}(T)$ then $T\cup\{x\}$ is also $\mathcal{F}$-independent.) Thus, $Q=f[\mathfrak{C}^\alpha]$ and R satisfy I_α. So, for the rest of the proof we will assume that $\beta>0$.

Next, assume that $0<\beta<\alpha$. Let $\mathcal{B}_\beta$ be a countable basis of $\mathfrak{C}^{\alpha\setminus\beta}$ consisting of non-empty clopen sets and assume that F satisfying (13) is $(n+1)$-ary. For every $B\in\mathcal{B}_\beta$ consider the set

$$K_B=\left\{z\in\mathfrak{C}^\beta\colon (\exists\langle x_1,\ldots,x_n\rangle\in P^n)\,(\forall y\in B)\ F(f(z\cup y),x_1,\ldots,x_n)\right\}.$$

It is easy to see that each set K_B is closed. Notice also that

$$\mathfrak{C}^\beta=\bigcup_{B\in\mathcal{B}_\beta}K_B. \tag{14}$$

To see this, fix a $z\in\mathfrak{C}^\beta$. By (13), there exists a finite set $S_z\subset\mathfrak{C}^\alpha$ such that $\mathfrak{C}^{\alpha\setminus\beta}=\bigcup_{x_1,\ldots,x_n\in f[S_z]}\{y\in\mathfrak{C}^{\alpha\setminus\beta}\colon F(f(z\cup y),x_1,\ldots,x_n)\}$. Since each set $\{y\in\mathfrak{C}^{\alpha\setminus\beta}\colon F(f(z\cup y),x_1,\ldots,x_n)\}$ is closed, one of them must contain a $B\in\mathcal{B}_\beta$, and so $z\in K_B$.

Thus, by (14), there exists a $B\in\mathcal{B}_\beta$ such that K_B has a non-empty interior. In particular, there is a non-empty clopen set $U\subset K_B$. But then for every $z\in U$ there exists a $g(z)=\langle g_1(z),\ldots,g_n(z)\rangle\in P^n$ such that $F(f(z\cup y),g_1(z),\ldots,g_n(z))$ holds for every $y\in B$. Now

$$T=\{\langle z,\bar p\rangle\in U\times P^n\colon (\forall y\in B)\ F(f(z\cup y),\bar p)\}$$

is a compact subset of $U\times P^n$ and g constitutes a selector of T. Thus, we can choose g to be Borel. In particular, there is a dense G_δ subset W of U such that $g\restriction W$ is continuous. So, by Claim 3.2, we can find a perfect cube $C\subset W\subset\mathfrak{C}^\beta$. Now, identifying C with $\mathfrak{C}^\beta$, we conclude that functions $g_1,\ldots,g_n\colon\mathfrak{C}^\beta\to P$ are continuous and that $F(f(z\cup y),g_1(z),\ldots,g_n(z))$ holds for every $z\in\mathfrak{C}^\beta$ and $y\in B$.

Since, by the inductive hypothesis, I_β is true, condition I^*_β holds as well. Thus, there exist an $E\in\mathbb{P}_\beta$ and a compact $\mathcal{F}$-independent set $R\subset P$ such that $\bigcup_{i=1}^n g_i[E]\subset\mathrm{cl}_{\mathcal{F}}(R)$. Since $Q=f[E\times B]$ is a subprism of P, we just need to show that $Q\subset\mathrm{cl}_{\mathcal{F}}(R)$. To see this it just note that for every $z\in E$ we have $f[\{z\}\times B]\subset F*\{g_1(z),\ldots,g_n(z)\}\subset\mathrm{cl}_{\mathcal{F}}\left(\bigcup_{i=1}^n g_i[E]\right)\subset\mathrm{cl}_{\mathcal{F}}(R)$. This finishes the proof of the case $0<\beta<\alpha$.

For the reminder of the proof we will assume that $\beta=\alpha$. This means that there is no $E\in\mathbb{P}_\alpha$ such that for some $F\in\mathcal{F}$ and $\beta<\alpha$

$$(\forall z\in\pi_\beta[E])\,(\exists R_z\in[P]^{<\omega})\ \ f[\{x\in E\colon z\subset x\}]\subset F*R_z. \tag{15}$$

For every n-ary $F \in \mathcal{F}$ let $F^* = \{\langle x_0, \ldots, x_{n-1}\rangle : F(f(x_0), \ldots, f(x_{n-1}))\}$ and let $\mathcal{F}^* = \{F^* : F \in \mathcal{F}\}$. We will apply Proposition 4.1 to find an $\mathcal{F}^*$-independent $E \in \mathbb{P}_\alpha$. Then $Q = f[E]$ is $\mathcal{F}$-independent subprism of P and together with $R = Q$ they satisfy the lemma.

To see that the assumptions of Proposition 4.1 are satisfied, first notice that unary relations in $\mathcal{F}^*$ are nowhere dense. Indeed, otherwise there is a unary relation $F^* \in \mathcal{F}^*$ and a non-empty clopen set $E \subset F^*$. But then E contradicts (15), as $f[E] \subset F * \emptyset$. Thus, we just need to show that the condition (ex) is satisfied.

