Topology 1, Math 581, Fall 2012: Notes and homework

Krzysztof Chris Ciesielski

Class of August 21:

Course and syllabus overview.

Topology is an abstract geometry, sometimes referred to as *Rubber Sheet Geometry.* Material, in this course, will be presented "from abstract definitions and results to specific examples."

Notation:

- Do not confuse $A \in \mathcal{A}$ (which reads "A is an element of $\mathcal{A}$") with $A \subset \mathcal{A}$ (which reads "A is a subset of $\mathcal{A}$" and means "every element of A is also an element of $\mathcal{A}$).

  Notice that $A \subset B \subset C$ implies $A \subset C$, but $A \in B \in C$ does not imply $A \in C$. You will never see in this course a pair A and B, for which we will have simultaneously $A \in B$ and $A \subset B$.

- Notation $f \colon X \to Y$ means that f is a function from a set X, domain of the function, into the set Y. For any set C (usually, $C \subset Y$), the preimage $f^{-1}(C)$ (of C under f) is defined as

$$f^{-1}(C) = \{x \in X \colon f(x) \in C\}.$$

Example 1 *$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ for every A, B, and function f.*

PROOF. $x \in f^{-1}(A \cap B) \Leftrightarrow f(x) \in A \cap B \Leftrightarrow f(x) \in A \;\&\; f(x) \in A$
$\Leftrightarrow x \in f^{-1}(A) \;\&\; x \in f^{-1}(B) \Leftrightarrow x \in f^{-1}(A) \cap f^{-1}(B)$. ■

Motivation:

Let $\mathbb{R}$ be the set of real numbers and for $x \in \mathbb{R}$ and $\varepsilon > 0$ let

$$B(x,\varepsilon) = \{r \in \mathbb{R} \colon |x - r| < \varepsilon\}.$$

We will refer to $B(x,\varepsilon)$ as an *open ball*, although for this case it is just an open interval $(x-\varepsilon, x+\varepsilon)$. Let $\mathcal{T}$ be the family of all subsets U of $\mathbb{R}$ such that for every $x \in U$ there is an $\varepsilon > 0$ such that $x \in B(x,\varepsilon) \subset U$:

$$\mathcal{T} = \{U \subset \mathbb{R} \colon \forall x \in U \exists \varepsilon > 0 (B(x,\varepsilon) \subset U)\}.$$

Latter, we will refer to $\mathcal{T}$ as the *standard topology* on $\mathbb{R}$ and its elements $U \in \mathcal{T}$ will be called *open sets*.

Theorem 2 *(Motivational) Let $f:\mathbb{R}\to\mathbb{R}$. The following two definitions of continuity of f are equivalent:*

(a) (Topological definition) $f^{-1}(U)\in\mathcal{T}$ for every $U\in\mathcal{T}$.

(b) (ε-δ definition) For every $x\in\mathbb{R}$ and every $\varepsilon>0$ there is a $\delta>0$ such that for every $r\in\mathbb{R}$, if $|x-r|<\delta$, then $|f(x)-f(r)|<\varepsilon$.

PROOF. (a)$\Longrightarrow$(b): Fix an $x\in\mathbb{R}$ and an $\varepsilon>0$. Using (a), we need to find a δ satisfying (b).

Let $U=B(f(x),\varepsilon)=(f(x)-\varepsilon,f(x)+\varepsilon)$. Notice that $U\in\mathcal{T}$. (This requires checking, that U satisfies the definition of sets in $\mathcal{T}$.) So, by (a), $f^{-1}(U)\in\mathcal{T}$. Note also, that $x\in f^{-1}(U)$, as $f(x)\in(f(x)-\varepsilon,f(x)+\varepsilon)=U$. Therefore, we have $x\in f^{-1}(U)\in\mathcal{T}$ and, by the definition of $\mathcal{T}$, there is a $\delta>0$ such that $B(x,\delta)\subset f^{-1}(U)$. We show, that this δ satisfies (b).

Indeed, let $r\in\mathbb{R}$ be such that $|x-r|<\delta$. Then, $r\in(x-\delta,x+\delta)=B(x,\delta)\subset f^{-1}(U)$. Therefore, $f(r)\in U=(f(x)-\varepsilon,f(x)+\varepsilon)$ and so, $|f(x)-f(r)|<\varepsilon$, as required.

(b)$\Longrightarrow$(a): Fix a $U\in\mathcal{T}$. We need to show that $f^{-1}(U)$ is in $\mathcal{T}$. For this, take an $x\in f^{-1}(U)$. We need to find a $\delta>0$ for which $B(x,\delta)\subset f^{-1}(U)$.

We have $f(x)\in U$, as $x\in f^{-1}(U)$. Since $U\in\mathcal{T}$, there exists an $\varepsilon>0$ for which $B(f(x),\varepsilon)\subset U$. Using (b) for this x and ε, we can find a $\delta>0$ such that $|f(x)-f(r)|<\varepsilon$ provided $|x-r|<\delta$. We will show that for this choice of δ we indeed have $B(x,\delta)\subset f^{-1}(U)$.

To see this, take an $r\in B(x,\delta)=(x-\delta,x+\delta)$. We need to show that $r\in f^{-1}(U)$. Since $r\in(x-\delta,x+\delta)$, we have $|x-r|<\delta$. So, by the choice of δ, $|f(x)-f(r)|<\varepsilon$. In particular, $f(r)\in(f(x)-\varepsilon,f(x)+\varepsilon)=B(f(x),\varepsilon)\subset U$. Thus, $r\in f^{-1}(U)$, as required. ∎

Reading assignment: Read Sections 1-7.

It is assumed that you are familiar with the material presented there. Therefore, we will not cover this material in class. (If necessary, we will be reviewing these notion on "as needed" basis.)

Written assignment: Write for the next class:

Exercise 1 *Prove that $f^{-1}(A\cup B)=f^{-1}(A)\cup f^{-1}(B)$ for every sets A and B, and a function $f:X\to Y$.*

Prove, for Tuesday, August 28, the following version of Theorem 2. Provide direct proof, that is, without using condition (b) of Theorem 2.
Note: Exercise 2 will be treated as ***bonus exercise***.

Exercise 2 *(Motivational Theorem Part 2) Let* $f:\mathbb{R} \to \mathbb{R}$. *The following two definitions of continuity of* f *are equivalent:*

(a) (Topological definition) $f^{-1}(U) \in \mathcal{T}$ *for every* $U \in \mathcal{T}$.

(c) (Sequential definition) $\lim_{n\to\infty} f(x_n) = f(x)$ *for every sequence* $\langle x_n \rangle_{n=1}^{\infty}$ *in* $\mathbb{R}$ *converging to* $x \in \mathbb{R}$.

Class of August 23:

For functions $f\colon X \to Y$ and $g\colon Y \to Z$ their composition $g \circ f\colon X \to Z$ is defined via formula: $(g \circ f)(x) = g(f(x))$ for every $x \in X$. Also, if $A \subset X$, then the image $f[A]$ of A under f is defined as $\{f(a)\colon a \in A\}$.

Theorem 3 *We have the following properties:*

(a) $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$

(b) $(g \circ f)[A] = g[f[A]]$

Proof. (a) $x \in (g \circ f)^{-1}(C) \Leftrightarrow (g \circ f)(x) \in C \Leftrightarrow g(f(x)) \in C \Leftrightarrow$ $f(x) \in g^{-1}(C) \Leftrightarrow x \in f^{-1}(g^{-1}(C))$.

Proof of the part (b) is left as an exercise. (Not homework assignment.)

■

The next theorem gives a motivation of defining continuity of a functions via property (a) of Theorem 1. Note, that the proof is considerably easier than a standard ε-δ proof.

Theorem 4 *If functions $f, g\colon \mathbb{R} \to \mathbb{R}$ are continuous, then so is their composition $g \circ f\colon \mathbb{R} \to \mathbb{R}$.*

Proof. Let $U \in \mathcal{T}$. By Theorem 1 it is enough to prove that $(g \circ f)^{-1}(U) \in \mathcal{T}$. By Theorem 3(a), $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$. Now, $W = g^{-1}(U) \in \mathcal{T}$ by the continuity of g and Theorem 1. Therefore, by the continuity of f (and Theorem 1 used once again), $(g \circ f)^{-1}(U) = f^{-1}(W) \in \mathcal{T}$, as required. ■

The same proof will work for arbitrary continuous functions defined via a general notion of defined below. (See section 12 in the text.)

Definition 1 Let X be an arbitrary set having at least two elements. A *topology* on X is any family $\mathcal{T}$ of subsets of X having the following properties:

(1) $\emptyset, X \in \mathcal{T}$.

(2) The union of the any subfamily of $\mathcal{T}$ is in $\mathcal{T}$, that is, $\bigcup \mathcal{U} \in \mathcal{T}$ for every $\mathcal{U} \subset \mathcal{T}$.

(3) The intersection of the any *finite* subfamily of $\mathcal{T}$ is in $\mathcal{T}$, that is, $\bigcap \mathcal{U} \in \mathcal{T}$ for every finite $\mathcal{U} \subset \mathcal{T}$.

The pair $\langle X, \mathcal{T} \rangle$ is called a *topological space*. For a fixed topological space $\langle X, \mathcal{T} \rangle$, the sets belonging to the family $\mathcal{T}$ will be refereed to as the *open sets* (with respect to this topology).

In the above definition, we used the following notation:

- Arbitrary unions and intersections of sets: Let $\mathcal{A}$ be a family of sets, say $\mathcal{A} = \{A_t : t \in T\}$. Then $\bigcup \mathcal{A} = \bigcup_{t \in T} A_t$ denotes the same set: $\{x : \exists A \in \mathcal{A}(x \in A)\}$, that is, $\{x : \exists t \in T(x \in A_t)\}$.
- Similarly, $\bigcap \mathcal{A} = \bigcap_{t \in T} A_t$ denotes the same set: $\{x : \forall A \in \mathcal{A}(x \in A)\}$, that is, $\{x : \forall t \in T(x \in A_t)\}$.

Remark 5 In the definition, condition (3) can be replaced with

(3') The intersection of the any two sets in $\mathcal{T}$ is in $\mathcal{T}$, that is, if $U, V \in \mathcal{T}$, the also $U \cap V \in \mathcal{T}$.

PROOF. Easy induction. ∎

Example 6 *Here are some examples of topological spaces $\langle X, \mathcal{T} \rangle$, where X is an arbitrary set.*

- $\mathcal{T} = \mathcal{P}(X)$, *where $\mathcal{P}(X)$ is the* **power set of** X, *that is, the family of all subsets of X. This topology is called the* **discrete topology**.
- $\mathcal{T} = \{\emptyset, X\}$. *This topology is called* **trivial** *or* **indiscrete topology**.
- **The standard topology** $\mathcal{T}$ *on $\mathbb{R}$, defined for Theorem 2.*

More examples:

Example 7 *Examples of topologies on a set X:*

- *For a three elements set $X = \{a, b, c\}$, there are many different possible topologies. (Nine are indicated in Example 1, page76). E.g. $\{\emptyset, \{a\}, \{a, b\}, X\}$. Other examples from the text, section 12.*
- **Finite complement topology** $\mathcal{T}_f = \{\emptyset\} \cup \{X \setminus F : F \text{ is finite}\}$. *Notice that $\langle X, \mathcal{T}_f \rangle$ is discrete, for finite X.*
- **Countable complement topology** $\mathcal{T}_C = \{\emptyset\} \cup \{X \setminus F : F \text{ is countable}\}$. *Notice that $\langle X, \mathcal{T}_C \rangle$ is discrete, for countable X.*

Definition of *finer* and *coarser* topologies.

Class of August 28:

Recall that a *topology* on X is a family $\mathcal{T}$ of subsets of X such that

(1) $\emptyset, X \in \mathcal{T}$;

(2) $\bigcup \mathcal{U} \in \mathcal{T}$ for every $\mathcal{U} \subset \mathcal{T}$;

(3) $\bigcap \mathcal{U} \in \mathcal{T}$ for every finite $\mathcal{U} \subset \mathcal{T}$.

Examples of topological spaces $\langle X, \mathcal{T} \rangle$:

- **Discrete topology** $\mathcal{T} = \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the **power set of** X.
- **Trivial** or **indiscrete topology** $\mathcal{T} = \{\emptyset, X\}$.
- **The standard topology** $\mathcal{T}$ on $\mathbb{R}$, defined for Theorem 2.
- **Finite complement topology** $\mathcal{T}_f = \{\emptyset\} \cup \{X \setminus F : F \text{ is finite}\}$. Notice that $\langle X, \mathcal{T}_f \rangle$ is discrete, for finite X.
- **Countable complement topology** $\mathcal{T}_C = \{\emptyset\} \cup \{X \setminus F : F \text{ is countable}\}$. Notice that $\langle X, \mathcal{T}_C \rangle$ is discrete, for countable X.

