Overview

Many interesting mathematical properties, especially those concerning real analysis, are known to be true in the iterated perfect set (Sacks) model, while they are false under the continuum hypothesis. However, the proofs that these facts are indeed true in this model are usually very technical and involve heavy forcing machinery. In this book we extract a combinatorial principle, an axiom similar to Martin's axiom, that is true in the model and show that this axiom implies the above-mentioned properties in a simple "mathematical" way. The proofs are essentially simpler than the original arguments.

It is also important that our axiom, which we call the *Covering Property Axiom* and denote by CPA, captures the essence of the Sacks model at least if it concerns most cardinal characteristics of continuum. This follows from a recent result of J. Zapletal [131], who proved that for a "nice" cardinal invariant κ, if $\kappa < \mathfrak{c}$ holds in any forcing extension, then $\kappa < \mathfrak{c}$ follows already from CPA. (In fact, $\kappa < \mathfrak{c}$ follows already from its weaker form, which we denote $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$.)

To follow all but the last chapter of this book only a moderate knowledge of set theory is required. No forcing knowledge is necessary.

The iterated perfect set model, also known as the iterated Sacks model, is a model of the set theory ZFC in which the continuum $\mathfrak{c} = \omega_2$ and many of the consequences of the continuum hypothesis (CH) fail. In this book we describe a combinatorial axiom of the form similar to Martin's axiom, which holds in the iterated perfect set model and represents a combinatorial core of this model – it implies all the "general mathematical statements" that are known (to us) to be true in this model.

It should be mentioned here that our axiom is more an axiom schema with the perfect set forcing being a "built-in" parameter. Similar axioms also hold for several other forcings (like iterated Miller and iterated Laver forcings; see, e.g., [131, sec. 5.1]). In this book, however, we concentrate only on the axiom associated with the iterated perfect set model. This is dictated by two reasons: The axiom has the simplest form in this particular model, and the iterated perfect set model is the most studied from the class of forcing models we are interested in — we have a good supply of statements against which we can test the power of our axiom. In particular, we use for this purpose the statements listed below as (A)–(H). The citations in the parentheses refer to the proofs that a given property holds in the iterated perfect set model. For the definitions see the end of the next section.

(A) For every subset S of $\mathbb{R}$ of cardinality $\mathfrak{c}$ there exists a (uniformly) continuous function $f\colon\mathbb{R}\to[0,1]$ such that $f[S]=[0,1]$. (A. Miller [95])

(B) Every perfectly meager set $S\subset\mathbb{R}$ has cardinality less than $\mathfrak{c}$. (A. Miller [95])

(C) Every universally null set $S\subset\mathbb{R}$ has cardinality less than $\mathfrak{c}$. (R. Laver [84])

(D) The cofinality of the ideal $\mathcal{N}$ of null (i.e., Lebesgue measure zero) sets is less than $\mathfrak{c}$. (Folklore, see, e.g., [97] or [4, p. 339])

(E) There exist selective ultrafilters on ω, and any such ultrafilter is generated by less than $\mathfrak{c}$ many sets. (J. Baumgartner and R. Laver [7])

(F) There is no Darboux Sierpiński-Zygmund function $f\colon\mathbb{R}\to\mathbb{R}$; that is, for every Darboux function $f\colon\mathbb{R}\to\mathbb{R}$ there is a subset Y of $\mathbb{R}$ of cardinality $\mathfrak{c}$ such that $f\restriction Y$ is continuous. (M. Balcerzak, K. Ciesielski, and T. Natkaniec [2])

(G) For every Darboux function $g\colon\mathbb{R}\to\mathbb{R}$ there is a continuous nowhere constant function $f\colon\mathbb{R}\to\mathbb{R}$ such that $f+g$ is Darboux. (J. Steprāns [123])

(H) The plane $\mathbb{R}^2$ can be covered by less than $\mathfrak{c}$ many sets, each of which is a graph of a differentiable function (allowing infinite derivatives) of either a horizontal or vertical axis. (J. Steprāns [124])

The counterexamples under CH for (B) and (C) are Luzin and Sierpiński sets. They have been constructed in [87][1] and [116], respectively. The negation of (A) is witnessed by either a Luzin or a Sierpiński set, as noticed in [116, 117]. The counterexamples under CH for (F) and (G) can be found in [2] and [78], respectively. The fact that (D), (E), and (H) are false under CH is obvious.

