ON ADDITIVE ALMOST CONTINUOUS FUNCTIONS UNDER $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$

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Abstract. We prove that the Covering Property Axiom $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$, which holds in the iterated perfect set model, implies that there exists an additive discontinuous almost continuous function $f\colon \mathbb{R}\to\mathbb{R}$ whose graph is of measure zero. We also show that, under $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$, there exists a Hamel basis H for which, $E^+(H)$, the set of all linear combinations of elements from H with positive rational coefficients, is of measure zero. The existence of both of these examples follows from Martin's axiom, while it is unknown whether either of them can be constructed in ZFC.

As a tool for the constructions we will show that $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ implies its seemingly stronger version, in which ω_1-many games are played simultaneously.

1. Preliminaries and axiom $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$

Our set theoretic terminology is standard and follows that of [1]. In particular, $|X|$ stands for the cardinality of a set X and $\mathfrak{c}=|\mathbb{R}|$. The Cantor set 2^ω will be denoted by a symbol $\mathfrak{C}$. We use term *Polish space* for a complete separable metric space `without isolated points`. For a Polish space X symbol $\mathrm{Perf}(X)$ will stand for a collection of all subsets of X homeomorphic to the Cantor set $\mathfrak{C}$. For a fixed $0<\alpha<\omega_1$ and $0<\beta\le\alpha$ a symbol π_β will stand for the projection from $\mathfrak{C}^\alpha$ onto $\mathfrak{C}^\beta$. In what follows we will consider $\mathbb{R}$ as a linear space over $\mathbb{Q}$. For $Z\subset\mathbb{R}$ its linear span with respect to this structure will be denoted by $\mathrm{LIN}(Z)$. A subset H of $\mathbb{R}$ is a *Hamel basis* provided it is a linear basis of $\mathbb{R}$ over $\mathbb{Q}$, that is, it is linearly independent and $\mathrm{LIN}(H)=\mathbb{R}$.

Axiom $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ was introduced by the authors in [5], where it is shown that it holds in the iterated perfect set model. Also, $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ is a simpler version of the axiom CPA which is described in a monograph [9]. For the reader's convenience, we will restate $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ in the next few paragraphs.

For $0<\alpha<\omega_1$ let $\Phi_{\mathrm{prism}}(\alpha)$ be the family of all continuous injections $f\colon\mathfrak{C}^\alpha\to\mathfrak{C}^\alpha$ with the property that

$$f(x)\restriction\beta=f(y)\restriction\beta \iff x\restriction\beta=y\restriction\beta \qquad \text{for all } \beta\in\alpha \text{ and } x,y\in\mathfrak{C}^\alpha.$$

Functions $\Phi_{\mathrm{prism}}(\alpha)$ are called *projection-keeping homeomorphisms.* (Compare [11].) Let $\mathbb{P}_\alpha=\{\mathrm{range}(f)\colon f\in\Phi_{\mathrm{prism}}(\alpha)\}$ and $\mathbb{P}_{\omega_1}=\bigcup_{0<\alpha<\omega_1}\mathbb{P}_\alpha$. We will refer to elements of $\mathbb{P}_{\omega_1}$ as *iterated perfect sets.* (In [17] the elements of $\mathbb{P}_\alpha$ are called I-perfect,

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where I is the ideal of countable sets.) The simplest elements of $\mathbb{P}_\alpha$ are *perfect cubes*, that is, the sets of the form $C = \prod_{\beta<\alpha} C_\beta$, where $C_\beta \in \mathrm{Perf}(\mathfrak{C})$ for each $\beta < \alpha$.

Claim 1.1. *Let $0 < \alpha < \omega_1$. If G is a Borel second category subset of $\mathfrak{C}^\alpha$ then G contains a perfect cube. In particular, if $\mathcal{G}$ is a Borel countable cover of $\mathfrak{C}^\alpha$ then there is a $G \in \mathcal{G}$ which contains an $E \in \mathbb{P}_\alpha$.*

An argument for the claim can be found in [4, claim 3.2] or [9, claim 1.1.5].

The only properties of the iterated perfect sets that we will use in this paper are listed in the next three lemmas.

Lemma 1.2. *For every $E \in \mathbb{P}_{\omega_1}$, a Polish space X, and a continuous function $f\colon E \to X$ there exists a $P \in \mathbb{P}_{\omega_1}$ such that $P \subset E$ and $f[P]$ is either a singleton or it is homeomorphic to the Cantor set.*

Lemma 1.2 follows immediately from [9, lemma 3.2.5] (see also [6, lemma 1.1] or [7, lemma 2.4]) which is a particular case of [11, thm. 20].

The next two lemmas will allow us to construct the elements of $\mathbb{P}_{\omega_1}$ by fusion arguments. They can be found, respectively, in [5, lemmas 4.3 and 4.4] or in [9, lemmas 3.1.1 and 3.1.2]. Here, for a fixed $0 < \alpha < \omega_1$ and $k < \omega$ we define $A_k = \{\langle \beta_i, n_i\rangle \colon i < k\}$, where $\{\langle \beta_k, n_k\rangle \colon k < \omega\}$ is a fixed enumeration of $\alpha \times \omega$.

Lemma 1.3. *Let $0 < \alpha < \omega_1$ and for $k < \omega$ let $\mathcal{E}_k = \left\{E_s \in \mathbb{P}_\alpha \colon s \in 2^{A_k}\right\}$. Assume that for every $k < \omega$, $s, t \in 2^{A_k}$, and $\beta < \alpha$ we have:*

- (i) *the diameter of E_s is less than or equal to 2^{-k},*
- (ii) *if $i < k$ then $E_s \subset E_{s\restriction i}$,*
- (ag) *(agreement) if $s \restriction (\beta \times \omega) = t \restriction (\beta \times \omega)$ then $\pi_\beta[E_s] = \pi_\beta[E_t]$,*
- (sp) *(split) if $s \restriction (\beta \times \omega) \neq t \restriction (\beta \times \omega)$ then $\pi_\beta[E_s] \cap \pi_\beta[E_t] = \emptyset$.*

Then $Q = \bigcap_{k<\omega} \bigcup \mathcal{E}_k$ belongs to $\mathbb{P}_\alpha$.

For a family $\mathcal{E} \subset \mathbb{P}_\alpha$ we say that an $\mathcal{E}_0 \subset \mathbb{P}_\alpha$ is a *refinement* of $\mathcal{E}$ provided $\mathcal{E}_0 = \{P_E \colon E \in \mathcal{E}\}$, where $P_E \subset E$ for every $E \in \mathcal{E}$. A family $\mathcal{D} \subset [\mathbb{P}_\alpha]^{<\omega}$ is *closed under refinements* if for each $\mathcal{E} \in \mathcal{D}$ every refinement of $\mathcal{E}$ also belongs to $\mathcal{D}$.

Lemma 1.4. *Let $0 < \alpha < \omega_1$ and $k < \omega$. If $\mathcal{E}_k = \left\{E_s \in \mathbb{P}_\alpha \colon s \in 2^{A_k}\right\}$ satisfies (ag) and (sp) then*

- (A) *there exists an $\mathcal{E}_{k+1} = \left\{E_s \in \mathbb{P}_\alpha \colon s \in 2^{A_{k+1}}\right\}$ such that (i), (ii), (ag), and (sp) hold for all $s, t \in 2^{A_{k+1}}$ and $r \in 2^{A_k}$.*

Moreover, if $\mathcal{D} \subset [\mathbb{P}_\alpha]^{<\omega}$ is a family of pairwise disjoint sets such that $\emptyset \in \mathcal{D}$, $\mathcal{D}$ is closed under refinements, and

- (†) *for every $\mathcal{E} \in \mathcal{D}$ and $E \in \mathbb{P}_\alpha$ which is disjoint with $\bigcup \mathcal{E}$ there exists an $E' \in \mathbb{P}_\alpha \cap \mathcal{P}(E)$ such that $\{E'\} \cup \mathcal{E} \in \mathcal{D}$*

then

- (B) *there exists a refinement $\mathcal{E}'_k \in \mathcal{D}$ of $\mathcal{E}_k$ satisfying (ag) and (sp),*
- (C) *there exists an $\mathcal{E}_{k+1}$ as in (A) such that $\mathcal{E}_{k+1} \in \mathcal{D}$.*