So, fix an $F \in \mathcal{F}$. For $0 < \beta < \alpha$ and $B \in \mathcal{B}_\beta$ let

$$K(B) = \left\{z \in \mathfrak{C}^\beta : (\exists R_z \in [P]^{<\omega}) \;\; f[\{z\} \times B] \subset F * R_z\right\}.$$

Clearly $K(B)$ is F_σ. Notice also that it is meager, since otherwise there would exist a non-empty clopen $U \subset K(B)$ and $E = U \times B$ would contradict (15). Thus, each set $K_\beta = \bigcup_{B \in \mathcal{B}_\alpha} K(B)$ is meager. Also, for every $z \in \mathfrak{C}^\beta \setminus K_\beta$ and for every finite $R \subset P$ the set $\left\{y \in \mathfrak{C}^{\alpha\setminus\beta} : f(z \cup y) \notin F * R\right\}$ is dense and open. In particular, if R is a finite F-independent subset of P then

$$W_R = \left\{y \in \mathfrak{C}^{\alpha\setminus\beta} : R \cup \{f(z \cup y)\} \text{ is } F\text{-independent}\right\} \tag{16}$$

is dense and open. Let

$$H = \bigcap_{0<\beta<\alpha} \left(\left(\mathfrak{C}^\beta \setminus K_\beta\right) \times \mathfrak{C}^{\alpha\setminus\beta}\right)$$

and notice that H is comeager since each K_β is meager in $\mathfrak{C}^\beta$. By Lemma 3.3 we can find a comeager set $G \subset H$ such that

$$G_{x\restriction\beta} = \left\{y \in \mathfrak{C}^{\alpha\setminus\beta} : (x \restriction \beta) \cup y \in G\right\}$$

is comeager every $x \in G$ and $\beta < \alpha$. To finish the proof it is enough to show that G satisfy (ex) for F^*. So, take an F^*-independent finite set $S \subset G$, an $x \in S$, and a $\beta < \alpha$.

First let us assume that $\beta > 0$. Then $x \in S \subset G \subset H$ implies that $z = x \restriction \beta \in \mathfrak{C}^\beta \setminus K_\beta$. In particular, the set $W_{f[S]}$ from (16) is comeager, and so is $W_{f[S]} \cap G_{x\restriction\beta}$. To get (ex) it is enough to notice that $W_{f[S]} \cap G_{x\restriction\beta}$ is a subset of $\left\{y \in \mathfrak{C}^{\alpha\setminus\beta} :\; S \cup \{y \cup z\} \subset G \text{ is } F^*\text{-independent}\right\}$.

Finally assume that $\beta = 0$. We need to show that the set

$$\{y \in G : S \cup \{y\} \text{ is } F^*\text{-independent}\}$$

is dense. But this set must be comeager, since otherwise its complement would contain a non-empty clopen set E which wold contradict (15) with $\beta = 0$. ■

PROOF OF LEMMA 2.1. Let $\mathcal{F} = \mathcal{F}_{\rm lin}$ be the linear independence family defined at (9) and let $\bar{M} = \langle M_n : n < \omega \rangle$ be an increasing family of compact sets such that $M = \bigcup_{n<\omega} M_n$. Let $\mathcal{F}_{\bar{M}} = \bigcup_{n<\omega} \mathcal{F}_{M_n}$, where each $\mathcal{F}_{M_n}$ is defined at (10), that is, $\mathcal{F}_{M_n}$ is the the collection of all possible projections of the relations from $\mathcal{F}$ along M_n.

If $M \cap P$ is of second category in P then we can choose a subprism Q of P with $Q \subset M$. Then Q and $R = \emptyset$ have the desired properties. On the other hand, if $M \cap P$ is of first category in P then, by Claim 3.2, we can find a subprism P_1 of P disjoint with M.

Now, applying Lemma 4.2 we can find a subprism Q of P_1 and a compact $\mathcal{F}_{\bar{M}}$-independent set $R \subset P_1 \subset P \setminus M$ such that $Q \subset {\rm cl}_{\mathcal{F}_{\bar{M}}}(R)$. But then $M \cup R$ is $\mathcal{F}$-independent, see (12). Moreover,

$$Q \subset {\rm cl}_{\mathcal{F}_{\bar{M}}}(R) = {\rm cl}_{\mathcal{F}}(M \cup R) = {\rm LIN}(M \cup R).$$

So, $M \cup Q \subset {\rm LIN}(M \cup R)$ proving that Q and R are as desired. ■

5 Remarks

It is worth to notice that in case when $M = \emptyset$ Lemma 2.1 can be proved easier, and in a stronger form.

Proposition 5.1 *Every prism P in $\mathbb{R}$ there is a subprism Q which is linearly independent.*

PROOF. This follows from Proposition 4.1 used with $\mathcal{F} = \mathcal{F}_{\rm lin}$. ■

Remark 5.2 Note that Proposition 5.1 is false if we require Q to be a subcube of prism P, that is, $Q = f[C]$, where C is a perfect cube in $\mathfrak{C}^\alpha$ and $f : \mathfrak{C}^\alpha \to P$ is a coordinate function making P a prism.

PROOF. Indeed, let P_1 and P_2 be disjoint perfect subsets of $\mathbb{R}$ such that $P_1 \cup P_1$ is linearly independent over $\mathbb{Q}$. Let $f : P_1 \times P_2 \to \mathbb{R}$ be defined by a formula $f(x_1, x_2) = x_1 + x_2$. Identifying P_1 and P_2 with $\mathfrak{C}$ we think about f as defined on $\mathfrak{C}^2$ and treat P as a prism. To see that P has no linearly independent subcube let $Q = Q_1 \times Q_2$ be a subcube of P and choose different $a_1, b_1 \in Q_1$ and $a_2, b_2 \in Q_2$. Then $\{a_1 + a_2, a_1 + b_2, b_1 + a_2, b_1 + a_2\} \subset Q$ and they are clearly linearly dependent. ∎

Remark 5.3 In Lemma 2.1 we cannot require $R = Q$.

PROOF. Let P_1, P_2, and f be as in Remark 5.2. If $M = P_2$ then P has no subprism Q such that $M \cup Q$ is linearly independent, since any vertical section of Q is a translation of a portion of M. ∎

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