[1] Constructed also a year earlier by Mahlo [88].

The book is organized as follows. Since our main axiom, which we call the *Covering Property Axiom* and denote by CPA, requires some extra definitions that are unnecessary for most of the applications, we will introduce the axiom in several approximations, from the easiest to state and use to the most powerful but more complicated. All the versions of the axiom will be formulated and discussed in the main body of the chapters. The sections that follow contain only the consequences of the axioms. In particular, most of the sections can be omitted in the first reading without causing any difficulty in following the rest of the material.

Thus, we start in Chapter 1 with a formulation of the simplest form of our axiom, $\mathrm{CPA}_{\mathrm{cube}}$, which is based on a natural notion of a cube in a Polish space. In Section 1.1 we show that $\mathrm{CPA}_{\mathrm{cube}}$ implies properties (A)–(C), while in Section 1.2 we present A. Nowik's proof [107] that $\mathrm{CPA}_{\mathrm{cube}}$ implies that

(I) Every uniformly completely Ramsey null $S \subset \mathbb{R}$ has cardinality less than $\mathfrak{c}$.

In Section 1.3 we prove that $\mathrm{CPA}_{\mathrm{cube}}$ implies property (D), that is, $\mathrm{cof}(\mathcal{N}) = \omega_1$, and Section 1.4 is devoted to the proof that $\mathrm{CPA}_{\mathrm{cube}}$ implies the following fact, known as *the total failure of Martin's axiom*:

(J) $\mathfrak{c} > \omega_1$ and for every nontrivial forcing $\mathbb{P}$ satisfying the countable chain condition (ccc), there exists ω_1 many dense sets in $\mathbb{P}$ such that no filter intersects all of them.

Recall that a forcing $\mathbb{P}$ is ccc provided it has no uncountable antichains, where $A \subset \mathbb{P}$ is an antichain in $\mathbb{P}$ provided no distinct elements of A have a common extension in $\mathbb{P}$. The consistency of (J) was first proved by J. Baumgartner [6] in a model obtained by adding Sacks reals side by side.

In Section 1.5 we show that $\mathrm{CPA}_{\mathrm{cube}}$ implies that every selective ultrafilter is generated by ω_1 sets (i.e., the second part of property (E)) and that

(K) $\mathfrak{r} = \omega_1$,

where $\mathfrak{r}$ is the *reaping* (or *refinement*) number, that is,

$$\mathfrak{r} = \min\{|\mathcal{W}| : \mathcal{W} \subset [\omega]^\omega \ \&\ \forall A \in [\omega]^\omega\ \exists W \in \mathcal{W}\ (W \subset A \text{ or } W \subset \omega \setminus A)\}.$$

In Section 1.6 we prove that $\mathrm{CPA}_{\mathrm{cube}}$ implies the following version of a theorem of S. Mazurkiewicz [91]:

(L) For each Polish space X and for every uniformly bounded sequence $\langle f_n: X \to \mathbb{R}\rangle_{n<\omega}$ of Borel measurable functions there are the sequences: $\langle P_\xi: \xi < \omega_1\rangle$ of compact subsets of X and $\langle W_\xi \in [\omega]^\omega: \xi < \omega_1\rangle$ such that $X = \bigcup_{\xi<\omega_1} P_\xi$ and for every $\xi < \omega_1$:

$\langle f_n \restriction P_\xi\rangle_{n\in W_\xi}$ is a monotone uniformly convergent sequence of uniformly continuous functions.

We also show that $\mathrm{CPA}_{\mathrm{cube}}$+"$\exists$ selective ultrafilter on ω" implies the following variant of (L):

(L*) Let X be an arbitrary set and let $f_n: X \to \mathbb{R}$ be a sequence of functions such that the set $\{f_n(x): n < \omega\}$ is bounded for every $x \in X$. Then there are the sequences: $\langle P_\xi: \xi < \omega_1\rangle$ of subsets of X and $\langle W_\xi \in \mathcal{F}: \xi < \omega_1\rangle$ such that $X = \bigcup_{\xi<\omega_1} P_\xi$ and for every $\xi < \omega_1$:

$\langle f_n \restriction P_\xi\rangle_{n\in W_\xi}$ is monotone and uniformly convergent.