To state $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ we need a few more definitions. For a fixed Polish space X let $\mathcal{F}_{\mathrm{prism}}$ stand for the family of all continuous injections from an $E \in \mathbb{P}_{\omega_1}$ onto perfect subsets of X. Each such injection f is called a *prism* and is considered as a coordinate system imposed on $P = \mathrm{range}(f)$.[1] We will usually abuse this

[1] In a language of forcing a coordinate function f is simply a nice name for an element from X.

terminology and refer to P itself as a *prism* (in X) and to f as a *witness function* for P. A function $g\in\mathcal{F}_{\rm prism}$ is *subprism* of f provided $g\subset f$. In the above spirit we call $Q=\mathrm{range}(g)$ a *subprism of a prism* P. Thus, when we say that Q *a subprism of a prism* $P\in\mathrm{Perf}(X)$ we mean that $Q=f[E]$, where f is a witness function for P and $E\subset\mathrm{dom}(f)$ is an iterated perfect set. Using the fact that $\Phi_{\rm prism}(\alpha)$ is closed under the composition it is easy to see that we can always assume that a witness function of a prism is always defined on the entire space $\mathfrak{C}^\alpha$ for an appropriate α.

Let $\mathrm{Perf}^*(X)$ stand for the family of all sets P such that either $P\in\mathrm{Perf}(X)$ or P is a singleton in X. In what follows we will consider singletons as *constant prisms*, that is, with the constant coordinate function from $\mathfrak{C}^\alpha$ onto the singleton. In particular, a subprism of a constant prism is the same singleton.

Consider the following game $\mathrm{GAME}_{\rm prism}(X)$ of length ω_1. The game has two players, Player I and Player II. At each stage $\xi<\omega_1$ of the game Player I can play an arbitrary prism $P_\xi\in\mathrm{Perf}^*(X)$ and Player II must respond with a subprism Q_ξ of P_ξ. The game $\langle\langle P_\xi,Q_\xi\rangle\colon\xi<\omega_1\rangle$ is won by Player I provided $\bigcup_{\xi<\omega_1}Q_\xi=X$; otherwise the game is won by Player II.

A strategy for Player II is any function S such that $S(\langle\langle P_\eta,Q_\eta\rangle\colon\eta<\xi\rangle,P_\xi)$ is a subprism of P_ξ, where $\langle\langle P_\eta,Q_\eta\rangle\colon\eta<\xi\rangle$ is any partial game. (We abuse here slightly the notation, since function S depends also on the implicitly given coordinate functions f_η making each P_η a prism.) A game $\langle\langle P_\xi,Q_\xi\rangle\colon\xi<\omega_1\rangle$ is played according to a strategy S for Player II when $Q_\xi=S(\langle\langle P_\eta,Q_\eta\rangle\colon\eta<\xi\rangle,P_\xi)$ for every $\xi<\omega_1$. A strategy S for Player II is a *winning strategy* for Player II provided Player II wins any game played according to the strategy S.

Here is the axiom.

$\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$**:** $\mathfrak{c}=\omega_2$ and for any Polish space X Player II has no winning strategy in the game $\mathrm{GAME}_{\rm prism}(X)$.

In what follows we will use the following prism density fact, which proof can be found in [8, lemma 2.1] or in [9, lemma 5.1.5].

Lemma 1.5. *Let $M\subset\mathbb{R}$ be a sigma-compact and linearly independent. Then for every prism P in $\mathbb{R}$ there exist a subprism Q of P and a compact subset R of $P\setminus M$ such that $M\cup R$ is a maximal linearly independent subset of $M\cup Q$.*

We will also use the following fact.

Fact 1.6. $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ *implies that* $\mathrm{cof}(\mathcal{M})=\omega_1$*, where* $\mathrm{cof}(\mathcal{M})$ *is the cofinality of the ideal of meager sets.*

PROOF. It is proved in [4, cor. 4.3] (see also [9, cor. 1.1.3]) that $\mathrm{CPA}_{\rm cube}$ implies that $\mathrm{cof}(\mathcal{N})$, the cofinality of the ideal of measure zero sets, is equal to ω_1, while it is well known that $\mathrm{cof}(\mathcal{N})=\omega_1$ implies that $\mathrm{cof}(\mathcal{M})=\omega_1$. To finish the argument, it is enough to recall that $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ implies $\mathrm{CPA}_{\rm cube}$. (See e.g. [5] or [9].) ∎

2. Multi-games

For a non-empty collection $\mathcal{X}$ of pairwise disjoint Polish spaces consider the following "simultaneous" two-player game $\mathrm{GAME}_{\rm prism}(\mathcal{X})$ of length ω_1. At each stage $\xi<\omega_1$ of the game Player I can play a prism $P_\xi\in\mathrm{Perf}^*(X)$ from an arbitrarily chosen $X \in \mathcal{X}$. Player II responds with a subprism Q_ξ of P_ξ. The game $\langle\langle P_\xi, Q_\xi\rangle : \xi < \omega_1\rangle$ is won by Player I provided

$$\bigcup_{\xi<\omega_1} Q_\xi = \bigcup \mathcal{X};$$

otherwise the game is won by Player II. Thus, for any Polish space X the games $\mathrm{GAME}_{\mathrm{prism}}(X)$ and $\mathrm{GAME}_{\mathrm{prism}}(\{X\})$ are identical.

Theorem 2.1. *Let $\mathcal{X}$ of size $\leq \omega_1$ be a non-empty collection of pairwise disjoint Polish spaces. Then* $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ *is equivalent to*

$\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}(\mathcal{X})$**:** *Player II has no winning strategy in* $\mathrm{GAME}_{\mathrm{prism}}(\mathcal{X})$.

Proof. We will leave the implication "$\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}(\mathcal{X})$ implies $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$" without a proof, since it will not be used in the sequel. Its proof can be found in [9].

To see the converse implication assume that $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ holds and let $I = [0, 1]$. Let $L = \{x_\xi : \xi < \omega_1\}$ be a Luzin set in I, that is, such that $|L \cap N| \leq \omega$ for every nowhere dense subset N of I. The existence of such a set under $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ follows from Fact 1.6.

Let $\kappa = |\mathcal{X}| \leq \omega_1$ and let $\{X_\eta : \eta < \kappa\}$ be an enumeration of $\mathcal{X}$. We will identify each X_η, $\eta < \kappa$, with a G_δ subset of $\{x_\eta\} \times I^\omega$ homeomorphic to it.

Now, let S_0 be a Player II strategy in the game $\mathrm{GAME}_{\mathrm{prism}}(\mathcal{X})$. We will modify it to a Player II strategy S in the game $\mathrm{GAME}_{\mathrm{prism}}(I \times I^\omega)$ in the following way. First, for every prism P in $I \times I^\omega$ let $R(P)$ be its subprism such that

$$\text{either } R(P) \subset X_\eta \text{ for some } \eta < \kappa \text{ or } R(P) \cap \bigcup \mathcal{X} = \emptyset.$$

To choose such $R(P)$ first choose subprism R of P such that its first coordinate projection $\pi[R]$ is nowhere dense in I. (This can be done, for example, applying Lemma 1.2.) So, $\pi[R]$ contains at most countably many points x_η. Thus, by Claim 1.1, there is a subprism R_1 of R such that either $\pi[R_1]$ is disjoint with L or there is an $\eta < \kappa$ such that $\pi[R_1] = \{x_\eta\}$. In the first case we put $R(P) = R_1$. In the second case we use Claim 1.1 to find a subprism $R(P)$ of R_1 such that either $R(P) \subset X_\eta$ or $R(P) \cap X_\eta = \emptyset$.