Section 13: Basis for a Topology

Definition 2 *Basis — Two related definitions*

FROM A BASIS TO TOPOLOGY — **Basis for a topology:** A collection $\mathcal{B}$ of a subsets of a set X such that

(B1) For every $x \in X$ there is a $B \in \mathcal{B}$ with $x \in B$ (i.e., $\bigcup \mathcal{B} = X$).

(B2) For every $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ there is a $B \in \mathcal{B}$ with $x \in B \subset B_1 \cap B_2$.

[FROM A TOPOLOGY TO ITS BASIS — **Basis for a given topology** $\mathcal{T}$**:**

Let $\langle X, \mathcal{T} \rangle$ be a fixed topological space. A basis for $\mathcal{T}$ is any collection $\mathcal{B} \subset \mathcal{T}$ such that for every $U \in \mathcal{T}$ and every $x \in U$ there exists a $B \in \mathcal{B}$ with $x \in B \subset U$.

The first of these notion is used to create new topologies. The second is used to easier deal with a given, fixed topology $\mathcal{T}$. This second notion is used considerably more often than the first one.

Fact 1 *If $\mathcal{B}$ satisfies (B1) and (B2), then the family*

$$\mathcal{T}(\mathcal{B}) = \{U \subset X : \forall x \in U \exists B \in \mathcal{B}(x \in B \subset U)\} = \left\{\bigcup \mathcal{U} : \mathcal{U} \subset \mathcal{B}\right\}$$

is a topology on X. The family $\mathcal{B}$ is a basis to the topology $\mathcal{T}(\mathcal{B})$.

Fact 2 *(Lemma 13.2) If $\mathcal{B}$ is a basis for a topology $\mathcal{T}$, then $\mathcal{T} = \mathcal{T}(\mathcal{B})$.*

Discuss examples 1–3.

Fact 3 *(Lemma 13.2) If $\mathcal{B}$ is a basis for a topology $\mathcal{T}$, then $\mathcal{T} = \mathcal{T}(\mathcal{B})$.*

Lemma 13.3 — stated, to be proved next class.

There may be more than one basis for a given topology: Example 4 (from Examples 1 and 2).

Example 8 *Two examples of topologies on $\mathbb{R}$:*

- **Standard topology**, *generated by basis $\mathcal{B}_{st} = \{(a, b) : a, b \in \mathbb{R}, a < b\}$, that is, the topology $\mathcal{T}_{st} = \mathcal{T}(\mathcal{B}_{st})$. We usually write just $\mathbb{R}$ for $\langle \mathbb{R}, \mathcal{T}_{st} \rangle$.*

  Notice, that this is the same topology that was used in Theorem 2.

- **Lower limit (or Sorgenfrey) topology** *$\mathcal{T}_\ell$ is generated by basis $\mathcal{B}_\ell = \{[a, b) : a, b \in \mathbb{R}, a < b\}$, that is, $\mathcal{T}_\ell = \mathcal{T}(\mathcal{B}_\ell)$. We usually write $\mathbb{R}_\ell$ for $\langle \mathbb{R}, \mathcal{T}_\ell \rangle$.*

Written assignment for Tuesday, September 4: Exercise 8, page 83. (In part (b), do not forget to prove, that $\mathcal{T}(\mathcal{C})$ is indeed a topology. Do you need to prove, in part (a), that $\mathcal{T}(\mathcal{B})$ is a topology?)

Class of August 30:

Recall that (rephrasing):

Basis for a given topology $\mathcal{T}$: Let $\langle X, \mathcal{T}\rangle$ be a fixed topological space. A basis for $\mathcal{T}$ is any collection $\mathcal{B} \subset \mathcal{T}$ such that for every $U \in \mathcal{T}$ and every $x \in U$ there exists a $B \in \mathcal{B}$ with $x \in B \subset U$.

Fact 4 *For a collection $\mathcal{B}$ of subsets of X, let*

$$\mathcal{T}(\mathcal{B}) = \{U \subset X : \forall x \in U \exists B \in \mathcal{B}(x \in B \subset U)\}.$$

If $\mathcal{B}$ satisfies the following two conditions:

(B1) For every $x \in X$ there is a $B \in \mathcal{B}$ with $x \in B$ (i.e., $\bigcup \mathcal{B} = X$).

(B2) For every $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ there is a $B \in \mathcal{B}$ with $x \in B \subset B_1 \cap B_2$.

then $\mathcal{T}(\mathcal{B})$ is a topology on X and $\mathcal{B}$ is a basis for $\mathcal{T}(\mathcal{B})$.

New material:

Restate and prove Lemma 13.3.

Example 9 *Three examples of topologies on $\mathbb{R}$, defined via bases:*

- **Standard topology**, *generated by basis $\mathcal{B}_{st} = \{(a, b) : a, b \in \mathbb{R}, a < b\}$, that is, the topology $\mathcal{T}_{st} = \mathcal{T}(\mathcal{B}_{st})$. We usually write just $\mathbb{R}$ for $\langle \mathbb{R}, \mathcal{T}_{st}\rangle$.*

  Notice, that this is the same topology that was used in Theorem 2.

- **Lower limit (or Sorgenfrey) topology** *$\mathcal{T}_\ell$ is generated by basis $\mathcal{B}_\ell = \{[a, b) : a, b \in \mathbb{R}, a < b\}$, that is, $\mathcal{T}_\ell = \mathcal{T}(\mathcal{B}_\ell)$. We usually write $\mathbb{R}_\ell$ for $\langle \mathbb{R}, \mathcal{T}_\ell\rangle$.*

- **K-topology** *$\mathcal{T}_K$: Let $K = \{1/n : n = 1, 2, 3, \ldots\}$. Then $\mathcal{T}_K$ is generated by basis $\mathcal{B}_K = \mathcal{B}_{st} \cup \{(a, b) \setminus K : a, b \in \mathbb{R}, a < b\}$, that is, $\mathcal{T}_K = \mathcal{T}(\mathcal{B}_K)$. We usually write $\mathbb{R}_K$ for $\langle \mathbb{R}, \mathcal{T}_K\rangle$.*

Fact 5 *(Lemma 13.4) $\mathcal{T}_\ell$ and $\mathcal{T}_K$ are strictly finer than $\mathcal{T}_{st}$.*

Definition of *subbasis* for a topology.

Note that $\mathcal{S} = \{(a, \infty) : a \in \mathbb{R}\} \cup \{(-\infty, b) : b \in \mathbb{R}\}$ is a subbasis for $\mathbb{R}$ (with the standard topology).

Go over exercises 1, 3, 6.

Section 14: Order Topology: For linearly ordered set $\langle X, \leq \rangle$, order topology is generated by subbasis $\mathcal{S} = \{(a, \infty) \colon a \in X\} \cup \{(-\infty, b) \colon b \in X\}$.

Describe basis for X. (Definition, page 84.)

Go over examples 1-4.

Class of September 4:

Section 15: Product Topology on $X \times Y$

Definition 3 For topological spaces $\langle X, \mathcal{T}_1\rangle$ and $\langle Y, \mathcal{T}_2\rangle$ let $\mathcal{B}_{\mathcal{T}_1,\mathcal{T}_2}$ be the family of all open rectangles, that is,

$$\mathcal{B}_{\mathcal{T}_1,\mathcal{T}_2} = \{U \times V : U \in \mathcal{T}_1 \ \& \ V \in \mathcal{T}_2\}.$$

Note that $\mathcal{B}_{\mathcal{T}_1,\mathcal{T}_2}$ satisfies conditions (B1) and (B2) for a topology on $X \times Y$. So, the family $\mathcal{T}(\mathcal{B}_{\mathcal{T}_1,\mathcal{T}_2})$ is a topology on $X \times Y$.

The topology $\mathcal{T}(\mathcal{B}_{\mathcal{T}_1,\mathcal{T}_2})$ is called the *product topology* on $X \times Y$.

Note that, in general,

$$\mathcal{T}(\mathcal{B}_{\mathcal{T}_1,\mathcal{T}_2}) \neq \mathcal{B}_{\mathcal{T}_1,\mathcal{T}_2},$$

since, $\mathcal{B}_{\mathcal{T}_1,\mathcal{T}_2}$ is not closed under unions, as, usually, $(U_1 \times V_1) \cup (U_2 \times V_2)$ is not a rectangle (so, itdoes not belong to $\mathcal{B}_{\mathcal{T}_1,\mathcal{T}_2}$).

Theorem 10 *If $\mathcal{B}_1$ is a basis for $\langle X, \mathcal{T}_1\rangle$ and $\mathcal{B}_2$ is a basis for $\langle Y, \mathcal{T}_2\rangle$, then the family*

$$\mathcal{B}_{\mathcal{B}_1,\mathcal{B}_2} = \{U \times V : U \in \mathcal{B}_1 \ \& \ V \in \mathcal{B}_2\}$$

is a basis for the product topology on $X \times Y$.

Corollary 11 *(Example 1) The family $\mathcal{B} = \{(a, b) \times (c, d) : a, b, c, d \in \mathbb{R}\}$ is a basis for the product topology on $\mathbb{R} \times \mathbb{R}$, where $\mathbb{R}$ is considered with the standard topology. Thus, the product topology on $\mathbb{R} \times \mathbb{R}$ coincides with the standard topology $\mathcal{T}(\mathcal{B})$ on $\mathbb{R} \times \mathbb{R}$.*

Definition 4 For the Cartesian product $X_1 \times X_2$ define the *projection function* $\pi_1 : X_1 \times X_2 \to X_1$ onto the first coordinate as $\pi_1(x_1, x_2) = x_1$. Similarly, the projection onto the second coordinate is the function $\pi_2 : X_1 \times X_2 \to X_2$ defined as $\pi_2(x_1, x_2) = x_2$.

Notice that for $U \subset X_1$ and $V \subset X_2$ we have

$$\pi_1^{-1}(U) = U \times X_2 \quad \text{and} \quad \pi_2^{-1}(V) = X_1 \times V.$$

In particular, for topological spaces $\langle X_1, \mathcal{T}_1\rangle$ and $\langle X_2, \mathcal{T}_2\rangle$, the family

$$\mathcal{S} = \{\pi_i^{-1}(W) : i \in \{1, 2\} \ \& \ W \in \mathcal{T}_i\}$$

forms a subbasis for the product topology on $X_1 \times X_2$, since we have the identity $\pi_1^{-1}(U) \cap \pi_2^{-1}(V) = U \times V$.

Section 16: Subspace Topology

Definition 5 Let $\langle X, \mathcal{T} \rangle$ be a topological space and Y be any subset of X (containing at least two points). Then the family

$$\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\}$$

forms a topology on Y called the *subspace topology*.

Lemma 12 *If $\mathcal{B}$ is a basis for a topological space $\langle X, \mathcal{T} \rangle$ and $Y \subset X$, then the family*

$$\mathcal{B}_Y = \{Y \cap B : B \in \mathcal{B}\}$$

is a basis for $\langle Y, \mathcal{T}_Y \rangle$.

Go over Lemma 16.2.

Theorem 13 *(Theorem 16.3) Let $\langle A, \mathcal{T}_A \rangle$ be a subspace of $\langle X, \mathcal{T}_1 \rangle$ and $\langle B, \mathcal{T}_B \rangle$ be a subspace of $\langle Y, \mathcal{T}_2 \rangle$. Then the following two topologies on $A \times B$ coincide:*

- *$\mathcal{T}_{A \times B}$, the subspace topology of the product topology on $X \times Y$;*
- *$\mathcal{T}(\mathcal{B}_{\mathcal{T}_A, \mathcal{T}_B})$, the product topology for the spaces $\langle A, \mathcal{T}_A \rangle$ and $\langle B, \mathcal{T}_B \rangle$.*

Go over Examples 1-3 and discuss Theorem 13.

Written assignment for Tuesday, September 11: Exercises 9 and 10, page 92.