It should be noted here that a result essentially due to W. Sierpiński (see Example 1.6.2) implies that (L*) is false under Martin's axiom.

In Section 1.7 we present some consequences of $\mathrm{cof}(\mathcal{N}) = \omega_1$ that seem to be related to the iterated perfect set model. In particular, we prove that $\mathrm{cof}(\mathcal{N}) = \omega_1$ implies that

(M) $\mathfrak{c} > \omega_1$ and there exists a Boolean algebra B of cardinality ω_1 that is not a union of a strictly increasing ω-sequence of subalgebras of B.

The consistency of (M) was first proved by W. Just and P. Koszmider [72] in a model obtained by adding Sacks reals side by side, while S. Koppelberg [79] showed that Martin's axiom contradicts (M).

The last section of Chapter 1 consists of remarks on a form and consistency of $\mathrm{CPA}_{\mathrm{cube}}$. In particular, we note that $\mathrm{CPA}_{\mathrm{cube}}$ is false in a model obtained by adding Sacks reals side by side.

In Chapter 2 we revise slightly the notion of a cube and introduce a cube-game $\mathrm{GAME}_{\mathrm{cube}}$ — a covering game of length ω_1 that is a foundation for our next (stronger) variant of the axiom, $\mathrm{CPA}_{\mathrm{cube}}^{\mathrm{game}}$. In Section 2.1, as its application, we show that $\mathrm{CPA}_{\mathrm{cube}}^{\mathrm{game}}$ implies that:

(N) $\mathfrak{c} > \omega_1$ and for every Polish space there exists a partition of X into ω_1 disjoint closed nowhere dense measure zero sets.

In Section 2.2 we show that $\mathrm{CPA}_{\mathrm{cube}}^{\mathrm{game}}$ implies that:

(O) $\mathfrak{c} > \omega_1$ and there exists a family $\mathcal{F} \subset [\omega]^\omega$ of cardinality ω_1 that is simultaneously maximal almost disjoint (MAD) and reaping.

Section 2.3 is devoted to the proof that, under $\mathrm{CPA}_{\mathrm{cube}}^{\mathrm{game}}$,

(P) there exists an uncountable γ-set.

Chapter 3 begins with a definition of a prism, which is a generalization of a notion of cube in a Polish space. This notion, perhaps the most important notion of this text, is then used in our next generation of the axioms, $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ and $\mathrm{CPA}_{\mathrm{prism}}$, which are prism (stronger) counterparts of axioms $\mathrm{CPA}_{\mathrm{cube}}^{\mathrm{game}}$ and $\mathrm{CPA}_{\mathrm{cube}}$. Since the notion of a prism is rather unknown, in the first two sections of Chapter 3 we develop the tools that will help us to deal with them (Section 3.1) and prove for them the main duality property that distinguishes them from cubes (Section 3.2). In the remaining sections of the chapter we discuss some applications of $\mathrm{CPA}_{\mathrm{prism}}$. In particular, we prove that $\mathrm{CPA}_{\mathrm{prism}}$ implies the following generalization of property (A):

(A*) There exists a family $\mathcal{G}$ of uniformly continuous functions from $\mathbb{R}$ to $[0,1]$ such that $|\mathcal{G}| = \omega_1$ and for every $S \in [\mathbb{R}]^{\mathfrak{c}}$ there exists a $g \in \mathcal{G}$ with $g[S] = [0,1]$.

We also show that $\mathrm{CPA}_{\mathrm{prism}}$ implies that:

(Q) $\mathrm{add}(s_0)$, the additivity of the Marczewski's ideal s_0, is equal to $\omega_1 < \mathfrak{c}$.

In Section 3.4 we prove that:

(N*) If $G \in G_{\omega_1}$, where G_{ω_1} is the family of the intersections of ω_1 many open subsets of a given Polish space X, and $|G| = \mathfrak{c}$, then G contains a perfect set; however, there exists a $G \in G_{\omega_1}$ that is not a union of ω_1 many closed subsets of X.

Thus, under $\mathrm{CPA}_{\mathrm{prism}}$, G_{ω_1} sets act to some extent as Polish spaces, but they fall short of having property (N). The fact that the first part of (N*) holds in the iterated perfect set model was originally proved by J. Brendle, P. Larson, and S. Todorcevic [12, thm. 5.10]. The second part of (N*) refutes their conjecture [12, conj. 5.11]. We finish Chapter 3 with several remarks on $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$. In particular, we prove that $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ implies axiom $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}(\mathcal{X})$, in which the game is played simultaneously over ω_1 Polish spaces.