Now strategy S is defined by induction on ξ, the step level of the game. Thus, if a sequences $\bar{T} = \langle\langle P_\eta, Q_\eta\rangle : \eta < \xi\rangle$ represents a "legal" sequence (a sequence that could have been produced by S defined so far) we define $S(\bar{T}, P_\xi)$ as follows. If $R(P_\xi) \cap \bigcup \mathcal{X} = \emptyset$ we just put $S(\bar{T}, P_\xi) = R(P_\xi)$. For the other case, define $J = \{\eta < \xi : R(\mathcal{P}_\eta) \subset X \text{ for some } X \in \mathcal{X}\}$ and let

$$S(\bar{T}, P_\xi) = S_0(\langle\langle R(P_\eta), Q_\eta\rangle : \eta \in J\rangle, R(P_\xi)),$$

where $\langle\langle R(P_\eta), Q_\eta\rangle : \eta \in J\rangle$ is identified with $\langle\langle R(P_{i(\eta)}), Q_{i(\eta)}\rangle : \eta < \alpha\rangle$, while i is an order isomorphism between an ordinal α and J.

Since, by $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$, S is not winning in $\mathrm{GAME}_{\mathrm{prism}}(I \times I^\omega)$ for Player II there is a game $\langle\langle P_\xi, Q_\xi\rangle : \xi < \omega_1\rangle$ played according to S in which Player I wins. To finish the proof put $K = \{\xi < \omega_1 : R(\mathcal{P}_\xi) \subset \bigcup \mathcal{X}\}$ and notice that $\langle\langle R(P_\xi), Q_\xi\rangle : \xi \in K\rangle$ is a game in $\mathrm{GAME}_{\mathrm{prism}}(\mathcal{X})$ played according to S_0 in which Player I wins. Thus, S_0 is not winning for Player II. ■

3. Additive almost continuous discontinuous function with measure zero graph

The construction presented here can be viewed as a "model example" of how some CH proofs can be modified to the proofs from $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$.

Recall that a function $f\colon \mathbb{R}\to\mathbb{R}$ is *almost continuous* provided any open subset U of $\mathbb{R}^2$ which contains the graph of f contains also a graph of a continuous function from $\mathbb{R}$ to $\mathbb{R}$. It is known that if f is almost continuous then its graph is connected in $\mathbb{R}^2$ (i.e., f is a connectivity function) and that f has the intermediate value property (i.e., f is Darboux). (See e.g. [16] or [3].) Recall also that a function $f\colon \mathbb{R}\to\mathbb{R}$ is *additive* provided $f(x+y)=f(x)+f(y)$ for every $x,y\in\mathbb{R}$. It is well known that every function defined on a Hamel basis can be uniquely extended to an additive function. (See e.g. [1, thm. 7.3.2].)

Our next goal will be to construct an additive discontinuous almost continuous function $f\colon \mathbb{R}\to\mathbb{R}$ whose graph is of measure zero. In fact, we will show that, under $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$, such an f can be found inside a set $(\mathbb{R}\times G)\cup(G\times\mathbb{R})$ for every G_δ subset G of $\mathbb{R}$ with $0\in G$. A first construction of such a function, under Martin's axiom, was given by K. Ciesielski in [2]. Although it can be shown that such a function (i.e., with graph being a subset of $(\mathbb{R}\times G)\cup(G\times\mathbb{R})$) does not exist in the Cohen model it is unknown whether the existence of an additive discontinuous almost continuous function with graph of measure zero can be proved in ZFC alone.

Now we are ready to state the theorem.

Theorem 3.1. $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ *implies that for every dense G_δ set $G\subset\mathbb{R}$ such that $0\in G$ there exists an additive discontinuous almost continuous function $f\colon\mathbb{R}\to\mathbb{R}$ whose graph is a subset of* $(\mathbb{R}\times G)\cup(G\times\mathbb{R})=(G^c\times G^c)^c$.

Using Theorem 3.1 with G of measure zero we obtain immediately the following corollary.

Corollary 3.2. $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ *implies that there exists a discontinuous, almost continuous, additive function $f\colon\mathbb{R}\to\mathbb{R}$ whose graph is of measure zero.*

Notice that if L_m, for $0<m<\omega$, is the collection of all functions $\ell\colon\mathbb{R}^m\to\mathbb{R}$ given by a formula

(1) $\quad \ell(x_0,\dots,x_{m-1})=\sum_{i<m} q_i x_i$, where $q_i\in\mathbb{Q}\setminus\{0\}$ for all $i<m$

then

$$\mathrm{LIN}(Z)=\bigcup_{0<m<\omega}\ \bigcup_{\ell\in L_m}\ell[Z^m].$$

Also $Z\subset\mathbb{R}$ is linearly independent (over $\mathbb{Q}$) provided $\ell(x_0,\dots,x_{m-1})\neq 0$ for every $\ell\in L_m$, $0<m<\omega$, and $\{x_0,\dots,x_{m-1}\}\in[Z]^m$.

Recall also that a function $f\colon\mathbb{R}\to\mathbb{R}$ is almost continuous if and only if it intersects every *blocking set*, that is, a closed set $K\subseteq\mathbb{R}^2$ which meets every continuous function from $\mathcal{C}(\mathbb{R})$ and is disjoint with at least one function from $\mathbb{R}^{\mathbb{R}}$. The domain of every blocking set contains a non-degenerate connected set. (See [12] or [16].) It is important for us that every blocking set contains a graph of a continuous function $g\colon G\to\mathbb{R}$, where G is a dense G_δ subset of some non-trivial interval. (See [13]. This follows from the fact that for every closed bounded set B with domain I, the mapping $I\ni x\mapsto\inf\{y\colon\langle x,y\rangle\in B\}$ is of first Baire class, so it is continuous when restricted to a dense G_δ subset.) Thus, in order to make sure that a function is almost continuous it is enough to insure that its graph intersects every function from the family

$$\mathcal{K} = \bigcup\{\mathcal{C}(G)\colon\ G \text{ is a } G_\delta \text{ second category subset of } \mathbb{R}\}. \tag{2}$$

In what follows we will use the following notation for $G, P \subset \mathbb{R}$:

$$G[P] = \{x \in \mathbb{R}\colon x - P \subset G\} = \bigcap_{p\in P}(p+G). \tag{3}$$

It is also convenient to note that

$$G[P]^c = G^c + P.$$

We start with noticing some simple properties of this operation.

Fact 3.3. *Let $P, S, G, G' \subset \mathbb{R}$.*

(a) *If $P \subset S$ and $G' \subset G$ then $G[P] \supset G'[S]$.*
(b) *If P is compact and G is open then $G[P]$ is open.*
(c) *If $P = \bigcup_{i<\omega} P_i$ and $G = \bigcap_{n<\omega} G_n$ then $G[P] = \bigcap_{i,n<\omega} G_n[P_i]$.*
(d) *If P is sigma compact and G is a G_δ set then $G[P]$ is also a G_δ set.*
(e) *If $G[P_n]$ is a dense G_δ set for every $n<\omega$ then so is $G[\bigcup_{n<\omega} P_n]$.*
(f) *$G[P][S] = G[P+S]$.*

Proof. (a) follows immediately from the second part of (3) while (b) from its first part. To see (c) notice that, by (3),

$$G[P] = \bigcap_{i<\omega}\{x \in \mathbb{R}\colon x - P_i \subset G\} = \bigcap_{i,n<\omega}\{x \in \mathbb{R}\colon x - P_i \subset G_n\} = \bigcap_{i,n<\omega} G_n[P_i].$$

So, (d) follows immediately from (b), while (e) is an easy consequence of (c). Note also that

$$G[P][S]^c = G[P]^c + S = G^c + P + S = G[P+S]^c$$

so (f) holds. ∎

Recall that for a Polish space X the space $\mathcal{C}(X)$ of continuous functions from X into $\mathbb{R}$ is considered with the metric of uniform convergence.

Lemma 3.4. *Let X be a Polish space and $\bar{x} \in \bar{K} \in \mathrm{Perf}(X)$. For every dense G_δ-set $G \subset \mathbb{R}$ and a prism P in $\mathcal{C}(X)$ there exist a subprism Q of P and a $K \in \mathrm{Perf}(\bar{K})$ with $\bar{x} \in K$ such that $G[\mathrm{LIN}(R_K(Q))]$ is a dense G_δ subset of $\mathbb{R}$, where $R_K(Q) = \{h(x)\colon h \in Q \ \&\ x \in K\}$.*

Proof. Let $\mathcal{U}$ be a countable family of open subsets of $\mathbb{R}$ with the property that $G = \bigcap \mathcal{U}$ and fix a countable basis $\mathcal{B}$ for $\mathbb{R}$. For $0 < m < \omega$ let L_m be the set of all functions ℓ defined as in (1) and put $L = \bigcup_{0<m<\omega} L_m$. In what follows for $\ell\colon X^m \to \mathbb{R}$ from L and $Z \subset X$ we will write $\ell[Z]$ in place of $\ell[Z^m]$.

