

SAMPLE TEST # 4

Solve the following exercises. **Show your work.**

Ex. 1. Set up the integral formulas, **including the limits of the integrations**, for the following problems. *Do not evaluate the integrals!* Where appropriate, use *polar, cylindrical, or spherical coordinates*.

- (a) The volume of the solid bounded by $z = x^2 + y^2$, $z = 0$, $x = 0$, $y = 0$, and $x + y = 1$.

Solution: If T is a triangle bounded by $x = 0$, $y = 0$, and $x + y = 1$ (i.e., $y = 1 - x$), then $V = \int \int \int_E 1 dV = \int \int_T \int_0^{x^2+y^2} 1 dz dA = \int_0^1 \int_0^{1-x} \int_0^{x^2+y^2} 1 dz dy dx$

- (b) The mass of the plane lamina bounded by $y = x^2$ and $y = 2x + 3$, with the density $\delta(x, y) = x^2$.

Solution: If $y = x^2$ and $y = 2x + 3$, then $x^2 = 2x + 3$, that is, $x^2 - 2x - 3 = 0$, so that $x = -3$ and $x = 1$. Then $mass = \int \int_R \delta(x, y) dA = \int_{-3}^1 \int_{x^2}^{2x+3} x^2 dy dx$.

- (c) The mass of the solid T with the density $\delta(x, y, z) = x^2 + e^z$ bounded by the surfaces: $6x + 2y + z = 12$, $x = 0$, $y = 0$, and $z = 0$.

Solution: The solid is a tetrahedron with a triangular base B on the xy -plane $z = 0$ bounded by $6x + 2y = 12$, $x = 0$, $y = 0$. The upper bound of T is $z = 12 - 6x - 2y$. So, $mass = \int \int \int_T \delta(x, y, z) dV = \int \int_B \int_0^{12-6x-2y} (x^2 + e^z) dz dA$.

Since the triangle side $6x + 2y = 12$ means that $y = 6 - 3x$, which equals 0 for $x = 2$, we get $mass = \int_0^2 \int_0^{6-3x} \int_0^{12-6x-2y} (x^2 + e^z) dz dy dx$.

Ex. 2. Evaluate the integrals:

- (a) $\int_0^1 \int_0^\pi \frac{1}{x+1} + \sin y dy dx =$

Solution: $int = \int_0^1 \left[\frac{1}{x+1} y - \cos y \right]_0^\pi dx = \int_0^1 \left(\frac{1}{x+1} \pi - (\cos \pi - \cos 0) \right) dx$. So

$$int = \int_0^1 \left(\frac{1}{x+1} \pi - (-1 - 1) \right) dx = \left[\pi \ln |x+1| + 2x \right]_0^1 = \pi(\ln 2 - \ln 1) + 2 = \pi \ln 2 + 2$$

- (b) $\int_{-2}^0 \int_0^y (x + 2y^2) dx dy =$

Solution: $int = \int_{-2}^0 \left[\frac{1}{2} x^2 + 2y^2 x \right]_{x=0}^{x=y} dy = \int_{-2}^0 \left(\frac{1}{2} y^2 + 2y^3 \right) dy = \left[\frac{1}{6} y^3 + \frac{1}{2} y^4 \right]_{-2}^0 = 0 - \left(\frac{1}{6}(-8) + \frac{1}{2}16 \right) = \frac{4}{3} - 8 = -6\frac{2}{3}$

(c) $\int \int_R \frac{dy dx}{\sqrt{9 - x^2 - y^2}}$, where R is the *second quadrant* region bounded by $x^2 + y^2 = 4$.

Solution: We use the polar coordinates, in which the region R is given as $0 \leq r \leq 2$ and $\pi/2 \leq \theta \leq \pi$. So, in the second equation using substitution $u = 9 - r^2$,

$$\begin{aligned} \text{int} &= \int_{\pi/2}^{\pi} \int_0^2 (9 - r^2)^{-1/2} r dr d\theta = \int_{\pi/2}^{\pi} \left[-(9 - r^2)^{1/2} \right]_0^2 d\theta = \\ &= \int_{\pi/2}^{\pi} \left[-\left((9 - 4)^{1/2} - 9^{1/2} \right) \right] d\theta = \left[3 - \sqrt{5} \right]_{\pi/2}^{\pi} = \frac{3 - \sqrt{5}}{2} \pi. \end{aligned}$$

Ex. 3. Find the mass of the solid bounded by the hemisphere $x^2 + y^2 + z^2 \leq R^2$, $z \geq 0$, with the density $\delta(x, y, z) = x^2 + y^2 + z^2$.