Chapters 4 and 5 deal with the applications of the axioms $\mathrm{CPA}_{\mathrm{prism}}$ and $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$, respectively. Chapter 4 contains a deep discussion of a problem of covering $\mathbb{R}^2$ and Borel functions from $\mathbb{R}$ to $\mathbb{R}$ by continuous functions of different smoothness levels. In particular, we show that $\mathrm{CPA}_{\mathrm{prism}}$ implies the following strengthening of property (H):

(H*) There exists a family $\mathcal{F}$ of less than continuum many $\mathcal{C}^1$ functions from $\mathbb{R}$ to $\mathbb{R}$ (i.e., differentiable functions with continuous derivatives) such that $\mathbb{R}^2$ is covered by functions from $\mathcal{F}$ and their inverses.

We also show the following covering property for the Borel functions:

(R) For every Borel function $f\colon\mathbb{R}\to\mathbb{R}$ there exists a family $\mathcal{F}$ of less than continuum many "$\mathcal{C}^1$" functions (i.e., differentiable functions with continuous derivatives, where the derivative can be infinite) whose graphs cover the graph of f.

We also examine which functions can be covered by less than $\mathfrak{c}$ many $\mathcal{C}^n$ functions for $n > 1$ and give examples showing that all of the covering theorems discussed are the best possible.

Chapter 5 concentrates on several specific applications of $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$. Thus, in Section 5.1 we show that $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ implies that:

(S) There is a family $\mathcal{H}$ of ω_1 pairwise disjoint perfect subsets of $\mathbb{R}$ such that $H=\bigcup\mathcal{H}$ is a Hamel basis, that is, a linear basis of $\mathbb{R}$ over $\mathbb{Q}$.

We also show that the following two properties are the consequences of (S):

(T) There exists a nonmeasurable subset X of $\mathbb{R}$ without the Baire property that is $\mathcal{N}\cap\mathcal{M}$-rigid, that is, such that $X\triangle(r+X)\in\mathcal{N}\cap\mathcal{M}$ for every $r\in\mathbb{R}$.

(U) There exists a function $f\colon\mathbb{R}\to\mathbb{R}$ such that for every $h\in\mathbb{R}$ the difference function $\Delta_h(x)=f(x+h)-f(x)$ is Borel; however, for every $\alpha<\omega_1$ there is an $h\in\mathbb{R}$ such that Δ_h is not of Borel class α.

The implication $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}\Longrightarrow$(T) answers a question related to the work of J. Cichoń, A. Jasiński, A. Kamburelis, and P. Szczepaniak [23]. The implication $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}\Longrightarrow$(U) shows that a recent construction of such a function from CH due to R. Filipów and I. Recław [57] (and answering a question of M. Laczkovich from [82]) can also be repeated with the help of our axiom. In Section 5.2 we show that $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ implies that:

(V) There exists a discontinuous, almost continuous, and additive function $f\colon\mathbb{R}\to\mathbb{R}$ whose graph is of measure zero.

The first construction of such a function, under Martin's axiom, was given by K. Ciesielski in [27]. It is unknown whether it can be constructed in ZFC. We also prove there that, under $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$:

(W) There exists a Hamel basis H such that $E^+(H)$ has measure zero.

Here $E^+(A)$ is a linear combination of $A \subset \mathbb{R}$ with nonnegative rational coefficients. This relates to the work of P. Erdős [54], H. Miller [98], and K. Muthuvel [100], who constructed such and similar Hamel bases under different set theoretical assumptions. It is unknown whether (W) holds in ZFC. In Section 5.3 we deduce from $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ that every selective ideal on ω can be extended to a maximal selective ideal. In particular, the first part of condition (E) holds and $\mathfrak{u} = \mathfrak{r}_\sigma = \omega_1$, where $\mathfrak{u}$ is the smallest cardinality of the base for a nonprincipal ultrafilter on ω. In Section 5.4 we prove that $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ implies that:

(X) There exist many nonselective P-points as well as a family $\mathcal{F} \subset [\omega]^\omega$ of cardinality ω_1 that is simultaneously independent and splitting.