Fix an enumeration $\{\langle U_k, \ell_k, B_k\rangle\colon k<\omega\}$ of $\mathcal{U}\times L\times\mathcal{B}$ and let $h \in \mathcal{F}_{\rm prism}(\mathcal{C}(X))$ be such that $P = h[\mathfrak{C}^\alpha]$. By induction on $k<\omega$ we will construct the sequences $\langle \mathcal{E}_k\colon k<\omega\rangle$ and $\langle \mathcal{K}_k\colon k<\omega\rangle$ such that for every $k<\omega$

(a) $\mathcal{K}_k$ is a family $\{K_t \in \mathrm{Perf}(\bar{K})\colon t \in 2^k\}$ of pairwise disjoint sets such that $\bar{x} \in \bigcup \mathcal{K}_k$,
(b) $K_s \subset K_t$ for each $t \in 2^k$ and $t \subset s \in 2^{k+1}$,
(c) $\mathcal{E}_k = \left\{E_s \in \mathbb{P}_\alpha\colon s \in 2^{A_k}\right\}$,

(d) $\mathcal{E}_k$ and $\mathcal{E}_{k+1}$ satisfy conditions (i), (ii), (ag), and (sp) from Lemma 1.3 for every $s,t\in 2^{A_{k+1}}$ and $r\in 2^{A_k}$,

(e) if $R_k=\{h(g)(x)\colon g\in\bigcup\mathcal{E}_k \ \&\ x\in\bigcup\mathcal{K}_k\}$ then $U_k(\ell_k[R_k])\cap B_k\neq\emptyset$.

Before we construct such sequences, note how this will complete the proof. Clearly, by (a) and (b), sequence $\langle\mathcal{E}_k\colon k<\omega\rangle$ satisfies the assumptions of Lemma 1.3. Thus, $E=\bigcap_{k<\omega}\bigcup\mathcal{E}_k$ belongs to $\mathbb{P}_\alpha$, so $Q=h[E]$ is a subprism of P. Also, if $K=\bigcap_{k<\omega}\bigcup\mathcal{K}_k$ then $\bar{x}\in K\in\mathrm{Perf}(\bar{K})$. To see that $G[\mathrm{LIN}(R_K(Q))]$ is a dense G_δ notice that $R_K\subset R_k$ for all $k<\omega$. So, by (e), we have $U_k(\ell_k[R_K])\cap B_k\neq\emptyset$. In particular, $U(\ell[R_K])$ is dense and open for every $U\in\mathcal{U}$ and $\ell\in L$. Thus, for every $U\in\mathcal{U}$ the set

$$\bigcap_{\ell\in L}U[\ell[R_K(Q)]]=U\left[\bigcup_{\ell\in L}\ell[R_K(Q)]\right]=U\left[\mathrm{LIN}(R_K(Q))\right]$$

is a dense G_δ-set, and so is $G[\mathrm{LIN}(R_K(Q))]=\bigcap_{U\in\mathcal{U}}U\left[\mathrm{LIN}(R_K(Q))\right]$, as desired.

To choose $\mathcal{E}_0=\{E_\emptyset\}$ and $\mathcal{K}_0=\{K_\emptyset\}$ pick $g_0\in\mathfrak{C}^\alpha$, put $y=h(g_0)(\bar{x})$, and let $\{z\}=\ell_0[\{y\}]$. Clearly $U_0[\{z\}]=\{x\in\mathbb{R}\colon x-\{z\}\subset U_0\}$ is open and dense, so there is a $b_0\in B_0$ such that $b_0-\{z\}\subset U_0$. Let $\varepsilon>0$ be such that $b_0-(z-\varepsilon,z+\varepsilon)\subset U_0$. Find a number $\delta>0$ such that $\ell_0[(y-2\delta,y+2\delta)]\subset(z-\varepsilon,z+\varepsilon)$ and a clopen subset $K_\emptyset$ of $\bar{K}$ containing $\bar{x}$ for which $h(g_0)[K_\emptyset]\subset(y-\delta,y+\delta)$. Also, let $\delta_0>0$ be such that the diameter of $h[B_\alpha(g,\delta_0)]$ is less than δ and put $E_\emptyset=B_\alpha(g_0,\delta_0)$. We just need to check (e). But for every $g\in E_\emptyset$ and $x\in K_\emptyset$ we have $|h(g)(x)-y|\leq|h(g)(x)-h(g_0)(x)|+|h(g_0)(x)-h(g_0)(\bar{x})|<2\delta$. So, $R_0\subset(y-2\delta,y+2\delta)$ and $b_0-\ell_0[R_0]\subset b_0-(z-\varepsilon,z+\varepsilon)\subset U_0$. Thus, $b_0\in U_0(\ell_0[R_0])\cap B_0$.

To make an inductive step assume that for some $k<\omega$ families $\mathcal{E}_k$ and $\mathcal{K}_k$ are already constructed. We will find appropriate $\mathcal{E}_{k+1}$ and $\mathcal{K}_{k+1}$. First use Lemma 1.4(A) to pick an $\mathcal{E}'_{k+1}=\left\{E'_s\in\mathbb{P}_\alpha\colon s\in 2^{A_{k+1}}\right\}$ such that (d) holds. For any $s\in 2^{A_{k+1}}$ choose a $g_s\in E'_s$ such that the family $\left\{\{g_s\}\colon s\in 2^{A_{k+1}}\right\}$ satisfies condition (ag). Also, for every $r\in 2^{k+1}$ choose an $x_r\in K_{r\restriction k}$ such that all points in $\bar{X}=\left\{x_r\colon r\in 2^{k+1}\right\}$ are distinct and $\bar{x}\in\bar{X}$. Put $Y=\bigcup\left\{h(g_s)[\bar{X}]\colon s\in 2^{A_{k+1}}\right\}$ and $Z=\ell_{k+1}[Y]$. Clearly $U_{k+1}[Z]=\{x\in\mathbb{R}\colon x-Z\subset U_{k+1}\}=\bigcap_{z\in Z}(z+U_{k+1})$ is open and dense since Z is finite. Thus there is a $b_{k+1}\in B_{k+1}$ such that $b_{k+1}-Z\subset U_{k+1}$. Let $\varepsilon>0$ be such that $b_{k+1}-B(Z,\varepsilon)\subset U_{k+1}$, where $B(Z,\varepsilon)$ is the set of all $x\in\mathbb{R}$ with distance from Z less than ε. Since Y is finite, ℓ_{k+1} is continuous, and $Z=\ell_{k+1}[Y]$, we can find a $\delta>0$ such that $\ell_{k+1}[B(Y,2\delta)]\subset B(Z,\varepsilon)$. Also, for every $r\in 2^{k+1}$ find a clopen subset K_r of $K_{r\restriction k}$ containing x_r such that $h(g_s)[K_r]\subset B(Y,\delta)$ for every $s\in 2^{A_{k+1}}$ and $\mathcal{K}_{k+1}=\{K_r\colon r\in 2^{k+1}\}$ is pairwise disjoint. This ensures (a) and (b). Let $\delta_0>0$ be such that for every $s\in 2^{A_{k+1}}$ the diameter of $h[B_\alpha(g_s,\delta_0)]$ is less than δ and put $E_s=B_\alpha(g_s,\delta_0)\cap E'_s$. It is easy to see that with $\mathcal{E}_{k+1}=\left\{E_s\colon s\in 2^{A_{k+1}}\right\}$ conditions (c) and (d) are satisfied. We just need to check (e). To see it notice that $R_{k+1}\subset B(Y,2\delta)$ since for every $h(g)(x)\in R_{k+1}$ there are $s\in 2^{A_{k+1}}$ and $r\in 2^{k+1}$ such that

$$|h(g)(x)-h(g_s)(x_r)|\leq|h(g)(x)-h(g_s)(x)|+|h(g_s)(x)-h(g_s)(x_r)|<2\delta,$$

while $h(g_s)(x_r)\in Y$. So, $b_{k+1}-\ell_{k+1}[R_{k+1}]\subset b_{k+1}-B(Z,\varepsilon)\subset U_{k+1}$. Thus, $b_{k+1}\in U_{k+1}(\ell_{k+1}[R_{k+1}])\cap B_{k+1}$. ■

As a corollary, needed in the proof but also interesting on its own, we conclude the following.