Class of September 6:

Recall that:

- For the topological spaces $\langle X, \mathcal{T}_1 \rangle$ and $\langle Y, \mathcal{T}_2 \rangle$, the *product topology* on $X \times Y$ is generated by a basis: $\mathcal{B}_{\mathcal{T}_1, \mathcal{T}_2} = \{U \times V : U \in \mathcal{T}_1 \ \& \ V \in \mathcal{T}_2\}$.
- If $\mathcal{B}_1$ is a basis for $\langle X, \mathcal{T}_1 \rangle$ and $\mathcal{B}_2$ is a basis for $\langle Y, \mathcal{T}_2 \rangle$, then the family $\mathcal{B}_{\mathcal{B}_1, \mathcal{B}_2} = \{U \times V : U \in \mathcal{B}_1 \ \& \ V \in \mathcal{B}_2\}$ is a basis for the product topology on $X \times Y$.
- If $\langle X, \mathcal{T} \rangle$ is a topological space and $Y \subset X$, then $\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\}$ is the *subspace topology* on Y.
- If $\mathcal{B}$ is a basis for a topological space $\langle X, \mathcal{T} \rangle$ and $Y \subset X$, then the family $\mathcal{B}_Y = \{Y \cap B : B \in \mathcal{B}\}$ is a basis for $\langle Y, \mathcal{T}_Y \rangle$.

Discuss briefly Theorem 16.4.
Go over Exercises 1 and 4.

Section 17: Closed sets

Definition 6 A set $A \subset X$ is *closed* in the topological space $\langle X, \mathcal{T} \rangle$ if its complement $X \setminus A$ is open.

Go over Examples 1-5.
Go over Theorem 17.1.
Go over Exercise 1.

Theorem 14 *(Theorem 17.2) Let Y be a subspace of X. Then, $A \subset Y$ is closed in Y iff $A = Y \cap F$ for some closed subset F of X.*

Go over Theorem 17.3.
Go over Exercises 2, 3, and 4.

Ex. 8. p. 92: If L is a straight line in the plane, describe the topology L inherits as a subspace of $\mathbb{R}_\ell \times \mathbb{R}$ and as a subspace of $\mathbb{R}_\ell \times \mathbb{R}_\ell$.
SOLUTION: We will use the following immediate consequence of Lemma 13.2:

($\star$) If $\mathcal{B}$ is a basis for a topology $\mathcal{T}$ on X and $\mathcal{B} \subset \mathcal{B}' \subset \mathcal{T}$, then $\mathcal{B}'$ is a basis for $\mathcal{T}$.

Recall that $\mathbb{R}$ is generated by a basis $\mathcal{B}_s = \{(a,b)\colon a,b \in \mathbb{R}\}$ and $\mathbb{R}_\ell$ by a basis $\mathcal{B}_\ell = \{[a,b)\colon a,b \in \mathbb{R}\}$. Therefore, by Lemma 16.1, $\mathbb{R}_\ell \times \mathbb{R}$ and $\mathbb{R}_\ell \times \mathbb{R}_\ell$ are generated, respectively, by the bases $\mathcal{B}_{\ell s} = \{[a,b) \times (c,d)\colon a,b,c,d \in \mathbb{R}\}$ and $\mathcal{B}_{\ell\ell} = \{[a,b) \times [c,d)\colon a,b,c,d \in \mathbb{R}\}$.

Let L be a straight line in the plane, considered with the positive direction: to the right, when L is not vertical; and upward if L is vertical. By Lemma 16.1, the subspace topology $\mathcal{T}_L$ on L is generated by a basis $\mathcal{B}_L = \{B \cap L\colon B \in \mathcal{B}\}$, $\mathcal{B}$ is equal to $\mathcal{B}_{\ell s}$ or $\mathcal{B}_{\ell\ell}$ for the cases of $\mathbb{R}_\ell \times \mathbb{R}$ and $\mathbb{R}_\ell \times \mathbb{R}_\ell$, respectively.

Case $\mathbb{R}_\ell \times \mathbb{R}$ and L is vertical: $\mathcal{B}_L$ consists of *all* open intervals in L, so the topology $\mathcal{T}_L$ generated by $\mathcal{B}_L$ is the standard topology on the line L.

Case $\mathbb{R}_\ell \times \mathbb{R}$ and L is horizontal: $\mathcal{B}_L$ consists of *all* left closed and right open intervals $[p,q)$ in L, so the topology $\mathcal{T}_L$ generated by $\mathcal{B}_L$ is the lower limit topology on the line L.

Case $\mathbb{R}_\ell \times \mathbb{R}$ and L is neither vertical nor horizontal: $\mathcal{B}_L$ consists of:

- *all* left closed and right open intervals $[p,q)$ in L, and
- *all* open intervals (r,s) in L,

Then, by ($\star$), $\mathcal{B}_L$ generates the lower limit topology $\mathcal{T}_\ell$ on the line L, since $\mathcal{B}_\ell \subset \mathcal{B}_L \subset \mathcal{T}_\ell$. (Recall that open intervals belong to $\mathcal{T}_s \subset \mathcal{T}_\ell$.)

Case $\mathbb{R}_\ell \times \mathbb{R}_\ell$ and L has negative slope: $\mathcal{B}_L$ consists of *all* possible intervals in L, including all singletons $\{c\} = [c,c]$. Then, by ($\star$), $\mathcal{B}_L$ generates the discrete topology $\mathcal{T}_d$ on the line L, since $\mathcal{B}_d \subset \mathcal{B}_L \subset \mathcal{T}_d$, where $\mathcal{B}_d$ consists of all singletons. (Recall that all subsets of L, so also all intervals on L, belong to $\mathcal{T}_d$.)

Case $\mathbb{R}_\ell \times \mathbb{R}_\ell$ and L does not have negative slope: $\mathcal{B}_L$ consists of *all* left closed and right open intervals $[p,q)$ in L, so the topology $\mathcal{T}_L$ generated by $\mathcal{B}_L$ is the lower limit topology on the line L. ∎

Solutions for an assignment of August 21, 2012

Let $f\colon \mathbb{R} \to \mathbb{R}$. The following two definitions of continuity of f are equivalent:

(a) (Topological definition) $f^{-1}(U) \in \mathcal{T}$ for every $U \in \mathcal{T}$.

(c) (Sequential definition) $\lim_{n\to\infty} f(x_n) = f(x)$ for every sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{R}$ converging to $x \in \mathbb{R}$.

PROOF. (a)$\Longrightarrow$(c): This is an easiest direction.

Fix a sequence $\langle x_n \rangle_{n=1}^{\infty}$ converging to an $x \in \mathbb{R}$. We need to show that $\lim_{n\to\infty} f(x_n) = f(x)$. For this, fix an $\varepsilon > 0$. We need to find an n_0 such that

($\star$) $|f(x_n) - f(x)| < \varepsilon$ for every $n > n_0$.

For this, we need to use (a).

Let $U = B(f(x), \varepsilon)$. Then, $U \in \mathcal{T}$ and, by (a), also $f^{-1}(U) \in \mathcal{T}$. Clearly $x \in f^{-1}(U)$, as $f(x) \in U$. Hence, $x \in f^{-1}(U) \in \mathcal{T}$ and, by the definition of $\mathcal{T}$, there exists a $\delta > 0$ such that $B(x, \delta) \subset f^{-1}(U)$. Since $\lim_{n\to\infty} x_n = x$, there exists an n_0 such that $|x_n - x| < \delta$ for every $n > n_0$. We will show, that this n_0 satisfies ($\star$).

Indeed, let $n > n_0$. Then $|x_n - x| < \delta$, so $x_n \in B(x, \delta) \subset f^{-1}(U)$. Therefore, $f(x_n) \in U = B(f(x), \varepsilon)$ and so $|f(x_n) - f(x)| < \varepsilon$.

(c)$\Longrightarrow$(a): This is a more difficult direction.

Fix $U \in \mathcal{T}$. We need to show that $f^{-1}(U) \in \mathcal{T}$. For this, fix an $x \in f^{-1}(U)$. By the definition of $\mathcal{T}$, it is enough to find a $\delta > 0$ such that $B(x, \delta) \subset f^{-1}(U)$.

Since $f(x) \in U \in \mathcal{T}$, there exists an $\varepsilon > 0$ with $f(x) \in B(f(x), \varepsilon) \subset U$. Thus, $x \in f^{-1}(B(f(x), \varepsilon)) \subset f^{-1}(U)$. Therefore, to find a $\delta > 0$ for which $B(x, \delta) \subset f^{-1}(U)$, it is enough to prove that

($**$) there exists a $\delta > 0$ such that $B(x, \delta) \subset f^{-1}(B(f(x), \varepsilon))$.

In other words, to finish the proof it is enough to show that (c)$\Longrightarrow$($**$). We will prove this last implication by contraposition: we will assume that ($**$) is false, and construct a sequence that contradicts (c).

So, assume, that ($**$) is false. Then, for every $n = 1, 2, 3, \ldots$, using the negation of ($**$) with $\delta = \frac{1}{n}$, we can find an $x_n \in B(x, \frac{1}{n}) \setminus f^{-1}(B(f(x), \varepsilon))$. Then the sequence $\langle x_n \rangle_{n=1}^{\infty}$ converges to x. However, $\langle f(x_n) \rangle_{n=1}^{\infty}$ does not converge to $f(x)$, since for every n we have: $x_n \notin f^{-1}(B(f(x), \varepsilon))$, so $f(x_n) \notin B(f(x), \varepsilon)$, that is, $|f(x_n) - f(x)| \geq \varepsilon$. ■

Class of September 11, 2012:

Recall, from the last class:

- A set $A \subset X$ is *closed* in the topological space $\langle X, \mathcal{T} \rangle$ if its complement $X \setminus A$ is open.
- (Theorem 17.2) Let Y be a subspace of X. Then, $A \subset Y$ is closed in Y iff $A = Y \cap F$ for some closed subset F of X.

Section 17: Closure and Interior of a Set

Definition 7 Let $A \subset X$ be a subset of a topological space $\langle X, \mathcal{T} \rangle$.

- The *interior* of A, denoted as $\mathrm{int}(A)$, is defined as a union of all open subsets contained in A, that is, $\mathrm{int}(A) = \bigcup\{U \in \mathcal{T} : U \subset A\}$.

  Notice that $\mathrm{int}(A)$ is open and that it is the largest open subset of A.

- The *closure* of A, denoted either as $\mathrm{cl}(A)$ or as $\bar{A}$, is defined as an intersection of all closed subsets containing in A, that is, $\mathrm{cl}(A) = \bigcap\{F \supset A : F \text{ is closed in } X\}$.

  Notice that $\mathrm{cl}(A)$ is closed and that it is the smallest closed set containing A.

We will sometimes use symbols $\mathrm{int}_X(A)$ and $\mathrm{cl}_X(A)$ in place of $\mathrm{int}(A)$ and $\mathrm{cl}(A)$ to stress that the operation is with respect to the given topology on X.

Theorem 15 *(Theorem 17.4) Let Y be a subspace of X and $A \subset Y$. Then $\mathrm{cl}_Y(A) = Y \cap \mathrm{cl}_X(A)$.*

Theorem 16 *(Theorem 17.5) Let $A \subset X$ be a subset of a topological space $\langle X, \mathcal{T} \rangle$ and $\mathcal{B}$ be a basis for X. Then*

$$x \in \mathrm{cl}(A) \textit{ if, and only if, } A \cap B \neq \emptyset \textit{ for every } B \in \mathcal{B} \textit{ with } x \in B.$$

In particular, the result is true with $\mathcal{B} = \mathcal{T}$.

Go over Examples 6 and 7.

Let $A = K \cup (2, 3)$, where $K = \{1/n : n \in \{1, 2, 3, \ldots\}\}$. Find the closures of A in: $\mathbb{R}$ (i.e., $\mathbb{R}$ with the standard topology), $\mathbb{R}_\ell$, $\mathbb{R}_d$ (i.e., $\mathbb{R}$ with the discrete topology), and $\mathbb{R}_K$.

Answer: $\mathrm{cl}_{\mathbb{R}}(A) = \{0\} \cup K \cup [2, 3]$; $\mathrm{cl}_{\mathbb{R}_\ell}(A) = \{0\} \cup K \cup [2, 3)$; $\mathrm{cl}_{\mathbb{R}_d}(A) = A$; $\mathrm{cl}_{\mathbb{R}_K}(A) = K \cup [2, 3]$;

Go over Exercise 6, 9.

Written assignment due Tuesday, Sept. 18: Ex. 8(b) page 101.