In particular, $\mathfrak{i} = \omega_1$, where $\mathfrak{i}$ is the smallest cardinality of an infinite maximal independent family. We finish the chapter with the proof that $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ implies that:

(Y) There exists a nonprincipal ultrafilter on $\mathbb{Q}$ that is crowded.

In Chapter 6 we formulate the most general form of our axiom, CPA, and show that it implies all the other versions of the axiom. In Section 6.1 we conclude from CPA that

(Z) $\mathrm{cov}(s_0) = \mathfrak{c}$.

In Section 6.2 we show that CPA implies the following two generalizations of property (F):

(F^*) For an arbitrary function h from a subset S of a Polish space X onto a Polish space Y there exists a uniformly continuous function f from a subset of X into Y such that $|f \cap h| = \mathfrak{c}$.

(F') For any function h from a subset S of $\mathbb{R}$ onto a perfect subset of $\mathbb{R}$ there exists a function $f \in$ "$\mathcal{C}_{\mathrm{perf}}^\infty$" such that $|f \cap h| = \mathfrak{c}$, and f can be extended to a function $\bar{f} \in$ "$\mathcal{C}^1(\mathbb{R})$" such that either $\bar{f} \in \mathcal{C}^1$ or $\bar{f}$ is an autohomeomorphism of $\mathbb{R}$ with $\bar{f}^{-1} \in \mathcal{C}^1$.

In Section 6.3 we show that (A)&(F^*)$\Longrightarrow$(G). In particular, (G) follows from CPA.

Finally, in Chapter 7 we show that CPA holds in the iterated perfect set model.

Preliminaries

Our set theoretic terminology is standard and follows that of [4], [25], and [81]. The sets of real, rational, and integer numbers are denoted by $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}$, respectively. If $a, b \in \mathbb{R}$ and $a < b$, then $(b, a) = (a, b)$ will stand for the open interval $\{x \in \mathbb{R}: a < x < b\}$. Similarly, $[b, a] = [a, b]$ is an appropriate closed interval. The Cantor set 2^ω will be denoted by the symbol $\mathfrak{C}$. In this text we use the term *Polish space* for a complete separable metric space **without isolated points**. A subset of a Polish space is *perfect* if it is closed and contains no isolated points. For a Polish space X, the symbol $\mathrm{Perf}(X)$ will denote the collection of all subsets of X homeomorphic to $\mathfrak{C}$; the closure of an $A \subset X$ will be denoted by $\mathrm{cl}(A)$; and, as usual, $\mathcal{C}(X)$ will stand for the family of all continuous functions from X into $\mathbb{R}$.

A function $f: \mathbb{R} \to \mathbb{R}$ is *Darboux* if a conclusion of the intermediate value theorem holds for f or, equivalently, when f maps every interval onto an interval; f is a *Sierpiński-Zygmund function* if its restriction $f \restriction Y$ is discontinuous for every subset Y of $\mathbb{R}$ of cardinality $\mathfrak{c}$; and f is *nowhere constant* if it is not constant on any nontrivial interval.

A set $S \subset \mathbb{R}$ is *perfectly meager* if $S \cap P$ is meager in P for every perfect set $P \subset \mathbb{R}$, and S is *universally null* provided for every perfect set $P \subset \mathbb{R}$ the set $S \cap P$ has measure zero with respect to every countably additive probability measure on P vanishing on singletons.

For an ideal $\mathcal{I}$ on a set X, its *cofinality* is defined by

$$\mathrm{cof}(\mathcal{I}) = \min\{|\mathcal{B}|: \mathcal{B} \text{ generates } \mathcal{I}\}$$

and its *covering* as

$$\mathrm{cov}(\mathcal{I}) = \min\left\{|\mathcal{B}|: \mathcal{B} \subset \mathcal{I} \ \&\ \bigcup \mathcal{B} = X\right\}.$$

The symbol $\mathcal{N}$ will stand for the ideal of Lebesgue measure zero subsets of $\mathbb{R}$. For a fixed Polish space X. the ideal of its meager subsets will be denoted by $\mathcal{M}$, and we will use the symbol s_0 (or $s_0(X)$) to denote the σ-ideal of Marczewski's s_0-sets, that is,

$$s_0 = \{S \subset X\colon (\forall P \in \mathrm{Perf}(X))(\exists Q \in \mathrm{Perf}(X))\ Q \subset P \setminus S\}.$$

For an ideal $\mathcal{I}$ on a set X we use the symbol $\mathcal{I}^+$ to denote its coideal, that is, $\mathcal{I}^+ = \mathcal{P}(X) \setminus \mathcal{I}$.