Lemma 3.5. *For every dense G_δ subset G of $\mathbb{R}$ and for every prism P in $\mathbb{R}$ there exists a subprism Q of P such that $G[\mathrm{LIN}(Q)]$ is a dense G_δ subset of $\mathbb{R}$.*

PROOF. Let $f \in \Phi_{\mathrm{prism}}(\alpha)$ be such that $P = f[\mathfrak{C}^\alpha]$ and let $h\colon \mathbb{R} \to \mathcal{C}(\mathbb{R})$ be given by $h(r)(x) = r + x$. Then $h[P]$ is a prism in $\mathcal{C}(\mathbb{R})$ witnessed by $h \circ f$. By Lemma 3.4 there exist a subprism $Q_0 = h \circ f[E]$ of $h[P]$ and a $K \in \mathrm{Perf}(\mathbb{R})$ with $0 \in K$ such that $Z = G[\mathrm{LIN}(\{g(x)\colon g \in Q_0 \ \&\ x \in K\})]$ is dense in $\mathbb{R}$. But then $Q = f[E] = h^{-1}(Q)$ is a subprism of P and, since $0 \in K$,

$$\begin{aligned} Z &= G[\mathrm{LIN}(\{h(r)(x)\colon r \in Q \ \&\ x \in K\})] \\ &= G[\mathrm{LIN}(\{r + x\colon r \in Q \ \&\ x \in K\})] \\ &\subset G[\mathrm{LIN}(\{r\colon r \in Q\})] \\ &= G[\mathrm{LIN}(Q)]. \end{aligned}$$

So, $G[\mathrm{LIN}(Q)]$ is dense. It is G_δ by Fact 3.3(d) since $\mathrm{LIN}(Q)$ is sigma compact. ■

We will also need the following fact about perfect sets.

Lemma 3.6. *Let G be a proper dense G_δ-subset of $\mathbb{R}$, W a second category G_δ-subset of $\mathbb{R}$, and let M be an F_σ-subset of $\mathbb{R}$ such that $G[\mathrm{LIN}(M)]$ is a dense G_δ-subset of $\mathbb{R}$. Then there exists a linearly independent set $K \in \mathrm{Perf}(W)$ such that $G[\mathrm{LIN}(M \cup K)]$ is dense, $\mathrm{LIN}(M) \cap \mathrm{LIN}(K) = \{0\}$, and $\mathrm{LIN}(M \cup K) \setminus \mathrm{LIN}(M) \subset G$.*

In particular, if M is linearly independent then so is $M \cup K$.

PROOF. First note that the density of $G[\mathrm{LIN}(M)]$ implies $\mathrm{LIN}(M) \neq \mathbb{R}$. So, $\mathrm{LIN}(M)$ must be of first category.

Replacing G with $\bigcap\{q\,G\colon q \in \mathbb{Q} \setminus \{0\}\}$, if necessary, we can assume that $q\,G = G$ for every $q \in \mathbb{Q} \setminus \{0\}$. Notice that then for every $q \in \mathbb{Q} \setminus \{0\}$ and linear subspace V of $\mathbb{R}$ we also have

$$q\,G[V] = \{q\,x\colon x - V \subset G\} = \{y\colon (y/q) - V \subset G\} = \{y\colon y - q\,V \subset q\,G\} = G[V].$$

Let J be a non-empty open interval such that W is dense in J and let $\langle G_k\colon k < \omega\rangle$ and $\langle W_k\colon k < \omega\rangle$ be the decreasing sequences of open subsets of $\mathbb{R}$ such that $G = \bigcap_{k<\omega} G_k$ and $W \cap J = \bigcap_{k<\omega} W_k$. Choose an increasing sequence $\langle M_k\colon k < \omega\rangle$ of compact sets such that $\mathrm{LIN}(M) = \bigcup_{k<\omega} M_k$, let $\mathcal{R}$ be a family of all triples $\langle \ell, m, n\rangle$ such that $m, n < \omega$, $n > 0$, and $\ell \in L_{m+n}$, where L_i's are as in (1), and fix a sequence $\langle\langle \ell_k, m_k, n_k\rangle \in \mathcal{R}\colon k < \omega\rangle$ with each triple appearing infinitely many times. We will construct, by induction on $k < \omega$, a sequence $\langle U_s\colon s \in 2^k \ \&\ k < \omega\rangle$ of non-empty open subsets of $\mathbb{R}$ such that $U_\emptyset = J$ and for every $0 < k < \omega$ and $s \in 2^{k-1}$ the following inductive conditions hold.

(a) $\mathrm{cl}(U_{s\hat{\ }0})$ and $\mathrm{cl}(U_{s\hat{\ }1})$ are disjoint subsets of $U_s \cap W_k$.
(b) $\ell_k(\bar{a}, x_1, \ldots, x_{n_k}) \in G_k \setminus M_k$ for every $\bar{a} \in (M_k)^{m_k}$ and x_j chosen from different U_t with $t \in 2^k$.

To see that such a sequence can be built assume that for some $0 < k < \omega$ the sets $\{U_s\colon s \in 2^k\}$ have been already constructed. Let $\{t_i\colon i < 2^k\}$ be an enumeration of 2^k and by induction on i choose

$$x_{t_i} \in U_{t_i\restriction k-1} \cap (W \setminus \mathrm{LIN}(M \cup \{x_{t_j}: j < i\}) \cap \bigcap_{y \in \mathrm{LIN}\{x_{t_j}\,:\, j<i\}} (y + G[\mathrm{LIN}(M)]).$$

The choice can be made since $U_{t_i\restriction k-1}$ is non-empty and open while the remaining sets are dense G_δ's in $U_{t_i\restriction k-1}\subset J$. Notice that the choice guarantees that

(4) (b) holds for x_j chosen as different elements of $\{x_t\colon t\in 2^k\}$.

To see it first notice that clearly $\{x_{t_i}\colon i<2^k\}$ is linearly independent and that $\mathrm{LIN}(M)\cap\mathrm{LIN}(\{x_{t_i}\colon i<2^k\})=\{0\}$. Also, $q\,x_{t_i}-\mathrm{LIN}\{x_{t_j}\colon j<i\}\subset G[\mathrm{LIN}(M)]$, that means, $q\,x_{t_i}-\mathrm{LIN}\{x_{t_j}\colon j<i\}-\mathrm{LIN}(M)\subset G$ for every $q\in\mathbb{Q}\setminus\{0\}$ and $i<2^k$. But if $\bar a\in (M_k)^{m_k}$ and $\{x_1,\ldots,x_{n_k}\}\in[\{x_{t_i}\colon i<2^k\}]^{n_k}$ then for appropriate $q\in\mathbb{Q}\setminus\{0\}$ and $i<2^k$ we have

$$\ell_k(\bar a,x_1,\ldots,x_{n_k})\in q\,x_{t_i}-\mathrm{LIN}\{x_{t_j}\colon j<i\}-\mathrm{LIN}(M)\subset G\setminus\mathrm{LIN}(M).$$

So, (4) is proved.

Now, by the compactness of M_k and continuity of ℓ_k, the set

$$Z=\{\langle x_1,\ldots,x_{n_k}\rangle\colon(\exists\bar a\in(M_k)^{m_k})\ \ell_k(\bar a,x_1,\ldots,x_{n_k})\in G_k\setminus M_k\}$$

is open and, by (4), contains all one-to-one sequences $\bar s$ of points from the set $\{x_t\colon t\in 2^k\}$. Since there is only finitely many such sequences $\bar s$ we can find disjoint basic clopen neighborhoods U_t of x_t such that (a) and (b) hold. This finishes the inductive construction.