Solution: We use the spherical coordinates. Since the solid, T , is the upper hemisphere, we get

$$\begin{aligned} \text{mass} &= \int \int \int_T \delta(x, y, z) dV = \int \int \int_T (x^2 + y^2 + z^2) dV = \int_0^{\pi/2} \int_0^{2\pi} \int_0^R (\rho^2) \rho^2 \sin \phi d\rho d\theta d\phi = \\ &= \int_0^{\pi/2} \int_0^{2\pi} \left[\frac{1}{5} \rho^5 \sin \phi \right]_0^R d\theta d\phi = \int_0^{\pi/2} \int_0^{2\pi} \frac{1}{5} R^5 \sin \phi d\theta d\phi = \int_0^{\pi/2} \left[\left(\frac{1}{5} R^5 \sin \phi \right) \theta \right]_0^{2\pi} d\phi = \\ &= \int_0^{\pi/2} \frac{2}{5} \pi R^5 \sin \phi d\phi = \left[\frac{2}{5} \pi R^5 (-\cos \phi) \right]_0^{\pi/2} = -\frac{2}{5} \pi R^5 (\cos(\pi/2) - \cos 0) = -\frac{2}{5} \pi R^5 (0 - 1) = \\ &= \frac{2}{5} \pi R^5 \end{aligned}$$

Ex. 4. Find the mass of the plane lamina bounded by $x = 0$ and $x = 9 - y^2$ with density $\delta(x, y) = x^2$.

Solution: Notice that $x = 0$ and $x = 9 - y^2$ when $9 - y^2 = 0$ that is, when $y = \pm 3$.

$$\begin{aligned} \text{mass} &= \int \int_R \delta(x, y) dA = \int_{-3}^3 \int_0^{9-y^2} x^2 dx dy = \int_{-3}^3 \left[\frac{1}{3} x^3 \right]_0^{9-y^2} dy = \int_{-3}^3 \frac{1}{3} (9 - y^2)^3 dy = \\ &= \int_{-3}^3 \frac{1}{3} (9^3 - 3 \cdot 9^2 (y^2) + 3 \cdot 9 (y^2)^2 - (y^2)^3) dy = \int_{-3}^3 (3^5 - 3^4 y^2 + 3^2 y^4 - \frac{1}{3} y^6) dy = \\ &= \left[3^5 y - 3^3 y^3 + \frac{3^2}{5} y^5 - \frac{1}{21} y^7 \right]_{-3}^3 = 3^5 (3 + 3) - 3^3 (3^3 + 3^3) + \frac{3^2}{5} (3^5 + 3^5) - \frac{1}{21} (3^7 + 3^7) = \\ &= 2 \cdot 3^6 - 2 \cdot 3^6 + \frac{2}{5} 3^7 - \frac{2}{21} 3^7 = 2 \left(\frac{1}{5} - \frac{1}{21} \right) 3^7 = 2 \frac{21-5}{105} 3^7 = 2 \frac{16}{35} 3^6 = \frac{32}{35} 3^6 \end{aligned}$$

Ex. 5. Evaluate $\int_C xy ds$, where C is the parametric curve for which $x = 3t$, $y = t^4$, and $0 \leq t \leq 1$.

Solution: Since $ds = \sqrt{(x'(t))^2 + (y'(t))^2} dt = \sqrt{(3)^2 + (4t^3)^2} dt = \sqrt{9 + 16t^6} dt$,

$$\int_C xy ds = \int_0^1 (3t)(t^4) \sqrt{9 + 16t^6} dt = \int_0^1 (9 + 16t^6)^{1/2} (3t^5 dt)$$

For $u = 9 + 16t^6$, we get $\frac{du}{dx} = 6 \cdot 16t^5$, and so $3t^5 dt = \frac{1}{32} du$.

Hence, $\int (9 + 16t^6)^{1/2} (3t^5 dt) = \int u^{1/2} \frac{1}{32} du = \frac{1}{3 \cdot 16} u^{3/2} + C = \frac{1}{48} (9 + 16t^6)^{3/2} + C$. Thus

$$\int_C xy ds = \left[\frac{1}{48} (9 + 16t^6)^{3/2} \right]_0^1 = \frac{1}{48} [(9 + 16)^{3/2} - 9^{3/2}] = \frac{1}{48} [125 - 27] = \frac{49}{24} = 2 \frac{1}{24}$$

Ex. 6. Evaluate the integral, where C is the graph of $y = x^3$ from $(-1, -1)$ to $(1, 1)$

$$\int_C y^2 dx + x dy =$$

Solution: Clearly x changes from -1 to 1 . Put $x = t$. Then $y(t) = t^3$ and $-1 \leq t \leq 1$ and

$$\begin{aligned} \int_C y^2 dx + x dy &= \int_{-1}^1 (y(t))^2 x'(t) dt + x(t) y'(t) dt = \int_{-1}^1 [(t^3)^2 1 + t(3t^2)] dt = \int_{-1}^1 (t^6 + 3t^3) dt = \\ &= \left[\frac{1}{7} t^7 + \frac{3}{4} t^4 \right]_{-1}^1 = \frac{1}{7}(1 + 1) + \frac{3}{4}(1 - 1) = \frac{2}{7} \end{aligned}$$

Ex. 7. Determine if the following vector field is conservative. Find potential function for a field, if it is conservative.