Class of September 13, 2012:

Recall, from the last class:

- The *interior* of A is $\operatorname{int}(A) = \bigcup\{U \in \mathcal{T}\colon U \subset A\}$.
- The *closure* of A is $\operatorname{cl}(A) = \bigcap\{F \supset A\colon F \text{ is closed in } X\}$.
- If Y be is subspace of X and $A \subset Y$, then $\operatorname{cl}_Y(A) = Y \cap \operatorname{cl}_X(A)$.
- If $A \subset X$ and $\mathcal{B}$ is a basis for X, then

$$x \in \operatorname{cl}(A) \text{ if, and only if, } A \cap B \neq \emptyset \text{ for every } B \in \mathcal{B} \text{ with } x \in B.$$

Section 17: Limit Points

Definition 8 Let $A \subset X$ be a subset of a topological space $\langle X, \mathcal{T}\rangle$. A point $x \in X$ is a *limit point* (or *accumulation point*) of A provided $x \in \operatorname{cl}(A \setminus \{x\})$. The set of all limit points of A is denoted as A'.

Go over Example 8.

Theorem 17 *(Theorem 17.6) Let A be a subset of a topological space $\langle X, \mathcal{T}\rangle$. Then $\operatorname{cl}(A) = A \cup A'$.*

Theorem 18 *(Theorem 17.7) Let A be a subset of a topological space $\langle X, \mathcal{T}\rangle$. Then A is closed in X if, and only if, $A' \subset A$.*

Section 17: Hausdorff spaces

Definition 9 Let $\langle X, \mathcal{T}\rangle$ be a topological space. We say that:

- X is *Hausdorff* (or a T_2 *space*) provided for every distinct $x, y \in X$ there exists disjoint open sets $U, V \subset X$ such that $x \in U$ and $y \in V$.
- X is a T_1 *space* provided for every distinct $x, y \in X$ there exists an open set $U \subset X$ such that $x \in U$ and $y \notin U$.
- X is a T_0 *space* provided for every distinct $x, y \in X$ there exists an open set $U \subset X$ such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$ (i.e., such that U contains precisely one of the points x and y).

Notice that if X is T_2 then it is also T_1, and if X is T_1 then it is also T_0.
Examples:

- A space X with a trivial topology $\mathcal{T} = \{\emptyset, X\}$ is not T_0.
- $X = \{0, 1\}$ with a topology $\mathcal{T} = \{\emptyset, \{0\}, X\}$ is T_0 but not T_1.
- $X = \mathbb{R}$ with a cofinite topology $\mathcal{T} = \{\emptyset\} \cup \{X \setminus F\colon F \text{ is finite}\}$ is T_1 but not T_2.
- The following spaces are T_2: any space with the discrete topology, $\mathbb{R}$ with the standard topology, $\mathbb{R}_\ell$, $\mathbb{R}_K$.

Theorem 19 *(Exercise 15) A space X is T_1 if, and only if, every finite subset of X is closed.*

Corollary 20 *(Theorem 17.8) Every finite subset in a Hausdorff space is closed.*

Theorem 21 *(Theorem 17.9) Let X be a T_1 topological space and $A \subset X$. Then $x \in A'$ if, and only if, $U \cap A$ is infinite for every open U containing x.*

Definition 10 Let X be a topological. We say that a sequence $\langle x_n \rangle_{n=1}^{\infty}$ of points of X *converges* to an $x \in X$ provided for every open set U containing x there exists an N such that $x_n \in U$ for every $n \geq N$.

If this is the case, we say also, that x is a *limit* of a sequence $\langle x_n \rangle_{n=1}^{\infty}$.

Theorem 22 *(Theorem 17.10) If X is a Hausdorff topological space, then any sequence $\langle x_n \rangle_{n=1}^{\infty}$ of points of X* converges *to at most one point in X.*

Example: (Exercise 14) Theorem 21 is false for T_1 spaces. For example, if $X = \mathbb{R}$ is considered with the cofinite topology (which is T_1) and $x_n = 1/n$ for every n, then every real number is a limit of $\langle x_n \rangle_{n=1}^{\infty}$.

Theorem 23 *(Theorem 17.11) The product of two Hausdorff topological spaces is a Hausdorff space. A subspace of a Hausdorff topological space is a Hausdorff space.*

Proof left as an exercise.
Go over Exercise 12.

Written assignment due Thursday, Sept. 20: Ex. 11, page 101.

Solutions for Homework of August 28, 2012

Ex. 1. (Ex. 8. p. 83)

(a) Apply Lemma 13.2 to show that the countable collection

$$\mathcal{B} = \{(a,b)\colon a < b \;\&\; a,b \in \mathbb{Q}\}$$

is a basis that generates the standard topology $\mathcal{T}$ on $\mathbb{R}$.

(b) Show that the collection

$$\mathcal{C} = \{[a,b)\colon a < b \;\&\; a,b \in \mathbb{Q}\}$$

is a basis that generates a topology different from the lower limit topology on $\mathbb{R}$.

Proof. (a) Let $U \in \mathcal{T}$ be arbitrary and let $x \in U$. By Lemma 13.2 it is enough to show that there exists a $B \in \mathcal{B}$ with $x \in B \subset U$.

Since (see page 81) the standard topology on $\mathbb{R}$ is generated by the basis $\{(c,d)\colon c < d \;\&\; a,b \in \mathbb{R}\}$, there are $c < d$ such that $x \in (c,d) \subset U$. In particular, $c < x < d$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there are $a, b \in \mathbb{Q}$ with $c < a < x < b < d$. Then $(a,b) \in \mathcal{B}$ and $x \in (a,b) \subset (c,d) \subset U$, finishing the proof.

(b) First notice that $\mathcal{C}$ indeed is a basis for $\mathbb{R}$ for a topology, say $\mathcal{T}_0$, on $\mathbb{R}$. Indeed, clearly every real number belongs to an element of $\mathcal{C}$, and a nonempty intersection of two sets from $\mathcal{C}$ still belongs to $\mathcal{C}$. (As $[p,q) \cap [r,s) = [\max\{p,r\}, \min\{q,s\})$.)

To see that $\mathcal{T}_0$ is different from the lower limit topology $\mathcal{T}_l$ notice that for any irrational number $d \in \mathbb{R}$ (take, e.g. $d = \sqrt{2}$) the set $[d, d+1)$ belongs to $\mathcal{T}_l$ (as it is an element of a basis generating $\mathcal{T}_l$), while $[d, d+1) \notin \mathcal{T}_0$.

To argue for $[d, d+1) \notin \mathcal{T}_0$ note that for every $[a,b) \in \mathcal{C}$ if $d \in [a,b)$ then $a < d$ (as a is rational while d is not) and so $[a,b) \not\subset [d, d+1)$. In other words, there is no $C \in \mathcal{C}$ with $d \in C \subset [d, d+1)$, and so $[d, d+1)$ does not belong to $\mathcal{T}_0$. Thus, $\mathcal{T}_0 \neq \mathcal{T}_l$.

It is also worth to mention that in fact $\mathcal{T}_0 \subset \mathcal{T}_l$, since $\mathcal{C} \subset \mathcal{T}_l$. (Compare Lemma 13.3.) ∎

Class of September 18, 2012:

Recall that for a topological space $\langle X, \mathcal{T} \rangle$:

- X is *Hausdorff* (or a T_2 *space*) provided for every distinct $x, y \in X$ there exists disjoint open sets $U, V \subset X$ such that $x \in U$ and $y \in V$.
- X is a T_1 *space* provided for every distinct $x, y \in X$ there exists an open set $U \subset X$ such that $x \in U$ and $y \notin U$. Equivalently, X is T_1 if, and only if, every singleton is closed in X.
- If X is a Hausdorff topological space, then any sequence $\langle x_n \rangle_{n=1}^{\infty}$ of points of X *converges* to at most one point in X.
- The product of two Hausdorff topological spaces is a Hausdorff space. A subspace of a Hausdorff topological space is a Hausdorff space.

Go over Exercises 10 and 13.

Section 18: Continuous functions

Definition 11 Let X and Y be the topological spaces. A function $f: X \to Y$ is *continuous* provided $f^{-1}(V)$ is open in X for every open subset V of Y.

Notice, that the definition agrees with (a) from Theorem 2.

Theorem 24 *Let X and Y be the topological spaces and $\mathcal{B}$ a basis for Y. Then $f: X \to Y$ is continuous if, and only if, $f^{-1}(B)$ is open in X for every $B \in \mathcal{B}$.*

Similarly, if $\mathcal{S}$ is a subbasis for Y, then $f: X \to Y$ is continuous if, and only if, $f^{-1}(S)$ is open in X for every $S \in \mathcal{S}$.

Example 3:

- $f: \mathbb{R} \to \mathbb{R}_\ell$, $f(x) = x$, is discontinuous, as $f^{-1}([0,1)) = [0,1)$ is not open in $\mathbb{R}$.
- $f: \mathbb{R}_\ell \to \mathbb{R}$ is continuous, as $f^{-1}(U) = U \in \mathcal{T}_{st} \subset \mathcal{T}_\ell$ for every $U \in \mathcal{T}_{st}$.

Go over Exercise 3(a).
Go over Theorem 18.1. (Very important!)
Stress continuity at a point.
Go over Exercise 2.

Class of September 20, 2012:

Recall:

Definition of continuous functions and it equivalent forms: Thm 18.1.

Section 18: Homeomorphisms

Definition 12 Let X and Y be the topological spaces and let $f: X \to Y$ be a bijection (i.e., one-to-one and onto). Then f is a *homeomorphism* (from X onto Y) provided both f and $f^{-1}: Y \to X$ are continuous.

Topological spaces X and Y are *homeomorphic* provided there is a homeomorphism from X onto Y.

A mapping $f: X \to Y$ is an *imbedding* provided f is injective (i.e., one-to-one), continuous, and $f^{-1}: f[X] \to X$ is also continuous. In such a case a mapping $f': X \to f[X]$, $f'(x) = f(x)$, is a homeomorphism (from X onto $f[X]$.

Go over Examples 4-6.

Go over Exercise 4-6.

Section 18: Constructing Continuous Functions

Go over Theorem 18.2.

Go over Theorem 18.3 (The pasting Lemma).

Go over Example 8.

Go over Theorem 18.4.

Class of September 25, 2012:

Written assignment for Tuesday, October 2: Exercise 8, page 111. Part (a) is treated as a bonus homework problem. (It is easy for $Y = \mathbb{R}$, but you are to show the result for any ordered space Y.) In part (b) you can use the result from the part (a), even if you did not completed this part of the assignment.

Go over Exercise 11.

Variant of Exercise 12, with $f(x,y) = \frac{xy^2}{x^2+y^4}$ for $\langle x,y\rangle \neq \langle 0,0\rangle$ and $f(0,0) = 0$. Show that it is discontinuous (on curve $y^2 = x$), but $f \restriction L$ is continuous for every straight line L.

Go over Exercise 13. Few words on its consequences: the cardinality of the set $C(\mathbb{R})$ of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$ is the same as of the set $\mathbb{R}^{\mathbb{Q}}$ of all functions from $\mathbb{Q}$ to $\mathbb{R}$, and also the same as the size of $\mathbb{R}$. (While the set $\mathbb{R}^{\mathbb{R}}$ of **all** functions from $\mathbb{R}$ to $\mathbb{R}$ has bigger size.)

Section 19: The product topology

Definition 13 For sets J and X let X^J denotes the family of all functions $f\colon J \to X$.

Let $\{A_\alpha\}_{\alpha\in J}$ be an arbitrary indexed family of sets and let $X = \bigcup_{\alpha\in J} A_\alpha$. (Notice that the index set J may be uncountable!) The *cartesian product* of the family $\{A_\alpha\}_{\alpha\in J}$, denoted by $\prod_{\alpha\in J} A_\alpha$, is defined as

$$\prod_{\alpha\in J} A_\alpha = \{f \in X^J\colon f(j) \in A_j \text{ for all } j \in J\}.$$

Elements of $\{A_\alpha\}_{\alpha\in J}$ will be also sometimes denotes as $\langle a_\alpha\rangle_{\alpha\in J}$ and referred to as *J-tuples*.

Notice that $X^J = \prod_{\alpha\in J} A_\alpha$, where $A_\alpha = X$ for every $\alpha \in J$.

Notice, that this definition agrees the definition of the finite cartesian product (over the set $J = \{1,\dots,n\}$) $\prod_{i=1}^n A_i = A_1 \times\cdots\times A_n$ as the set of all sequences $\langle a(1),\dots,a(n)\rangle$ with $a(i) \in A_i$, since any such sequence can be considerd as a function $a\colon \{1,\dots,n\} \to X$. Similar agreement is also for $J = \{1,2,3,\dots\}$.