Let $K_0=\bigcap_{k<\omega}\bigcup_{t\in 2^k}U_t$. By (a), K_0 is a perfect subset of W. Notice also that, by condition (b),

$$T=\bigcup_{m,n<\omega}\{\ell(\bar a,x_0,\ldots,x_n)\colon\bar a\in M^m\ \&\ \{x_0,\ldots,x_n\}\in[K_0]^{n+1}\ \&\ \ell\in L_{n+m}\}$$

is a subset of $G\setminus\mathrm{LIN}(M)$. Clearly $0\in\mathrm{LIN}(M)$, so $0\notin T$. Thus K_0 is linearly independent and $\mathrm{LIN}(M)\cap\mathrm{LIN}(K_0)=\{0\}$. So, $\mathrm{LIN}(M\cup K_0)\setminus\mathrm{LIN}(M)=T\subset G$.

Now fix an $x\in K_0$ and $K\in\mathrm{Perf}(K_0\setminus\{x\})$. Then for every $q\in\mathbb{Q}\setminus\{0\}$ and $v\in\mathrm{LIN}(M\cup K)$ we have $q\,x-v\in\mathrm{LIN}(M\cup K_0)\setminus\mathrm{LIN}(M)\subset G$. Thus, $q\,x-\mathrm{LIN}(M\cup K)\subset G$ and so, $G[\mathrm{LIN}(M\cup K)]$ contains a set $\{q\,x\colon q\in\mathbb{Q}\setminus\{0\}\}$, which is clearly dense. Thus, K is as desired. ∎

We will also need the following strengthening of [8, thm. 1.1]. (See also [9, thm. 5.1.7].)

Proposition 3.7. $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ *implies that for every dense G_δ subset G of $\mathbb{R}$ there is a family $\mathcal{H}$ of compact pairwise disjoint sets such that $H=\bigcup\mathcal{H}$ is a Hamel basis and for every non-meager G_δ subset B of $\mathbb{R}$ and every countable $\mathcal{H}_0\subset\mathcal{H}$ there exists an uncountable $H\in\mathcal{H}\setminus\mathcal{H}_0$ such that $H\subset B$ and $\mathrm{LIN}(H\cup\bigcup\mathcal{H}_0)\setminus\mathrm{LIN}(\bigcup\mathcal{H}_0)\subset G$.*

Proof. First notice that if $\mathcal{G}_\delta$ stands for the family of all G_δ second category subsets of $\mathbb{R}$ then, assuming $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$, there exists a $\mathcal{B}\in[\mathcal{G}_\delta]^{\omega_1}$ coinitial with $\mathcal{G}_\delta$, that is, such that

(5) for every $G\in\mathcal{G}_\delta$ there exists a $B\in\mathcal{B}$ such that $B\subset G$.

Indeed, since $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ implies $\mathrm{cof}(\mathcal{M})=\omega_1$ (see [4, cor. 4.3] or [9, cor. 1.3.3]) there exists a decreasing sequence $\langle G_\xi\colon\xi<\omega_1\rangle$ of dense G_δ subsets of $\mathbb{R}$ such that for every dense G_δ-set $W\subset\mathbb{R}$ there exists a $\xi<\omega_1$ with $G_\xi\subset W$. It is easy to see that $\mathcal{B}=\{G_\xi\cap(p_0,p_1)\colon\xi<\omega_1\ \&\ p_0,p_1\in\mathbb{Q}\ \&\ p_0<p_1\}$ satisfies (5).

Decreasing set G, if necessary, we can assume that $G\neq\mathbb{R}$. Fix a sequence $\langle B_\xi\in\mathcal{B}\colon\xi<\omega_1\rangle$ in which each $B\in\mathcal{B}$ is listed ω_1-many times. For a sequence $\langle P_\xi\colon\xi<\omega_1\rangle$ of prisms in $\mathbb{R}$ representing potential play of Player I construct a sequence $\langle\langle Q_\xi, R^0_\xi, R^1_\xi\rangle \colon \xi < \omega_1\rangle$ such that the following inductive conditions hold for every $\xi < \omega_1$, where $R_\xi = \bigcup_{\eta<\xi}(R^0_\eta \cup R^1_\eta)$.

(i) Sets $\{R^i_\eta \colon \eta \le \xi \ \&\ i < 2\}$ are compact, pairwise disjoint.
(ii) $R_{\xi+1} = \bigcup\{R^i_\eta \colon \eta \le \xi \ \&\ i < 2\}$ is linearly independent over $\mathbb{Q}$.
(iii) Q_ξ is a subprism of P_ξ and $Q_\xi \subset \mathrm{LIN}(R_{\xi+1})$.
(iv) $R^0_\xi \in \mathrm{Perf}(B_\xi)$ and $\mathrm{LIN}(R_\xi \cup R^0_\xi) \setminus \mathrm{LIN}(R_\xi) \subset G$.
(v) $G[\mathrm{LIN}(R_{\xi+1})]$ is a dense G_δ in $\mathbb{R}$.

To make an inductive step assume that for some $\xi < \omega_1$ the required sequence $\langle\langle Q_\zeta, R^0_\zeta, R^1_\zeta\rangle \colon \zeta < \xi\rangle$ is already constructed. So, R_ξ is already defined and, by the inductive assumption, R_ξ is clearly linearly independent. Next notice that

$$G[\mathrm{LIN}(R_\xi)] \text{ is a dense } G_\delta.$$

If $\xi = \eta + 1$ then it follows from (v) for η. On the other hand, if ξ is a limit ordinal then $G\left[\mathrm{LIN}(R_\xi)\right] = G\left[\bigcup_{\eta<\xi}\mathrm{LIN}(R_{\eta+1})\right] = \bigcap_{\eta<\xi} G\left[\mathrm{LIN}(R_{\eta+1})\right]$ so it follows from the inductive assumption as well.

We define R^0_ξ as a K from Lemma 3.6 applied to $W = B_\xi$ and $M = R_\xi$. This guarantees (iv), $R_\xi \cap R^0_\xi = \emptyset$, density of $G[\mathrm{LIN}(R_\xi \cup R^0_\xi)]$, and linear indpendence of $R_\xi \cup R^0_\xi$.

Next use Lemma 3.5 to prism P_ξ and $G[\mathrm{LIN}(R_\xi \cup R^0_\xi)]$ to find a subprism Q' of P_ξ such that

$$\begin{aligned} G[\mathrm{LIN}(R_\xi \cup R^0_\xi)][\mathrm{LIN}(Q')] &= G[\mathrm{LIN}(R_\xi \cup R^0_\xi) + \mathrm{LIN}(Q')] \\ &= G[\mathrm{LIN}(R_\xi \cup R^0_\xi \cup Q')] \end{aligned}$$

is a dense G_δ, where the first equation follows from Fact 3.3(f). Further, apply Lemma 1.5 to $M = R_\xi \cup R^0_\xi$ and prism $P = Q'$ to find a subprism Q_ξ of Q' and a compact R^1_ξ subset of $Q' \setminus M$ such that $M \cup R^1_\xi$ is a maximal linearly independent subset of $M \cup Q_\xi$.

The maximality immediately implies $Q_\xi \subset \mathrm{LIN}(M \cup R^1_\xi) = \mathrm{LIN}(R_{\xi+1})$ so (iii) holds. We also clearly have (i) and (ii). Condition (v) follows from the density of $G[\mathrm{LIN}(R_\xi \cup R^0_\xi \cup Q')]$ and the fact that $R^1_\xi \subset Q'$. This finishes the inductive construction.

