

**Solutions for SAMPLE TEST # 1**

Solve the following exercises. **Show your work.** (No credit will be given for an answer with no supporting work shown.)

**Ex. 1.** Evaluate

$$(a) \quad 3 \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 11 & -1 \end{bmatrix} - 5 \begin{bmatrix} 0 & 8 \\ 4 & -2 \\ 5 & 1 \end{bmatrix} =$$

$$\text{Sol:} \quad \begin{bmatrix} 6 & 9 \\ 12 & 15 \\ 33 & -3 \end{bmatrix} + \begin{bmatrix} 0 & -40 \\ -20 & 10 \\ -25 & -5 \end{bmatrix} = \begin{bmatrix} 6 & -31 \\ -8 & 25 \\ 8 & -8 \end{bmatrix}.$$

$$(b) \quad \begin{bmatrix} 1 & 2 & 1 \\ -1 & 6 & 3 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 2 & 3 \end{bmatrix} =$$

$$\text{Sol:} \quad \begin{bmatrix} 1+0+2 & 1-2+3 \\ -1+0+6 & -1-6+9 \\ 0+0+4 & 0-1+6 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 5 & 2 \\ 4 & 5 \end{bmatrix}.$$

$$(c) \quad \begin{bmatrix} 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 7 \\ -1 \end{bmatrix} =$$

$$\text{Sol:} \quad \begin{bmatrix} 1 - 14 - 3 \end{bmatrix} = \begin{bmatrix} -16 \end{bmatrix}.$$

$$(d) \quad \begin{bmatrix} 1 \\ 7 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \end{bmatrix} =$$

$$\text{Sol:} \quad \begin{bmatrix} 1 & -2 & 3 \\ 7 & -14 & 21 \\ -1 & 2 & -3 \end{bmatrix}.$$

**Ex. 2.** Solve each of the following systems of linear equations by representing as augmented matrix and transforming it to the row reduced echelon form. If the system inconsistent, give a reason for it explain the meaning of this fact in terms of solutions. If it is consistent, express

its solution in the vertical vector form as  $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$  or  $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 5 \\ 11 \end{bmatrix}$ .

$$(a) \begin{cases} a + b - c = 0 \\ a - 4b + 2c = -1 \\ 2a - 3b + c = 1 \end{cases}$$

**Sol:**

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & -4 & 2 & -1 \\ 2 & -3 & 1 & 1 \end{bmatrix} \xrightarrow[-2R_1]{-R_1} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -5 & 3 & -1 \\ 0 & -5 & 3 & 1 \end{bmatrix} \xrightarrow[-R_2]{-R_2} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -5 & 3 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

The last row leads to the equation  $0a + 0b + 0c = 2$ , which is never satisfied.

Answer: System is inconsistent, no solution.

$$(b) \begin{cases} a + b - c = 0 \\ a - 4b + 2c = 1 \\ 2a - 3b + c = 1 \end{cases}$$

**Sol:**

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & -4 & 2 & -1 \\ 2 & -3 & 1 & 1 \end{bmatrix} \xrightarrow[-2R_1]{-R_1} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -5 & 3 & 1 \\ 0 & -5 & 3 & 1 \end{bmatrix} \xrightarrow[-R_2]{\times -\frac{1}{5}} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -5 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[-R_2]{-R_2} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -\frac{3}{5} & -\frac{1}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We get:  $a - \frac{2}{5}c = \frac{1}{5}$ , that is,  $a = \frac{1}{5} + \frac{2}{5}c$ ;  $b - \frac{3}{5}c = -\frac{1}{5}$ , that is,  $b = -\frac{1}{5} + \frac{3}{5}c$

Ans: Consistent, infinitely many solutions:  $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{1}{5} + \frac{2}{5}c \\ -\frac{1}{5} + \frac{3}{5}c \\ c \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ -\frac{1}{5} \\ 0 \end{bmatrix} + c \begin{bmatrix} \frac{2}{5} \\ \frac{3}{5} \\ 1 \end{bmatrix}$ .

$$(c) \begin{cases} a + b - c = 0 \\ a + 2c = 1 \\ 2a - 3b + c = 1 \end{cases} \quad \text{Sol:}$$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 0 & 2 & 1 \\ 2 & -3 & 1 & 1 \end{bmatrix} \xrightarrow[-2R_1]{-R_1} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & -5 & 3 & 1 \end{bmatrix} \xrightarrow[-5R_2]{+R_2} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & -12 & -4 \end{bmatrix} \xrightarrow[\times \frac{-1}{12}]{\times -1} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & \frac{1}{3} \end{bmatrix} \xrightarrow[-2R_3]{+3R_3} \begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{3} \end{bmatrix}$$

Answer: System is consistent, it has the unique solution:  $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 0 \\ \frac{1}{3} \end{bmatrix}$ .