Definition 14 Let $\{X_\alpha\}_{\alpha\in J}$ be an indexed family of topological spaces. Then, on the product space $X = \prod_{\alpha\in J} X_\alpha$, we define the following two kinds of topologies.

box topology: Generated by a basis $\mathcal{B}_{box}$ formed by all sets of the form

$$\prod_{\alpha\in J} U_\alpha \quad \text{where each } U_\alpha \text{ is open in } X_\alpha.$$

product topology: Generated by a subbasis $\mathcal{S}$ formed by all sets of the form

$$\pi_\beta^{-1}(U_\beta) \text{ for all } \beta \in J \text{ and open subsets } U_\beta \text{ of } X_\beta,$$

where $\pi_\beta\colon X \to X_\beta$ is the *projection* onto βth coordinate, that is, defined as $\pi_\beta(x) = x(\beta)$.

Notice that $\pi_\beta^{-1}(U_\beta) = \prod_{\alpha\in J} U_\alpha$, where $U_\alpha = X_\alpha$ for all $\alpha \neq \beta$.

A natural basis, $\mathcal{B}_{prod}$ associated with $\mathcal{S}$ is formed by finite intersections of sets from $\mathcal{S}$, that is, all sets of the form $\prod_{\alpha\in J} U_\alpha$ where each U_α is open in X_α and *the set* $\{\alpha \in J\colon U_\alpha \neq X_\alpha\}$ *is finite*.

Proved Theorem 19.1.

Stated, without proofs, Theorems 19.2, 19.3, 19.4, and 19.5

Stated and proved Theorem 19.6.

Stated Example 2.

Class of September 27, 2012:

Recall that for an indexed family $\{X_\alpha\}_{\alpha\in J}$ of topological spaces we defined two topologies on $X = \prod_{\alpha\in J} X_\alpha$:

box topology: Generated by a basis $\mathcal{B}_{box}$ formed by all sets of the form

$$\prod_{\alpha\in J} U_\alpha \quad \text{where each } U_\alpha \text{ is open in } X_\alpha.$$

product topology: Generated by a subbasis $\mathcal{S}$ formed by all sets

$$\pi_\beta^{-1}(U_\beta) \text{ for all } \beta \in J \text{ and open subsets } U_\beta \text{ of } X_\beta,$$

where $\pi_\beta \colon X \to X_\beta$ is the *projection* onto βth coordinate.

Restate, without proofs, Theorems 19.3 and 19.4.
Restate and prove Theorem 19.5.
Restate Theorem 19.6.
Restate and prove Example 2.
Go over Exercises 7 and 8.

Solutions for Homework of September 4, 2012

Ex. 2. (Ex. 9. p. 92) Show that the dictionary order topology $\mathcal{T}_{\preceq}$ on $\mathbb{R}\times\mathbb{R}$ coincides with the product topology $\mathcal{T}_{ds}$ of $\mathbb{R}_d\times\mathbb{R}$. Compare this topology with the standard topology $\mathcal{T}_{st}$ on $\mathbb{R}^2$.

PROOF. In the proof, we will use the following two facts, mentioned many times in class. (For notation, see lecture for Section 13.)

(i) If $\mathcal{B}_0\subset\mathcal{B}_1\subset\mathcal{P}(X)$, then $\mathcal{T}(\mathcal{B}_0)\subset\mathcal{T}(\mathcal{B}_1)$.

(ii) If $\mathcal{T}_0$ is a topology (on X), then $\mathcal{T}(\mathcal{T}_0)=\mathcal{T}_0$.

Property (i) holds, as $\mathcal{T}(\mathcal{B}_0)=\{\bigcup\mathcal{B}\colon\mathcal{B}\subset\mathcal{B}_0\}\subset\{\bigcup\mathcal{B}\colon\mathcal{B}\subset\mathcal{B}_1\}=\mathcal{T}(\mathcal{B}_1)$. Property (ii) holds, since the family $\mathcal{B}_0=\mathcal{T}_0$ is a basis for $\mathcal{T}_0$, and so $\mathcal{T}(\mathcal{T}_0)=\mathcal{T}(\mathcal{B}_0)=\mathcal{T}_0$.

Next, notice that, by Thm 15.1, $\mathcal{B}_{ds}=\{\{x\}\times(p,q)\colon x,p,q\in\mathbb{R}\}$ is a basis for $\mathcal{T}_{ds}$. Also, $\mathcal{B}_{\preceq}=\{(\langle a,b\rangle,\langle c,d\rangle)\colon a,b,c,d\in\mathbb{R}\}$ is a basis for $\mathcal{T}_{\preceq}$, where $(\langle a,b\rangle,\langle c,d\rangle)=\{\langle x,y\rangle\in\mathbb{R}^2\colon\langle a,b\rangle\prec\langle x,y\rangle\prec\langle c,d\rangle\}$. (Here $\preceq$ is the dictionary order on $\mathbb{R}\times\mathbb{R}$.)

To show $\mathcal{T}_{ds}\subset\mathcal{T}_{\preceq}$, notice that $\mathcal{B}_{ds}\subset\mathcal{B}_{\preceq}$, as $\{x\}\times(p,q)=(\langle x,p\rangle,\langle x,q\rangle)\in\mathcal{B}_{\preceq}$. Therefore, by (i), $\mathcal{T}_{ds}=\mathcal{T}(\mathcal{B}_{ds})\subset\mathcal{T}(\mathcal{B}_{\preceq})=\mathcal{T}_{\preceq}$.

To show $\mathcal{T}_{\preceq}\subset\mathcal{T}_{ds}$, first notice that $\mathcal{B}_{\preceq}\subset\mathcal{T}_{ds}$. So, take a non-empty $(\langle a,b\rangle,\langle c,d\rangle)\in\mathcal{B}_{\preceq}$. If $a=c$, then $(\langle a,b\rangle,\langle c,d\rangle)=\{a\}\times(b,d)\in\mathcal{B}_{ds}\subset\mathcal{T}_{ds}$. Otherwise $a<c$ and $(\langle a,b\rangle,\langle c,d\rangle)$ is a union of the following sets from $\mathcal{T}_{ds}$: $\{a\}\times(b,\infty)$, $\{c\}\times(-\infty,d)$, and $\{z\}\times\mathbb{R}$, where $a<z<c$. Therefore, once again, $(\langle a,b\rangle,\langle c,d\rangle)\in\mathcal{B}_{ds}\subset\mathcal{T}_{ds}$. Hence, indeed, $\mathcal{B}_{\preceq}\subset\mathcal{T}_{ds}$.

Now, $\mathcal{T}_{\preceq}\subset\mathcal{T}_{ds}$ follows from (i) and (ii): $\mathcal{T}_{\preceq}=\mathcal{T}(\mathcal{B}_{\preceq})\subset\mathcal{T}(\mathcal{T}_{ds})=\mathcal{T}_{ds}$.

To finish the exercise, we will show that $\mathcal{T}_{st}\subsetneq\mathcal{T}_{ds}$. Indeed, to see the inclusion, recall that the family $\mathcal{B}_{st}=\{(a,b)\times(c,d)\colon a,b,c,d\in\mathbb{R}\}$ is a basis for $\mathcal{T}_{st}$. Also, any set from $\mathcal{B}_{st}$ belongs to the standard basis $\mathcal{B}_{pr}$ for $\mathbb{R}_d\times\mathbb{R}$: $\mathcal{B}_{pr}=\{U\times V\colon U\text{ open in }\mathbb{R}_d\text{ and }V\text{ open in }\mathbb{R}\}$. Therefore, $\mathcal{B}_{st}\subset\mathcal{B}_{pr}\subset\mathcal{T}_{ds}$ and, by (i) and (ii), $\mathcal{T}_{st}=\mathcal{T}(\mathcal{B}_{st})\subset\mathcal{T}(\mathcal{T}_{ds})=\mathcal{T}_{ds}$.

To see that the inclusion is strict, it is enough to notice that, for example, a set $W=\{0\}\times(0,1)$ belongs to $\mathcal{T}_{ds}$ but it does not belong to $\mathcal{T}_{st}$. ∎

Ex. 3. (Ex. 10. p. 92) Let $I = [0, 1]$. Compare the following topologies on I^2: the standard product topology τ_{st}, the dictionary order topology $\tau_{\preceq}$, and the subspace topology $\tau_{\preceq}^*$ of $\mathcal{T}_{\preceq}$.

PROOF. We will use notation and results used above, for Ex. 9. p. 92. We will prove, that the only inclusions between the topologies are $\tau_{st} \subset \tau_{\preceq}^*$ and $\tau_{\preceq} \subset \tau_{\preceq}^*$.

By Lemma 16.1, the family $\mathcal{D}_{st} = \{B \cap I^2 \colon B \in \mathcal{B}_{st}\}$ forms a basis for τ_{st}. Also, since $\mathcal{B}_{pr}$ is a basis for $\mathcal{T}_{ds} = \mathcal{T}_{\preceq}$, the family $\mathcal{D}_{pr} = \{B \cap I^2 \colon B \in \mathcal{B}_{pr}\}$ is a basis for $\tau_{\preceq}^*$. Therefore, since $\mathcal{B}_{st} \subset \mathcal{B}_{pr}$, we have $\mathcal{D}_{st} \subset \mathcal{D}_{pr}$ and so, by (i), $\tau_{st} = \mathcal{T}(\mathcal{D}_{pr}) \subset \mathcal{T}(\mathcal{D}_{pr}) = \tau_{\preceq}^*$. So, indeed, $\tau_{st} \subset \tau_{\preceq}^*$.

To prove $\tau_{\preceq} \subset \tau_{\preceq}^*$, notice that $\mathcal{D}_{\preceq} = \{(\langle a, b\rangle, \langle c, d\rangle) \cap I^2 \colon a, b, c, d \in I\}$ is a basis for $\tau_{\preceq}$ (straight from the definition of order topology) while, by Lemma 16.1, $\mathcal{D}_{\preceq}^* = \{(\langle a, b\rangle, \langle c, d\rangle) \cap I^2 \colon a, b, c, d \in \mathbb{R}\}$ is a basis for $\tau_{\preceq}^*$. Clearly, $\mathcal{D}_{\preceq} \subset \mathcal{D}_{\preceq}^*$. Therefore, by (i), $\tau_{\preceq} = \mathcal{T}(\mathcal{D}_{\preceq}) \subset \mathcal{T}(\mathcal{D}_{\preceq}^*) = \tau_{\preceq}^*$, as desired.

To finish the argument, we need to show that the topologies τ_{st} and $\tau_{\preceq}$ and not comparable. Indeed, $\tau_{st} \not\subset \tau_{\preceq}$ since a set $[0, 1)^2 = (-1, 1)^2 \cap I^2 \in \tau_{st}$ but it does not belong to $\tau_{\preceq}$ since there is no $J \in \mathcal{D}_{\preceq}$ with $\langle .5, 0\rangle \in J \subset [0, 1)^2$ (as any $J \in \mathcal{D}_{\preceq}$ containing $\langle .5, 0\rangle$ must contain also $\langle x, 1\rangle$ for some $x \in (0, .5)$).

Similarly, $\tau_{\preceq} \not\subset \tau_{st}$, as $\{0\} \times (0, 1) = (\langle 0, 0\rangle, \langle 0, 1\rangle) \in \tau_{\preceq}$ does not belong to τ_{st}. ∎

Solutions for Homework of September 11, 2012

Ex. 4. (Ex. 8(b). p. 101) Let X be topological space and $A_\alpha \subset X$ for $\alpha \in J$. Determine weather the equality $\mathrm{cl}\left(\bigcap_{\alpha\in J} A_\alpha\right) = \bigcap_{\alpha\in J} \mathrm{cl}\,(A_\alpha)$ holds. If equality fails, show which inclusion holds and give counterexample for the other one.

Solution: We will use the fact that

($\star$) $S \subset T$ implies $\mathrm{cl}(S) \subset \mathrm{cl}(T)$.

Inclusion $\mathrm{cl}\left(\bigcap_{\alpha\in J} A_\alpha\right) \subset \bigcap_{\alpha\in J} \mathrm{cl}\,(A_\alpha)$ holds, as $\bigcap_{\alpha\in J} A_\alpha \subset \bigcap_{\alpha\in J} \mathrm{cl}\,(A_\alpha)$ and so, by ($\star$), $\mathrm{cl}\left(\bigcap_{\alpha\in J} A_\alpha\right) \subset \mathrm{cl}\left(\bigcap_{\alpha\in J} \mathrm{cl}\,(A_\alpha)\right) = \bigcap_{\alpha\in J} \mathrm{cl}\,(A_\alpha)$, where the last equation is true, since $\bigcap_{\alpha\in J} \mathrm{cl}\,(A_\alpha)$ is closed, as an intersection of closed sets.