Now, if S is a Player II strategy associated with our construction, then by $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$, there exists a game $\langle\langle P_\xi, Q_\xi\rangle \colon \xi < \omega_1\rangle$ played according to S in which $\mathbb{R} = \bigcup_{\xi<\omega_1} Q_\xi$. Let $\langle\langle R^0_\xi, R^1_\xi\rangle \colon \xi < \omega_1\rangle$ be a sequence associated with this game. Then $\mathcal{H} = \{R^i_\xi \colon \xi < \omega_1 \ \&\ i < 2\}$ is as desired. ■

Proof of Theorem 3.1. Let $\mathcal{X} = \{\mathcal{C}(B) \colon B \in \mathcal{B}\}$ where $\mathcal{B}$ is as in (5). We will play $\mathrm{GAME}_{\mathrm{prism}}(\mathcal{X})$ in which, by Theorem 2.1, Player II has no winning strategy. Notice that since each $g \in \mathcal{K}$, where $\mathcal{K}$ is defined as in (2), contains some function from $\bigcup\mathcal{X}$, every function f intersecting each $g \in \bigcup\mathcal{X}$ is almost continuous.

Let $\mathcal{H} = \{H_\xi \colon \xi < \omega_1\}$ be as in Proposition 3.7. We also fix a sequence $\bar{P} = \langle P_\xi \colon \xi < \omega_1\rangle$ such that each P_ξ represents a prism in some $\mathcal{C}(B) \in \mathcal{X}$. Sequence $\bar{P}$ represents potential play for Player I in $\mathrm{GAME}_{\mathrm{prism}}(\mathcal{X})$ and we will construct, by induction, a strategy S for Player II which will describe a game played according to S in response to $\bar{P}$. To make S a legitimate strategy its value at stage $\xi < \omega_1$ will depend only on $\bar{P}_\xi = \langle P_\eta \colon \eta \le \xi\rangle$.

So, construct a sequence $\langle\langle H_\xi^0, H_\xi^1, Q_\xi, K_\xi, R_\xi, Y_\xi\rangle\colon \xi<\omega_1\rangle$ of subsets of $\mathbb{R}$ such that for every $\xi<\omega_1$ the following inductive conditions are satisfied, where $B_\xi\in\mathcal{B}$ is such that $P_\xi\subset\mathcal{C}(B_\xi)$ and $\mathcal{H}_\xi=\{H_\eta^i\colon \eta<\xi\ \&\ i<2\}$.

(I) H_ξ^0 and H_ξ^1 are distinct elements of $\mathcal{H}\setminus\bigcup\mathcal{H}_\xi$.
(II) $H_\xi^0\in[B_\xi]^{\mathfrak{c}}$ and $\mathrm{LIN}(H_\xi^0\cup\bigcup\mathcal{H}_\xi)\setminus\mathrm{LIN}(\bigcup\mathcal{H}_\xi)\subset G$.
(III) $H_\xi\in\{H_\eta^i\colon \eta\le\xi\ \&\ i<2\}$.

We can choose such H_ξ^0 and H_ξ^1 since $\mathcal{H}$ was taken from Proposition 3.7. Also, if $F_\xi=\bigcup_{\eta<\xi}(R_\eta\cup Y_\eta)$ and $U_\xi=G[\mathrm{LIN}(F_\xi)]$ then

(IV) U_ξ is a dense G_δ in $\mathbb{R}$,
(V) $K_\xi\in\mathrm{Perf}(H_\xi^0)$, Q_ξ is a subprism of P_ξ, $R_\xi=\{h(x)\colon h\in Q_\xi,\ x\in K_\xi\}$, and $U_\xi[\mathrm{LIN}(R_\xi)]$ is dense G_δ in $\mathbb{R}$, and
(VI) $Y_\xi\in\mathrm{Perf}(\mathbb{R})$ is a linearly independent set such that $G[\mathrm{LIN}(F_{\xi+1})]$ is dense, $\mathrm{LIN}(F_\xi\cup R_\xi)\cap\mathrm{LIN}(Y_\xi)=\{0\}$, and $\mathrm{LIN}(F_\xi\cup R_\xi\cup Y_\xi)\setminus\mathrm{LIN}(F_\xi\cup R_\xi)$ is a subset of G.

Assuming that (IV) holds the possibility of a choice of Q_ξ, K_ξ, and R_ξ as in (V) follows directly from Lemma 3.4. Next, since by Fact 3.3(f)

$$\begin{aligned} U_\xi[\mathrm{LIN}(R_\xi)] &= G[\mathrm{LIN}(F_\xi)][\mathrm{LIN}(R_\xi)]\\ &= G[\mathrm{LIN}(F_\xi)+\mathrm{LIN}(R_\xi)]\\ &= G[\mathrm{LIN}(F_\xi\cup R_\xi)]\end{aligned}$$

we can apply Lemma 3.6 to our G, $W=\mathbb{R}$, and $M=F_\xi\cup R_\xi$ to find a linearly independent $K\in\mathrm{Perf}(\mathbb{R})$ for which $G[\mathrm{LIN}(F_\xi\cup R_\xi\cup K)]$ is a dense G_δ subset of $\mathbb{R}$, $\mathrm{LIN}(F_\xi\cup R_\xi\cup K)\setminus\mathrm{LIN}(F_\xi\cup R_\xi)\subset G$, and $\mathrm{LIN}(F_\xi\cup R_\xi)\cap\mathrm{LIN}(K)=\{0\}$. Then put $Y_\xi=K$ and notice that (VI) is satisfied, since $F_{\xi+1}=F_\xi\cup R_\xi\cup Y_\xi$.

To finish the construction it is enough to argue that (IV) is preserved. But if $\xi=\eta+1$ is a successor ordinal then it follows immediately from (VI) for η. But if ξ is a limit ordinal then (IV) follows easily from the density of sets U_η for $\eta<\xi$ since $U_\xi = G\left[\bigcup_{\eta<\xi}\mathrm{LIN}\left(\bigcup_{\zeta<\eta}(R_\zeta\cup\{y_\zeta\})\right)\right]=\bigcap_{\eta<\xi}U_\eta$. This finishes the inductive construction of the sequence.

We define a strategy S for Player II by $S(\langle\langle P_\eta,Q_\eta\rangle\colon\eta<\xi\rangle,P_\xi)=Q_\xi$. By Theorem 2.1 this is not a winning strategy, so there exists a game $\langle\langle P_\xi,Q_\xi\rangle\colon\xi<\omega_1\rangle$ played according to S in which $\bigcup\mathcal{X}=\bigcup_{\xi<\omega_1}Q_\xi$. We will use the sequence $\langle\langle H_\xi^0,H_\xi^1,Q_\xi,K_\xi,R_\xi,y_\xi\rangle\colon\xi<\omega_1\rangle$ associated with this game to construct the desired function f.

Since, by (I) and (III), $\{H_\xi^i\colon\xi<\omega_1\ \&\ i<2\}=\mathcal{H}$, it is enough to define f on each H_ξ^i and extend it to a unique additive function. So, for each $\xi<\omega$ define f on H_ξ^1 as a one-to-one function with values in Y_ξ. On each H_ξ^0 we define f such that $f[H_\xi^0]\subset R_\xi$ and f intersects every $g\in Q_\xi$ on a set K_ξ. It remains to prove that f is as advertised.

Certainly f is additive. It is also not difficult to see that f defined that way cannot be continuous. To see that it is almost continuous it is enough to notice that every $g\in\bigcup\mathcal{X}$ belongs to some Q_ξ, so it is intersected by f. To finish the proof it is enough to show that $f\subset(\mathbb{R}\times G)\cup(G\times\mathbb{R})$. So, define f_ξ as $f\restriction\mathrm{LIN}(\bigcup\mathcal{H}_\xi)$. Since $f=\bigcup_{\xi<\omega_1}f_\xi$, it is enough to prove that

$$f_\eta\subset(\mathbb{R}\times G)\cup(G\times\mathbb{R}) \tag{6}$$

for every $\eta < \omega_1$. This will be proved by induction.