**Ex. 3.** Find the determinant of the matrix. Each time you expand the the matrix, you **must** expand it over a row or column that has the largest number of zeros. If necessary, use the row (or column) reduction method to create additional zeros.

$$A = \begin{bmatrix} -1 & 2 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 2 & 0 & 1 \\ 0 & 3 & 1 & 2 \end{bmatrix}$$

**Sol:** If we subtract from row # 4 the row # 2 and expand by the third column, we get

$$|A| = \begin{vmatrix} -1 & 2 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 2 & 0 & 1 \\ -1 & 4 & 0 & 3 \end{vmatrix} = (-1) \cdot 1 \begin{vmatrix} -1 & 2 & 0 \\ 1 & 2 & 1 \\ -1 & 4 & 3 \end{vmatrix}$$

Next, subtracting from row # 3 three times the row # 2 and expanding again by the third column, we get

$$|A| = (-1) \begin{vmatrix} -1 & 2 & 0 \\ 1 & 2 & 1 \\ -1 & 4 & 3 \end{vmatrix} = (-1) \begin{vmatrix} -1 & 2 & 0 \\ 1 & 2 & 1 \\ -4 & -2 & 0 \end{vmatrix} = (-1)(-1) \begin{vmatrix} -1 & 2 \\ -4 & -2 \end{vmatrix} = 2 - (-8) = 10.$$

**Ex. 4.** Find the inverse matrix of

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

**Sol:** We need to transform  $[A; I]$  to  $[I; B]$ . Then  $B = A^{-1}$ .

$$\begin{aligned} [A; I] &= \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{+R_1} \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-R_2} \\ &\begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & 0 & -4 & -1 & -1 & 1 \end{bmatrix} \xrightarrow{\times -\frac{1}{4}} \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix} \xrightarrow{-R_3} \\ &\begin{bmatrix} 1 & 0 & 1 & -\frac{3}{4} & -\frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix} \xrightarrow{-3R_3} \\ \text{Answer: } A^{-1} &= \begin{bmatrix} -\frac{3}{4} & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix}. \end{aligned}$$

**Ex. 5.** Let  $A$  be as below. Show that it is its own inverse, that is, that  $A^{-1} = A$ .

$$A = \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

**Sol:** Matrix  $B$  is an inverse of  $A$ ,  $B = A^{-1}$ , precisely when  $AB = I$ . Thus, for  $A = A^{-1}$ , we must have  $AA = I$ . Here is the checking:

$$AA = \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & -1 \end{bmatrix} =$$

$$\begin{bmatrix} 1/4 + 3/4 + 0 & \sqrt{3}/4 - \sqrt{3}/4 + 0 & 0 + 0 + 0 \\ \sqrt{3}/4 - \sqrt{3}/4 + 0 & 3/4 + 1/4 + 0 & 0 + 0 + 0 \\ 0 + 0 + 0 & 0 + 0 + 0 & 0 + 0 + 1 \end{bmatrix} = I, \text{ as required.}$$

**Ex. 6.** Let  $\mathbf{a} = \langle 0, 1, 2 \rangle$ ,  $\mathbf{b} = \langle -1, 0, 7 \rangle$ , and  $\mathbf{c} = \langle 2, 3, -1 \rangle$ . Evaluate:  $2\mathbf{a} - \mathbf{b} + \mathbf{c}$ ,  $|\mathbf{c}|$ , and  $(\mathbf{a} \cdot \mathbf{b})(\mathbf{b} \times \mathbf{c})$ . (Do not confuse vectors with numbers. No partial credit for solutions with such errors.)

**Sol:**

$$2\mathbf{a} - \mathbf{b} + \mathbf{c} = 2\langle 0, 1, 2 \rangle - \langle -1, 0, 7 \rangle + \langle 2, 3, -1 \rangle = \langle 0, 2, 4 \rangle + \langle 1, 0, -7 \rangle + \langle 2, 3, -1 \rangle = \langle 3, 5, -4 \rangle$$

$$|\mathbf{c}| = \sqrt{2^2 + 3^2 + (-1)^2} = \sqrt{4 + 9 + 1} = \sqrt{14}$$

As  $\mathbf{a} \cdot \mathbf{b} = \langle 0, 1, 2 \rangle \cdot \langle -1, 0, 7 \rangle = 0 + 0 + 14 = 14$  and

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 7 \\ 2 & 3 & -1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 0 & 7 \\ 3 & -1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -1 & 7 \\ 2 & -1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 0 \\ 2 & -3 \end{vmatrix} =$$

$\mathbf{i}(0 - 21) - \mathbf{j}(1 - 14) + \mathbf{k}(-3 - 0) = \langle -21, 13, -3 \rangle$ , we have

$$(\mathbf{a} \cdot \mathbf{b})(\mathbf{b} \times \mathbf{c}) = 14 \langle -21, 13, -3 \rangle = \langle -14 \cdot 21, 14 \cdot 13, -3 \cdot 14 \rangle.$$