For an ideal $\mathcal{I}$ on ω containing all finite subsets of ω we use the following generalized selectivity terminology. We say (see I. Farah [55]) that an ideal $\mathcal{I}$ is *selective* provided for every sequence $F_0 \supset F_1 \supset \cdots$ of sets from $\mathcal{I}^+$ there exists an $F_\infty \in \mathcal{I}^+$ (called a *diagonalization* of this sequence) with the property that $F_\infty \setminus \{0, \ldots, n\} \subset F_n$ for all $n \in F_\infty$. Notice that this definition agrees with the definition of selectivity given by S. Grigorieff in [65, p. 365]. (The ideals selective in the above sense Grigorieff calls *inductive*, but he also proves [65, cor. 1.15] that the inductive ideals and the ideals selective in his sense are the same notions.)

For $A, B \subset \omega$ we write $A \subseteq^* B$ when $|A \setminus B| < \omega$. A set $\mathcal{D} \subset \mathcal{I}^+$ is *dense* in $\mathcal{I}^+$ provided for every $B \in \mathcal{I}^+$ there exists an $A \in \mathcal{D}$ such that $A \subseteq^* B$, and the set $\mathcal{D}$ is *open* in $\mathcal{I}^+$ if $B \in \mathcal{D}$ provided there is an $A \in \mathcal{D}$ such that $B \subseteq^* A$. For $\bar{\mathcal{D}} = \langle \mathcal{D}_n \subset \mathcal{I}^+\colon n < \omega\rangle$ we say that $F_\infty \in \mathcal{I}^+$ is a *diagonalization* of $\bar{\mathcal{D}}$ provided $F_\infty \setminus \{0, \ldots, n\} \in \mathcal{D}_n$ for every $n < \omega$. Following I. Farah [55] we say that an ideal $\mathcal{I}$ on ω is *semiselective* provided for every sequence $\bar{\mathcal{D}} = \langle \mathcal{D}_n \subset \mathcal{I}^+\colon n < \omega\rangle$ of dense and open subsets of $\mathcal{I}^+$ the family of all diagonalizations of $\bar{\mathcal{D}}$ is dense in $\mathcal{I}^+$.

Following S. Grigorieff [65, p. 390] we say that $\mathcal{I}$ is *weakly selective* (or *weak selective*) provided for every $A \in \mathcal{I}^+$ and $f\colon A \to \omega$ there exists a $B \in \mathcal{I}^+$ such that $f \restriction B$ is either one to one or constant. (I. Farah, in [55, sec. 2], terms such ideals as *having the Q^+-property.* Note also that J. Baumgartner and R. Laver, in [7], call such ideals selective, despite the fact that they claim to use Grigorieff's terminology from [65].)

We have the following implications between these notions (see I. Farah [55, sec. 2]):

$$\mathcal{I} \text{ is selective} \implies \mathcal{I} \text{ is semiselective} \implies \mathcal{I} \text{ is weakly selective}$$

All these notions represent different generalizations of the properties of the ideal $[\omega]^{<\omega}$. In particular, it is easy to see that $[\omega]^{<\omega}$ is selective.

We say that an ideal $\mathcal{I}$ on a countable set X is selective (weakly selective) provided it is such upon an identification of X with ω via an arbitrary bijection. A filter $\mathcal{F}$ on a countable set X is selective (semiselective, weakly selective) provided the same property has its dual ideal $\mathcal{I} = \{X \setminus F\colon F \in \mathcal{F}\}$.

It is important to note that a maximal ideal (or an ultrafilter) is selective if and only if it is weakly selective. This follows, for example, directly from the definitions of these notions as in S. Grigorieff [65]. Recall also that the existence of selective ultrafilters cannot be proved in ZFC. (K. Kunen [80] proved that there are no selective ultrafilters in the random real model. This also follows from the fact that every selective ultrafilter is a P-point, while S. Shelah proved that there are models with no P-points; see, e.g., [4, thm. 4.4.7].)

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