Clearly $f_0 = \{\langle 0,0\rangle\} \subset (\mathbb{R}\times G)\cup(G\times\mathbb{R})$ since $0\in G$. So assume that for some $0<\eta<\omega_1$ condition (6) holds for every $\zeta<\eta$. If η is a limit ordinal then $f_\eta = \bigcup_{\zeta<\eta} f_\zeta$ so (6) clearly holds. So assume that $\eta=\xi+1$ and notice that

$$f^*_\eta = f_\eta \restriction \mathrm{LIN}(H^0_\xi \cup \bigcup \mathcal{H}_\xi) \text{ is a subset of } (\mathbb{R}\times G)\cup(G\times\mathbb{R}).$$

This is the case since $f_\xi \subset (\mathbb{R}\times G)\cup(G\times\mathbb{R})$ by the inductive assumption while $f^*_\eta \setminus f_\xi \subset (G\times\mathbb{R})$ since $\mathrm{dom}(f^*_\eta\setminus f_\xi) = \mathrm{LIN}(H^0_\xi\cup\bigcup\mathcal{H}_\xi)\setminus\mathrm{LIN}(\bigcup\mathcal{H}_\xi) \subset G$ is guaranteed by (II).

Thus, to finish the proof, it is enough to show that

$$f_\eta \setminus f^*_\eta \subset (\mathbb{R}\times G).$$

To see it first note that, from our construction, $\mathrm{range}(f^*_\eta)\subset \mathrm{LIN}(F_\xi\cup R_\xi)$. Now, if $x\in\mathrm{dom}(f_\eta\setminus f^*_\eta) = \mathrm{LIN}(\bigcup\mathcal{H}_{\xi+1})\setminus\mathrm{LIN}(H^0_\xi\cup\bigcup\mathcal{H}_\xi)$ then $x=v+w$ for some $v\in\mathrm{LIN}(H^1_\xi)\setminus\{0\}$ and $w\in\mathrm{LIN}(H^0_\xi\cup\bigcup\mathcal{H}_\xi)$. Hence, by the definition of f and condition (VI),

$$\begin{aligned} f_\eta(x) = f_\eta(v)+f_\eta(w) &\in (\mathrm{LIN}(Y_\xi)\setminus\{0\}) + \mathrm{LIN}(F_\xi\cup R_\xi)\\ &= \mathrm{LIN}(F_\xi\cup R_\xi\cup Y_\xi)\setminus\mathrm{LIN}(F_\xi\cup R_\xi)\subset G. \end{aligned}$$

This completes the proof. ■

4. Hamel basis

For a subset A of $\mathbb{R}$ we define $E^+(A)$ as

$$E^+(A) = \left\{\sum_{i=0}^k q_i a_i \colon k<\omega \;\&\; a_i\in A \;\&\; q_i\in\mathbb{Q}\cap[0,\infty) \text{ for every } i\le k\right\}.$$

In [10] P. Erdős proved that under the continuum hypothesis there exists a Hamel basis H for which $E^+(H)$ is a Luzin set. In particular, such $E^+(H)$ is of measure zero. K. Muthuvel [15], answering a question of H. Miller [14], generalized Erdős' result by proving that, under Martin's axiom, there exists a Hamel basis H for which $E^+(H)$ is simultaneously of measure zero and first category. However, it is unknown whether there is a ZFC example of a Hamel basis H for which $E^+(H)$ is of measure zero. In what follows we show that the existence of such a Hamel basis is a consequence of $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$.

Theorem 4.1. $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ *implies that for every dense G_δ subset G of $\mathbb{R}$ with $0\in G$ there exists an $A\subset\mathbb{R}$ such that $\mathrm{LIN}(A)=\mathbb{R}$ and $E^+(A)\subset G$.*

Using Theorem 4.1 with G of measure zero and the fact that every set A spanning $\mathbb{R}$ contains a Hamel basis we obtain immediately the following corollary.

Corollary 4.2. $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ *implies that there exists a Hamel basis H such that $E^+(H)$ has measure zero.*

Proof of Theorem 4.1. Decreasing G, if necessary, we can assume that $qG=G$ for every non-zero $q\in\mathbb{Q}$. Since G has a Polish metric, we can use $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ for $\mathrm{GAME}_{\mathrm{prism}}(X)$ with $X=G$.

Fix a sequence $\bar{P} = \langle P_\xi\colon \xi<\omega_1\rangle$ such that each P_ξ represents a prism in X. Sequence $\bar{P}$ represents a potential play for Player I. We will construct, by induction, a strategy S for Player II which will describe a game played according to S in response to $\bar{P}$. The value of S at stage $\xi<\omega_1$ will depend only on $\bar{P}_\xi = \langle P_\eta\colon \eta\le\xi\rangle$.

For this, we will construct a sequence $\langle\langle Q_\xi, A_\xi\rangle\colon \xi<\omega_1\rangle$ of pairs of sigma-compact subsets of $\mathbb{R}$ such that for every $\zeta\le\xi<\omega_1$

(I) Q_ξ is a subprism of P_ξ,
(II) $A_\zeta\subset A_\xi$ and $\bigcup_{\eta\le\xi} Q_\eta\subset \mathrm{LIN}(A_\xi)$,
(III) set $G[E^+(A_\xi)]$ is dense and $E^+(A_\xi)\subset G$.

Assume that for some $\xi<\omega_1$ the desired sequence $\langle\langle Q_\eta, A_\eta\rangle\colon \eta<\xi\rangle$ is already constructed. Let $B_\xi=\bigcup_{\eta<\xi} A_\eta$. Then $E^+[B_\xi]=\bigcup_{\eta<\xi} E^+[A_\eta]$ is sigma-compact and $G_\xi=G[E^+(B_\xi)]=\bigcap_{\eta<\xi} G[E^+(A_\eta)]$ is a dense G_δ. Thus, by Lemma 3.5, we can find a subprism Q_ξ of P_ξ such that $G_\xi[\mathrm{LIN}(Q_\xi)]$ is a dense G_δ subset of $\mathbb{R}$. Since

$$G_\xi[\mathrm{LIN}(Q_\xi)]=G[E^+(B_\xi)][\mathrm{LIN}(Q_\xi)]=G[E^+(B_\xi)+\mathrm{LIN}(Q_\xi)]$$

there exists an $x\in\mathbb{R}$ such that $x+E^+(B_\xi)+\mathrm{LIN}(Q_\xi)\subset G$. Therefore, we have also $qx+E^+(B_\xi)+\mathrm{LIN}(Q_\xi)\subset G$ for every non-zero $q\in\mathbb{Q}$. Let us define $C_\xi=x+\mathrm{LIN}(Q_\xi)$ and put $A_\xi=B_\xi\cup C_\xi$. This clearly ensures (II). To see $E^+(A_\xi)\subset G$ notice that every element of $E^+(A_\xi)$ either belongs to $E^+(B_\xi)\subset G$ or to $qx+E^+(B_\xi)+\mathrm{LIN}(Q_\xi)\subset G$ for some positive $q\in\mathbb{Q}$. The density of $G[E^+(A_\xi)]$ follows from

$$\begin{aligned}
G[E^+(A_\xi)] &= G[E^+(B_\xi)+E^+(C_\xi)]\\
&= G[E^+(B_\xi)][E^+(C_\xi)]\\
&= G_\xi[E^+(C_\xi)]\\
&= G_\xi\left[\bigcup_{q\in\mathbb{Q}^+}(qx+\mathrm{LIN}(Q_\xi))\right]\\
&= \bigcap_{q\in\mathbb{Q}^+} G_\xi\left[qx+\mathrm{LIN}(Q_\xi)\right]\\
&= \bigcap_{q\in\mathbb{Q}^+}(qx+G_\xi\left[\mathrm{LIN}(Q_\xi)\right]),
\end{aligned}$$

where $\mathbb{Q}^+=\mathbb{Q}\cap(0,\infty)$, since $G_\xi\left[\mathrm{LIN}(Q_\xi)\right]$ is a dense G_δ. This finishes the inductive construction.

Let S be a strategy of Player II given by the above inductive construction. Since S is not winning, there is a game $\langle\langle P_\xi, Q_\xi\rangle\colon \xi<\omega_1\rangle$ played according to S in which $G=X=\bigcup_{\xi<\omega_1} Q_\xi$. Thus, for $A=\bigcup_{\xi<\omega_1} A_\xi$ condition (III) implies that $E^+(A)\subset G$, while by (II) we have $\mathbb{R}=\mathrm{LIN}(G)=\mathrm{LIN}\left(\bigcup_{\xi<\omega_1} Q_\xi\right)\subset\mathrm{LIN}(A)$. ■

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