Inclusion $\mathrm{cl}\left(\bigcap_{\alpha\in J} A_\alpha\right) \supset \bigcap_{\alpha\in J} \mathrm{cl}\,(A_\alpha)$ is false. Take, e.g., $X = \mathbb{R}$, $J = \{0, 2\}$, $A_0 = (0, 1)$, $A_1 = \{1\}$. Then $\bigcap_{\alpha\in J} \mathrm{cl}\,(A_\alpha) = \mathrm{cl}(A_0) \cap \mathrm{cl}(A_1) = [0, 1] \cap \{1\} = \{1\} \not\subset \emptyset = \mathrm{cl}(\emptyset) = \mathrm{cl}(A_0 \cap A_1) = \mathrm{cl}\left(\bigcap_{\alpha\in J} A_\alpha\right)$. ∎

Solutions for Homework of September 13, 2012

Ex. 5. (Ex. 11. p. 101) Show that the product of two Hausdorff spaces is Hausdorff.

Solution: Let X and Y be Hausdorff. Let $p_1 = \langle x_1, y_1\rangle$ and $p_2 = \langle x_2, y_2\rangle$ be distinct points from $X \times Y$. We need to find disjoint open subsets W_1 and W_2 of $X \times Y$ such that $p_1 \in W_1$ and $p_2 \in W_2$.

If $x_1 \neq x_2$, then, since X is Hausdorff, there are disjoint open subsets U_1 and U_2 of X such that $x_1 \in U_1$ and $x_2 \in U_2$. Then, $W_1 = U_1 \times Y$ and $W_2 = U_2 \times Y$ are as desired.

If $x_1 = x_2$, then $y_1 \neq y_2$, since $p_1 \neq p_2$. Then, since Y is Hausdorff, there are disjoint open subsets V_1 and V_2 of Y such that $y_1 \in V_1$ and $y_2 \in V_2$. Then, $W_1 = X \times V_1$ and $W_2 = X \times V_2$ are as desired. ∎

Class of October 2, 2012:

Section 20: The Metric Topology

Define a *metric* (*distance*) on X as a function $d: X \times X \to [0, \infty)$.

A *metric space* is a pair $\langle X, d\rangle$, where d is a metric on X.

In a metric space $\langle X, d\rangle$, define an *open ball* (centered at $x \in X$ with radius $\varepsilon > 0$) as $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$.

Prove that a family $\mathcal{B}_d = \{B(x, \varepsilon) : x \in X \ \&\ \varepsilon > 0\}$ is a basis for a topology on X.

Define a metric topology for a metric space $\langle X, d\rangle$ as $\mathcal{T}(\mathcal{B}_d)$, that is, as a topology generated by the family of all open balls in $\langle X, d\rangle$.

Go over Example 1 (discrete metric) and 2 (standard metric on $\mathbb{R}$).

Go over Exercise 3(a): $d: X \times X \to \mathbb{R}$ is continuous in X^2, where X is considered with the metric topology.

Proof. Let $B = (a, b)$ be basic open set in $\mathbb{R}$. Need to prove that $d^{-1}(B)$ is open in X^2.

Fix $\langle x, y\rangle \in d^{-1}(B)$. So, $d(x, y) \in B$. We need to find an open set U in X^2 with $\langle x, y\rangle \in U \subset d^{-1}(B)$. Let $\varepsilon > 0$ be such that $(d(x, y)-\varepsilon, d(x, y)+\varepsilon) \subset B$. Define $U = B(x, \varepsilon/2) \times B(y, \varepsilon/2)$. It is open in X^2 and contains $\langle x, y\rangle$.

So, fix $\langle z, t\rangle \in U$. Then $d(x, z) < \varepsilon/2$ and $d(y, t) < \varepsilon/2$. By the triangle inequality we get $d(z, x) + d(x, y) + d(y, t) \geq d(z, t)$, so

$$d(z, x) + d(y, t) \geq d(z, t) - d(x, y).$$

Similarly, $d(x, z) + d(z, t) + d(t, y) \geq d(x, y)$, so

$$d(x, z) + d(t, y) \geq d(x, y) - d(z, t).$$

Hence, $|d(z, t) - d(x, y)| \leq d(x, z) + d(t, y) < \varepsilon/2 + \varepsilon/2$ and so we have $d(z, t) \in (d(x, y) - \varepsilon, d(x, y) + \varepsilon) \subset B$, as required.

Definition 15 A topological space $\langle X, \tau\rangle$ is *metrizable* provided there exists a metric d on X such that $\tau = \mathcal{T}(\mathcal{B}_d)$.

Go over Exercise 2.

Define: bounded set and its diameter.

Go over Theorem 20.1. (So, boundedness is not a topological property!)

Class of October 4, 2012:

Recall that:

- A *metric space* is a pair $\langle X, d \rangle$, where d is a metric on X.
- $B_d(x, \varepsilon) = \{y \in X \colon d(x, y) < \varepsilon\}$ is an *open ball* in $\langle X, d \rangle$.
- $\mathcal{B}_d = \{B(x, \varepsilon) \colon x \in X \ \& \ \varepsilon > 0\}$ is a basis for a topology on X.
- $\mathcal{T}(\mathcal{B}_d)$ is the metric topology on X (for metric d).

New material

Define Euclidean metric and square metric on $\mathbb{R}^n$.

Go over Theorem 20.3, using Lemma 20.2.

Define uniform metric on $\mathbb{R}^J$.

Go over Exercise 6.

Go over Theorem 20.4.

Class of October 9, 2012:

Recall

- uniform metric on $\mathbb{R}^J$ and uniform topology;
- relations between box, uniform, and product topologies on $\mathbb{R}^J$.

New material

Go over Theorem 20.5. (Countable product of metric spaces is metrizable.)

Go over Exercise 6, page 118, and Exercise 4(b) page 127.

Written assignment for Thursday, October 18: Exercise 4(a), p. 127.

Section 21: The Metric Topology continued

- Subspace of a metric space is metric.
- No relation between ordered topologies and metric topologies.
- Every metrizable space is Hausdorff.
- Finite and countable product of metric spaces is metrizable.

State Theorem 21.1: for metric spaces, ε-δ definition of continuity is equivalent to topological definition of continuity. (This is an obvious generalization of Theorem 2.)

Solutions for Homework of October 2, 2012

Exercise 8(a), page 111: Let Y be an ordered set in the order topology. Let $f, g\colon X \to Y$ be continuous. Show that the set $C = \{x \in X\colon f(x) \leq g(x)\}$ is closed in X.

PROOF. First notice that

- for every $y < z$ from Y there exist open sets $U \ni y$ and $V \ni z$ in Y such that $U < V$, that is, that $u < v$ for every $u \in U$ and $v \in V$.

To see this, first notice that for every $y \in Y$ the set $[< y] = \{x\colon x < y\}$ is open in Y. Indeed, if Y has the smallest element, say m, then $[< y] = [m, y)$ is a basic open set. On the other hand, if Y does not have the smallest element, then $[< y] = \bigcup_{z<y}(z, y)$ is clearly open.

Similarly, we prove that $[> y] = \{x\colon x > y\}$ is open in Y.

To show • take $y < z$ from Y. If there is a $t \in Y$ with $y < t < z$, then $U = [< t]$ and $V = [> t]$ work. Otherwise we can take $U = [< z]$ and $V = [> y]$.

To prove that C is closed in X it is enough to show that $T = X \setminus C$ is open in X. For this take an $x \in T$. It is enough to find an open set W in X for which $x \in W \subset T$.

Since $x \in T$, we know that $y = g(x)$ is less than $z = f(x)$. Let U and V be as in • and let $W = g^{-1}(U) \cap f^{-1}(V)$. We will show that W is as desired.

Clearly W is open in X, since $g^{-1}(U)$ is open in X as a preimage of an open set U with respect to a continuous function g, while $f^{-1}(V)$ is open in X as a preimage of an open set V with respect to a continuous function f.

Also, we have $x \in g^{-1}(U)$, since $g(x) = y \in U$ and $x \in f^{-1}(V)$, as $f(x) = z \in V$. Thus, $x \in W$.

To finish the proof it is enough to show that $W \subset T$. So, take a $w \in W$. We need to show that $g(w) < f(w)$. But $w \in g^{-1}(U)$ implies that $g(w) \in U$, while $w \in f^{-1}(V)$ ensures that $f(w) \in V$. Since $U < V$, we have indeed $g(w) < f(w)$. This completes the proof. ∎

Alternative approach to the proof of Exercise 8(a), page 112: You definitely need to use some version of property $\bullet$. Some of you tried to argue for this by using the fact that Y is Hausdorff: take $y < z$ in Y (you were taking $y = g(x)$ and $z = f(x)$ for some $x \in X \setminus C$), disjoint basic open neighborhoods $U \ni y$ and $V \ni z$, and then argue that $U < V$. First of all, this may be false, if the word "basic" is not there. But, as is stated, it is true and it follows from the property:

($*$) for every $y < z$ and disjoint intervals $U \ni y$ and $V \ni z$ in Y, $U < V$.

The problem is, that the proof of ($*$) is more difficult that the proof of Hausdorff property for Y (as well as of $\bullet$). Here the proof for open intervals. (It is also true for arbitrary intervals.)

Proof. Let m and M be the largest and smallest elements of Y, if they exist. Then U is of the form (a, b) or $[m, b)$. (It cannot be of the form $(a, M]$, since then we would have $a < y < z \leq M$, so $z \in (a, M] \cap V$, contradicting the fact that $U \cap V = \emptyset$.) In fact, we must have $b \leq z$, since otherwise $y < z < b$ and we would have $z \in (y, b) \cap V \subset U \cap V$.

Similarly, V is of the form (c, d) or $(c, M]$, as it cannot be of the form $[m, d)$, and we have $y \leq c$.

Now consider two cases:

Case 1: $b \leq c$. Then clearly $U < V$, since for every $u \in U$ and $v \in V$ we have $u < b \leq c < v$.

Case 2: $c < b$. (Yes, it is still possible!) Then we have $y \leq c < b \leq z$ and $(c, b) = U \cap V = \emptyset$. (Situation $(c, b) = \emptyset$ for $c < b$ is possible, if there are no points in Y between c and b.) Therefore $U \subset (-\infty, c]$ and $V \subset [b, \infty)$. In particular, for every $u \in U$ and $v \in V$ we have $u \leq c < b \leq v$, so $U < V$. ■

Exercise 8(b), page 112: Let Y be an ordered set in the order topology. Let $f, g\colon X \to Y$ be continuous functions. Define $h\colon X \to Y$ by a formula $h(x) = \min\{f(x), g(x)\}$. Show that h is continuous.

Proof. Let $A = \{x \in X\colon f(x) \leq g(x)\}$ and $B = \{x \in X\colon g(x) \leq f(x)\}$. By part (a) of the exercise, these two sets are closed in X. Also, functions $f \restriction A\colon A \to Y$ and $g \restriction B\colon B \to Y$ are continuous, by Theorem 18.2. Notice also that $f(x) = g(x)$ for all $x \in A \cap B$. Thus, by the pasting lemma applied to the functions $f \restriction A$ and $g \restriction B$, the function $\bar{h}\colon X \to Y$, defined as $\bar{h}(x) = f(x)$ for $x \in A$ and $\bar{h}(x) = g(x)$ for $x \in B$, is continuous. To finish the proof it is enough to notice that $h = \bar{h}$. ∎

Class of October 11, 2012:

Hand the exercises for the mid term test, to be held, in class, on October 16, 2012.

Go over Exercise 5, page 127. Note, that this implies that, on $\mathbb{R}^\omega$, box, uniform, and product topologies are distinct.

Definition 16 Let $\langle X, \tau \rangle$ be a topological space.

- A family $\mathcal{B}_x \subset \tau$ is a *basis (for X) at x* provided for every open set $U \ni x$ there is a $B \in \mathcal{B}_x$ with $x \in B \subset U$.
- A topological space X is *first countable* (or *satisfies the first countability axiom*) provided for every $x \in X$ there exists a countable basis $\mathcal{B}_x$ of X at x.