(a) $\mathbf{F} = \left(x^3 + \frac{y}{x}\right) \mathbf{i} + (y^2 + \ln x) \mathbf{j}$

Solution: We have $P = x^3 + \frac{y}{x}$ and $Q = y^2 + \ln x$. So $\frac{\partial P}{\partial y} = \frac{1}{x}$ and $\frac{\partial Q}{\partial x} = \frac{1}{x}$. Since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, the field is conservative and we can find the potential function $f(x, y)$. We have

$$f(x, y) = \int P dx = \int x^3 + \frac{y}{x} dx = \frac{1}{4} x^4 + y \ln(x) + K(y).$$

Taking partial derivative, in terms of y , of both side we get

$$\ln(x) + K'(y) = \frac{\partial f}{\partial y} = Q = y^2 + \ln x, \text{ so that } K'(y) = y^2 \text{ and } K(y) = \frac{1}{3} y^3 + C.$$

Answer: The potential function $f(x, y) = \frac{1}{4} x^4 + y \ln(x) + \frac{1}{3} y^3 + C$.

(b) $\mathbf{F} = (y \cos x + \ln y) \mathbf{i} + \left(\frac{x}{y} + e^y\right) \mathbf{j}$

Solution: We have $P = y \cos x + \ln y$ and $Q = \frac{x}{y} + e^y$. So $\frac{\partial P}{\partial y} = \cos x + \frac{1}{y}$ and $\frac{\partial Q}{\partial x} = \frac{1}{y}$. Since $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, the field is not conservative and the potential function does not exist.

Ex. 8. Find a potential function of the vector field and use the fundamental theorem for line integrals to evaluate

$$\int_{(\pi/2, \pi/2)}^{(\pi, \pi)} (\sin y + y \cos x) dx + (\sin x + x \cos y) dy =$$

Solution: We have $P = \sin y + y \cos x$ and $Q = \sin x + x \cos y$. It is easy to see that $\frac{\partial P}{\partial y} = \cos y + \cos x = \frac{\partial Q}{\partial x}$ so indeed we can find the potential function $f(x, y)$. We have

$$f(x, y) = \int P dx = \int \sin y + y \cos x dx = x \sin y + y \sin x + K(y).$$

Taking partial derivative, in terms of y , of both side we get

$$x \cos y + \sin x + K'(y) = \frac{\partial f}{\partial y} = Q = \sin x + x \cos y, \text{ so that } K'(y) = 0 \text{ and } K(y) = C.$$

So, the potential function $f(x, y) = x \sin y + y \sin x + C$ and

$$\begin{aligned} \text{int} &= [f(x, y)]_{(\pi/2, \pi/2)}^{(\pi, \pi)} = [x \sin y + y \sin x]_{(\pi/2, \pi/2)}^{(\pi, \pi)} = (\pi \sin \pi + \pi \sin \pi) - \left(\frac{\pi}{2} \sin \frac{\pi}{2} + \frac{\pi}{2} \sin \frac{\pi}{2}\right) = \\ &= (0 + 0) - \left(\frac{\pi}{2} + \frac{\pi}{2}\right) = -\pi \end{aligned}$$

Ex. 9. Apply Green's theorem to evaluate the following integral, where the simple closed curve C , with counter clockwise direction, is the boundary of the circle $x^2 + y^2 = 1$.

$$\oint_C (\sin x - x^2 y) dx + xy^2 dy =$$

Solution: Let D denoted the disk $x^2 + y^2 \leq 1$.

By Green's theorem $\text{int} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$, where $P = \sin x - x^2 y$ and $Q = xy^2$. So,

$$\text{int} = \iint_D (y^2 - (-x^2)) dA = \iint_D (x^2 + y^2) dA$$

Changing to the polar coordinates, we get

$$\text{int} = \int_0^{2\pi} \int_0^1 r^2 r dr d\theta = \int_0^{2\pi} \left[\frac{1}{4}r^4\right]_0^1 d\theta = \int_0^{2\pi} \frac{1}{4} d\theta = \left[\frac{1}{4}\theta\right]_0^{2\pi} = \frac{1}{2}\pi$$