Proposition 25 *Every metrizable space is first countable.*

Note that for first countable spaces, a countable basis $\{B_n: n = 1, 2, 3, \ldots\}$ can be chosen monotone: $B_1 \supset B_2 \supset B_3 \supset \cdots$.

Go over Lemma 21.2, version for first countable spaces:

Lemma 26 *Let X be a first countable topological space and let $A \subset X$. Then $x \in \text{cl}(A)$ if, and only if, there is a sequence of points of A converging to x. Moreover, the implication "$\Longleftarrow$" does not require the assumption of first countability.*

Go over Theorem 21.3, version for first countable spaces:

Theorem 27 *Let X and Y topological spaces and let $f: X \to Y$. Assume also that X is first countable. Then f is continuous if, and only if, for every sequence $\langle x_n \rangle_n$ in X converging to an $x \in X$, $\langle f(x_n) \rangle_n$ converges to $f(x)$.*

Moreover, the implication "$\Longrightarrow$" does not require the assumption of first countability.

Go over Lemma 21.4 (no proof).

Go over Theorem 21.5.

Class of October 16, 2012:

In class mid term test.

Class of October 18, 2012:

Discussion of Mid Term Test.

Definition 17 Let $\langle Y, d\rangle$ be a metric space, X any set, and $f_n: X \to Y$ be a sequence of functions. We say that the sequence $\langle f_n\rangle_n$ *converges uniformly* to an $f: X \to Y$ provided for every $\varepsilon > 0$ there exists an N (independent of x) such that for every $x \in X$

$$d(f_n(x), f(x)) < \varepsilon \text{ for all } n > N.$$

State Theorem 21.6: uniform limit of continuous functions is continuous.

Go over Exercise 6: *uniform* convergence assumption in Theorem 21.6 is essential.

Prove Theorem 21.6.

Discuss Exercise 9: the implication in Theorem 21.6 cannot be reversed.

Go (briefly) over Example 1: $\mathbb{R}^\omega$ with the box topology is not first countable. In particular, it is not metrizable.

Class of October 23, 2012:

Recall

- Theorem 21.6: Uniform limit of continuous functions is continuous.
- Example 1: $\mathbb{R}^\omega$ with the box topology is not first countable. In particular, it is not metrizable.

Go (briefly) over Example 2: uncountable product $\mathbb{R}^J$, considered with the product topology, is not first countable. In particular, it is not metrizable.

Go over Ex. 7 page 134.

Skip the rest of Chapter 2, that is, section 22.

Chapter 3: Connectedness and Compactness

Stress usability of these notions to the proofs of three classical calculus theorems: *Intermediate Value Theorem*, *Maximum Value Theorem*, and *Uniform Continuity Theorem.*

Intermediate Value Theorem is a consequence of *connectedness* property.

The other two theorems are the consequences of *compactess* property.

Section 23: Connected spaces

Definition 18 Let X be a topological space. A *separation* of X is any pair U, V, of open, non-empty disjoint sets with $X = U \cup V$. A topological space X is *connected* provided it **does not** exist a separation of X.

Example 1: Any X with indiscreet topology is connected.

Any X with discreet topology is *disconnected*, that is, not connected.

Fact: A space is connected, when $\emptyset$ and X are its only subsets that are simultaneously closed and open.

Definition 19 Let Y be a subspace of X. A *separation* of Y is any pair $A, B \subset Y$ non-empty sets such that $Y = A \cup B$ and $\mathrm{cl}(A) \cap B = A \cap \mathrm{cl}(B) = \emptyset$.

Lemma 28 *A subspace Y of X is connected is, and only if, there is no separation of Y.*

Go over Examples 2, 3, 4, and 5.

Lemma 29 *Assume that sets C and D forms separation of X. If a subspace Y of X is connected, then either $Y \subset C$ or $Y \subset D$.*

Theorem 30 (Star Lemma) *Let $\{A_\alpha\}_{\alpha \in J}$ be a family of connected subspaces of X. If $\bigcap_{\alpha \in J} A_\alpha \neq \emptyset$, then $\bigcup_{\alpha \in J} A_\alpha$ is connected.*

Class of October 25, 2012:

Recall

- A *separation* of a topological space X is any pair U, V, of open, non-empty disjoint sets with $X = U \cup V$. A topological space X is *connected* provided it **does not** exist a separation of X.
- Assume that sets C and D forms separation of X. If a subspace Y of X is connected, then either $Y \subset C$ or $Y \subset D$.
- **(Star Lemma)** Let $\{A_\alpha\}_{\alpha \in J}$ be a family of connected subspaces of X. If $\bigcap_{\alpha \in J} A_\alpha \neq \emptyset$, then $\bigcup_{\alpha \in J} A_\alpha$ is connected.

New material

Theorem 31 (Theorem 23.4) *Let A be a connected subspace of X. If $A \subset B \subset \mathrm{cl}(A)$, then B is connected.*

Theorem 32 (Theorem 23.5) *Continuous image of connected space is connected.*

This, together with the fact that intervals are connected, is the Intermediate Value Theorem.

Theorem 33 (Theorem 23.6) *Finite product of connected spaces is connected.*

Actually, arbitrary product of connected spaces, considered with the product topology, is connected. We show this only for $\mathbb{R}^\omega$, Example 7. (In general, this is Exercise 10.)

Example 6: $\mathbb{R}^\omega$ with the box topology is disconnected.

Go over Exercise 7, 2, 5, page 152.

Suggestion to students: Look over Exercises 3 and 9, page 152. (Exercises 4 and 8 are also of interest.)

Section 24: Connected spaces of the Real Line

Recall that $\mathbb{R}$ has the *least upper bound property*:

- every non-empty bounded above subset A of $\mathbb{R}$ has an upper bound $\sup(A) \in \mathbb{R}$.

Theorem 34 (Theorem 24.1, for $\mathbb{R}$ only) *A subset A of $\mathbb{R}$ (considered with the standard topology) is connected if, and only if, A is an interval (possible degenerated).*

Go over Exercise 1.

Go over the Intermediate Value Theorem, Theorem 24.3.

Class of October 30, 2012:

Recall

- A closure of a connected space is connected.
- Continuous image of connected space is connected.
- Finite product of connected spaces is connected.
- $A \subset \mathbb{R}$ is connected if, and only if, A is an interval.

New material

Define *path connectedness*.

Note that every path connected space is connected.

Go over Examples 3, 4, and 6.

Go over Examples 7, *topologists sine curve*: it is connected but not path connected.

Go over Exercises 2 and 3.

Suggestion to students: Look over Exercises 9, **10**, and 11.

Written assignment for Thursday, November 8: Exercise 8, p. 158.

Go (briefly) over Section 25 Define components and path components.
Go over Examples 1 and 2.
Define locally connected spaces and locally path connected spaces.
Go over Theorems 25.3 and 25.4.
Briefly discuss Exercise 10: quasi components.

Class of November 1, 2012:

Sections 26 and 27 (with mixed order): compactness

Definition 20 Let Y be a subset of a topological space X. A family $\mathcal{U}$ of subsets of X is a *covering* of Y provided $Y \subset \bigcup \mathcal{U}$. A covering $\mathcal{U}$ of Y is an *open covering* of Y provided every $U \in \mathcal{U}$ is open in X.

Definition 21 A topological space X is *compact* provided every open cover $\mathcal{U}$ of X there exists a finite subfamily $\mathcal{U}_0$ of $\mathcal{U}$ that covers X (i.e., $\mathcal{U}_0 \subset \mathcal{U}$ is finite and $X = \bigcup \mathcal{U}_0$). Such a family $\mathcal{U}_0$ will be referred to as a (finite) *subcover* of $\mathcal{U}$.

Note: Although subcover $\mathcal{U}_0$ of $\mathcal{U}$ is defined in term of a union $\bigcup \mathcal{U}_0$, this union usually does not belong to $\mathcal{U}$!

Go over Examples 1 and 4: Neither $\mathbb{R}$ nor $(0, 1]$ are compact.

Go over Examples 2 and 3: Every finite space X is compact. So is $X = \{L\} \cup \{a_n : n = 1, 2, 3, \ldots\} \subset \mathbb{R}$, provided $\lim_n a_n = L$.

Lemma 35 (Lemma 26.1)

Theorem 36 (Theorem 26.2) *Closed subspace of compact space is compact.*

Theorem 37 (Theorem 26.3) *Every compact subspace of a Hausdorff space is closed.*

Go over Example 6. Proof of the theorem is based on:

Lemma 38 (Lemma 26.4) *Let X be Hausdorff. For every compact subspace Y of X and avery $x \in X \setminus Y$ there exists disjoint open sets U and V in X such that $x \in U$ and $Y \subset V$.*

Theorem 39 (Theorem 27.1) *Every closed interval $[a, b]$ in $\mathbb{R}$ is compact.*

Corollary 40 (Corollary 27.3 for $\mathbb{R}$) *A subspace X of $\mathbb{R}$ is compact if, and only if, it is closed and bounded.*

Theorem 41 (Theorem 26.5) *A continuous image of a compact space is compact.*

Corollary 42 (Thm 27.4: Extreme Value Theorem for Intervals) *For every continuous function $f : [a, b] \to \mathbb{R}$ there exist $c, d \in [a, b]$ such that $f(c) \le f(x) \le f(d)$ for every $x \in [a, b]$.*

Solutions for Homework of October 9, 2012

Ex. 6. (Ex. 4(a). p. 127) Consider the product, uniform, and box topologies on $\mathbb{R}^\omega$. In which topologies are the following functions from $\mathbb{R}$ to $\mathbb{R}^\omega$ continuous?

$$f(t) = \langle t, 2t, 3t, \ldots \rangle, \quad g(t) = \langle t, t, t, \ldots \rangle, \quad h(t) = \langle t, \frac{1}{2}t, \frac{1}{3}t, \ldots \rangle$$

Solution, version 1.

Product topology on $\mathbb{R}^\omega$: All these functions are continuous, since so are their coordinate functions $k(t) = at$, $a \in \mathbb{R}$.

Box topology on $\mathbb{R}^\omega$: None these functions is continuous (at $t = 0$). Indeed, for a basic open set $U = \prod_{n=1}^\infty (-1/n^2, 1/n^2)$ in the box topology, $h^{-1}(U) = \bigcap_{n=1}^\infty \{t \in \mathbb{R} \colon \frac{1}{n}t \in (-1/n^2, 1/n^2)\} = \bigcap_{n=1}^\infty (-1/n, 1/n) = \{0\}$ is not open in $\mathbb{R}$. Similarly, we have $g^{-1}(U) = \{0\}$ and $f^{-1}(U) = \{0\}$.

Uniform topology on $\mathbb{R}^\omega$: Function f is not continuous (at $t = 0$) since for the open ball B centered at $\theta = \langle 0, 0, 0, \ldots \rangle$ with radius 1 we have $f^{-1}(B) = \{0\}$. Indeed, $0 \in f^{-1}(B)$ and

$$f^{-1}(B) \subset f^{-1}((-1,1)^\omega) = \bigcap_{n=1}^\infty \{t \in \mathbb{R} \colon nt \in (-1,1)\} = \bigcap_{n=1}^\infty (-1/n, 1/n) = \{0\}.$$

On the other hand functions g and h are continuous, as they satisfy the ε-δ condition (Thm 21.1) for every $t_0 \in \mathbb{R}$ and $\varepsilon > 0$. Indeed, for $\delta = \varepsilon$ and any $t \in \mathbb{R}$ with $|t - t_0| < \delta = \varepsilon$ we have

$$\bar{\rho}(h(t), h(t_0)) \le \sup_n \left| \frac{1}{n}t - \frac{1}{n}t_0 \right| \le |t - t_0| < \varepsilon$$

and, similarly, $\bar{\rho}(g(t), g(t_0)) \le |t - t_0| < \varepsilon$.

Solution, version 2.

Notice that each of the functions is of the form $F(t) = \langle F_n(t) \rangle_n = \langle c_n t \rangle_n$ for some numbers $c_n > 0$. In particular, for every $y = \langle F_n(t) \rangle_n \in F[\mathbb{R}^\omega]$ and $\delta_n > 0$ we have

$$F^{-1}\left(\prod_{n=1}^{\infty}(F_n(t)-\delta_n, F_n(t)+\delta_n)\right) = \bigcap_{n=1}^{\infty}\left(t-\frac{\delta_n}{c_n}, t+\frac{\delta_n}{c_n}\right) \tag{1}$$

since, for $W = \prod_{n=1}^{\infty}(F_n(t)-\delta_n, F_n(t)+\delta_n)$,

$$\begin{aligned} F^{-1}(W) &= \left\{t \in \mathbb{R} \colon \langle F_n(t)\rangle_n \in \prod_{n=1}^{\infty}(F_n(t)-\delta_n, F_n(t)+\delta_n)\right\} \\ &= \bigcap_{n=1}^{\infty}\{t \in \mathbb{R} \colon F_n(t) \in (F_n(t)-\delta_n, F_n(t)+\delta_n)\} \\ &= \bigcap_{n=1}^{\infty}\{t \in \mathbb{R} \colon c_n t \in (c_n t-\delta_n, c_n t+\delta_n)\} \\ &= \bigcap_{n=1}^{\infty}\left(t-\frac{\delta_n}{c_n}, t+\frac{\delta_n}{c_n}\right). \end{aligned}$$

1) All functions f, g, and h are continuous with range $\langle \mathbb{R}^\omega, \mathcal{T}_p \rangle$. Indeed, all coordinate functions f_n, g_n, and h_n are continuous (as linear) for any n. So, by Theorem 19.6, f, g, and h are indeed continuous.

2) None of the functions f, g, and h is continuous with range $\langle \mathbb{R}^\omega, \mathcal{T}_b \rangle$. Indeed, consider the set $W = \prod_{n=1}^{\infty}(-\frac{1}{n^2}, \frac{1}{n^2})$. Thus, to prove our claim, it is enough to show that its pre-image, under each of functions f, g, and h, is not open.

By (1), we have $F^{-1}(W) = \bigcap_{n=1}^{\infty}\left(-\frac{1}{c_n n^2}, \frac{1}{c_n n^2}\right)$. In particular, if $c_n \geq \frac{1}{n}$ for every n, then $0 \in F^{-1}(W) \subset \bigcap_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$. However, we indeed have $c_n \geq \frac{1}{n}$ for the functions f, g, and h. Therefore, $f^{-1}(W) = g^{-1}(W) = h^{-1}(W) = \{0\}$, certainly not open in $\mathbb{R}$.

3) Case of the range $\langle \mathbb{R}^\omega, \mathcal{T}_u \rangle$.

First, we will show that functions g and h are continuous for this case. So, let F be one of these two functions and let $U \in \mathcal{T}_u$. We need to show that $F^{-1}(U)$. So, choose a $t \in F^{-1}(U)$. It is enough to find a $\delta > 0$ such that $(t-\delta, t+\delta) \subset F^{-1}(U)$.

Since U is open and $F(t) \in U$, there is an $\varepsilon > 0$ such that $B(F(t), \varepsilon) \subset U$. Recall that, by Exercise 6, $B(F(t), \varepsilon) = \bigcup_{\delta<\varepsilon} W(F(t), \delta)$, where $W(F(t), \delta) = \prod_{n=1}^{\infty}(F_n(t) - \delta, F_n(t) + \delta)$. Fix a $\delta < \varepsilon$. Since F is one of the functions g and h, for which we have $c_n \leq 1$, by (1) we obtain that

$$F^{-1}(U) \supset F^{-1}\left(\prod_{n=1}^{\infty}(F_n(t) - \delta, F_n(t) + \delta)\right) = \bigcap_{n=1}^{\infty}\left(t - \frac{\delta}{c_n}, t + \frac{\delta}{c_n}\right) \supset (t-\delta, t+\delta),$$

as desired.

Finally, we will show that f is discontinuous, by showing that the pre-image $f^{-1}(B(f(0), 1)) = \{0\}$, that is, that the pre-image of an open set $B(f(0), 1)$ is not open. Indeed, using (1) with $F = f$ and noticing that, in this case, $c_n = n$, we get

$$0 \in h^{-1}(B(f(0), 1)) \subset F^{-1}\left(\prod_{n=1}^{\infty}(-1, 1)\right) = \bigcap_{n=1}^{\infty}\left(-\frac{1}{c_n}, \frac{1}{c_n}\right) = \{0\},$$

completing the proof.

Class of November 8, 2012:

Recall that:

- X is *compact* provided every open cover of X has a finite subcover.
- For a subspace Y of a compact Hausdorff space

$$Y \text{ is compact if, and only if, } Y \text{ is closed in } X.$$

- A subspace X of $\mathbb{R}$ is compact if, and only if, it is closed and bounded.
- **(Theorem 27.4: Extreme Value Theorem for Intervals)** For every continuous function $f\colon [a,b] \to \mathbb{R}$

New material:

Go over the exercises from section 26: Exercises 4, 5, 6.

Theorem 43 (Theorem 26.6) *If $f\colon X \to Y$ is a continuous bijection, X is compact, and Y is Hausdorff, then f is a homeomorphism.*

Theorem 44 (Thm 26.7) *Finite product of compact spaces is compact.*

Remark: Actually, arbitrary product of compact spaces is compact. This is Tychonoff Theorem. But its proof is more difficult.

Proof of Theorem 44 based on

Lemma 45 (Lem 26.8: The Tube Lemma) *Let Y be compact and $x \in X$. If an open set W of $X \times Y$ contains $\{x\} \times Y$, then there is an open set U in X such that $\{x\} \times Y \subset U \times Y \subset W$.*

Corollary 46 (Corollary 27.3 for $\mathbb{R}^n$) *A subspace X of $\mathbb{R}^n$ is compact if, and only if, it is closed and bounded.*

Corollary 47 (Extreme Value Theorem for $\mathbb{R}^n$) *If R is a closed bounded subset of $\mathbb{R}^n$, then for every continuous function $f\colon R \to \mathbb{R}$ there exist $c, d \in R$ such that $f(c) \le f(x) \le f(d)$ for every $x \in R$.*

Go over Exercise 8 section 26.

Suggested exercises from section 26: Exercises 1, 7.

Written assignment for Thursday, November 15: Exercise 9, p. 171.

Class of November 13, 2012:

Recall that:

- **The Tube Lemma:** Let Y be compact and $x \in X$. If an open set W of $X \times Y$ contains $\{x\} \times Y$, then there is an open set U in X such that $\{x\} \times Y \subset U \times Y \subset W$.
- Finite product of compact spaces is compact.
- A subspace X of $\mathbb{R}^n$ is compact if, and only if, it is closed and bounded.
- **(Extreme Value Theorem for $\mathbb{R}^n$)** If R is a closed bounded subset of $\mathbb{R}^n$, then for every continuous function $f\colon R \to \mathbb{R}$ there exist $c, d \in R$ such that $f(c) \leq f(x) \leq f(d)$ for every $x \in R$.

New material:

Definition 22 A collection $\mathcal{C}$ of subsets of X has *finite intersection property, fip,* provided $\bigcap \mathcal{C}_0 \neq \emptyset$ for every finite $\mathcal{C}_0 \subset \mathcal{C}$.

Theorem 48 (Thm 26.9) *X is compact if, and only if, $\bigcap \mathcal{C} \neq \emptyset$ for every family $\mathcal{C}$ of closed subsets of X having fip.*

Definition 23 A function f from a metric space $\langle X, d\rangle$ into a metric space $\langle Y, \rho\rangle$ is said to be *uniformly continuous* provided for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $x_1, x_2 \in X$

$$d(x_0, x_1) < \delta \Longrightarrow \rho(f(x_0), f(x_1)) < \varepsilon.$$

Theorem 49 *Let f be a continuous function from a compact metric space $\langle X, d\rangle$ into a metric space $\langle Y, \rho\rangle$. Then f is uniformly continuous.*

Proved using the Lebesgue number lemma, where a *diameter* of a subset D of a metric space $\langle X, d\rangle$ is defined as $\operatorname{diam}(D) = \sup\{d(x, y)\colon x, y \in D\}$.

Lemma 50 *Let $\mathcal{A}$ be an open cover of a metric space $\langle X, d\rangle$. If X is compact, then there exists a $\delta > 0$, known as a* **Lebesgue number***, such that for every $D \subset X$ of diameter $< \delta$, there exists an $A \in \mathcal{A}$ with $D \subset A$.*

Class of November 15, 2012:

Recall that:

- A collection $\mathcal{C}$ of subsets of X has *finite intersection property, fip,* provided $\bigcap \mathcal{C}_0 \neq \emptyset$ for every finite $\mathcal{C}_0 \subset \mathcal{C}$.
- X is compact if, and only if, $\bigcap \mathcal{C} \neq \emptyset$ for every family $\mathcal{C}$ of closed subsets of X having fip.
- If $\mathcal{A}$ is an open cover of a compact metric space $\langle X, d \rangle$, then there exists a $\delta > 0$, **Lebesgue number**, such that for every $D \subset X$ of diameter $< \delta$, there exists an $A \in \mathcal{A}$ with $D \subset A$.
- If f is a continuous function from a compact metric space $\langle X, d \rangle$ into a metric space $\langle Y, \rho \rangle$, then f is uniformly continuous.

New material:

Define an isolated point.

Theorem 51 (Thm 27.7) *If X is compact, Hausdorff, and has no isolated points, then X is uncountable.*

This implies that every interval $[a, b]$, $a < b$, is uncountable.

Go over Ex. 2 page 177.

Section 28: Limit Point Compactness

Definition 24 A space X is *limit point compact* provided every infinite subset of X has a limit point.

Theorem 52 (Thm 28.1) *If X is compact, then X is limit point compact, but not conversely.*

Definition 25 A space X is *sequentially compact* provided every sequence in X has a convergent subsequence.

Theorem 53 (Thm 28.2) *For a metrizable space X, the following are equivalent:*

(1) X is compact.

(2) X is limit point compact.

(3) X is sequentially compact.

Class of November 27, 2012:

Recall that:

For a metrizable space X, the following are equivalent:

(1) X is compact.

(2) X is limit point compact.

(3) X is sequentially compact.

New material:

Section 29: Local Compactness

Definition 26 A space X is *locally compact* provided every $x \in X$ there is an open set $U \ni x$ such that $\mathrm{cl}(U)$ is compact.

Compact implies locally compact.
Go over Examples 1 and 2. $\mathbb{Q}$ is not locally compact.
State and prove Theorem 29.1.

Next block of material, mainly for the next semester

Countability axioms

- (already seen) A topological space X is *first countable* (or *satisfies the first countability axiom*) provided for every $x \in X$ there exists a countable basis $\mathcal{B}_x$ of X at x.
- (new) A topological space X is *second countable* (or *satisfies the second countability axiom*) provided X has a countable basis.
- (new) A topological space X is *separable* provided X has a countable dense subset D, that is, such that $\mathrm{cl}(D) = X$.
- (new) A topological space X is *Lindelöf* provided every open cover of X has a countable subcover.

Separation axioms

- (already seen) X is a T_0 *space* provided for every distinct $x, y \in X$ there exists an open set $U \subset X$ such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$ (i.e., such that U contains precisely one of the points x and y).
- (already seen) X is a T_1 *space* provided for every distinct $x, y \in X$ there exists an open set $U \subset X$ such that $x \in U$ and $y \notin U$.
- (already seen) X is *Hausdorff* (or a T_2 *space*) provided for every distinct $x, y \in X$ there exist disjoint open sets $U, V \subset X$ such that $x \in U$ and $y \in V$.
- (new) X is *regular* (or a T_3 *space*) provided it is a T_1 space and for every closed set K in X and $x \in X \setminus K$ there exist disjoint open sets $U, V \subset X$ such that $x \in U$ and $K \subset V$.
- (new) X is *normal* (or a T_4 *space*) provided it is a T_1 space and for every disjoint closed sets K and L in X there exist disjoint open sets $U, V \subset X$ such that $K \subset U$ and $L \subset V$.
- (new) X is *completely regular* (or a $T_{3\frac{1}{2}}$ *space*) provided it is a T_1 space and for every closed set K in X and $x \in X \setminus K$ there exist a continuous function $f\colon X \to [0,1]$ such that $f(x) = 0$ and $f[K] \subset \{1\}$.

Important related theorems

- *Urysohn Lemma*: Every T_4 space is a $T_{3\frac{1}{2}}$ space.
- *Tietze Extension Theorem*: If X is normal, $K \subset X$ is closed, and $f\colon K \to [0,1]$ is continuous, then f can be extended to a continuous $F\colon X \to [0,1]$.
- *Urysohn Metrization Theorem*: If X is regular and second countable, then it is metrizable.

The Tychonoff Theorem

- *The Tychonoff Theorem*: Arbitrary product of compact spaces is compact.