Topology 1, Math 581, Fall 2014: Notes and homework

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Class of August 19:

Course and syllabus overview.

Topology is an abstract geometry, sometimes referred to as *Rubber Sheet Geometry.* Material, in this course, will be presented "from abstract definitions and results to specific examples."

Notation:

- Do not confuse $A \in \mathcal{A}$ (which reads "A is an element of $\mathcal{A}$") with $A \subset \mathcal{A}$ (which reads "A is a subset of $\mathcal{A}$" and means "every element of A is also an element of $\mathcal{A}$).

  Notice that $A \subset B \subset C$ implies $A \subset C$, but $A \in B \in C$ does not imply $A \in C$. You will never see in this course a pair A and B, for which we will have simultaneously $A \in B$ and $A \subset B$.

- Notation $f\colon X \to Y$ means that f is a function from a set X, domain of the function, into the set Y. For any set C (usually, $C \subset Y$), the preimage $f^{-1}(C)$ (of C under f) is defined as

$$f^{-1}(C) = \{x \in X\colon f(x) \in C\}.$$

Example 1 *$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ for every A, B, and function f.*

PROOF. $x \in f^{-1}(A \cap B) \Leftrightarrow f(x) \in A \cap B \Leftrightarrow f(x) \in A \;\&\; f(x) \in A$
$\Leftrightarrow x \in f^{-1}(A) \;\&\; x \in f^{-1}(B) \Leftrightarrow x \in f^{-1}(A) \cap f^{-1}(B)$. ■

Motivation:

Let $\mathbb{R}$ be the set of real numbers and for $x \in \mathbb{R}$ and $\varepsilon > 0$ let

$$B(x, \varepsilon) = \{r \in \mathbb{R}\colon |x - r| < \varepsilon\}.$$

We will refer to $B(x, \varepsilon)$ as an *open ball*, although for this case it is just an open interval $(x - \varepsilon, x + \varepsilon)$. Let $\mathcal{T}$ be the family of all subsets U of $\mathbb{R}$ such that for every $x \in U$ there is an $\varepsilon > 0$ such that $x \in B(x, \varepsilon) \subset U$:

$$\mathcal{T} = \{U \subset \mathbb{R}\colon \forall x \in U \exists \varepsilon > 0 (B(x, \varepsilon) \subset U)\}.$$

Latter, we will refer to $\mathcal{T}$ as the *standard topology* on $\mathbb{R}$ and its elements $U \in \mathcal{T}$ will be called *open sets.*

Theorem 2 *(Motivational) Let $f\colon \mathbb{R} \to \mathbb{R}$. The following two definitions of continuity of f are equivalent:*

(a) (Topological definition) $f^{-1}(U) \in \mathcal{T}$ for every $U \in \mathcal{T}$.

(b) (ε-δ definition) For every $x \in \mathbb{R}$ and every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $r \in \mathbb{R}$, if $|x - r| < \delta$, then $|f(x) - f(r)| < \varepsilon$.

Proof. Latter today.

For functions $f\colon X \to Y$ and $g\colon Y \to Z$ their composition $g \circ f\colon X \to Z$ is defined via formula: $(g \circ f)(x) = g(f(x))$ for every $x \in X$. Also, if $A \subset X$, then the image $f[A]$ of A under f is defined as $\{f(a)\colon a \in A\}$.

Theorem 3 *We have the following properties:*

(a) $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$

(b) $(g \circ f)[A] = g[f[A]]$

Proof. (a) $x \in (g \circ f)^{-1}(C) \Leftrightarrow\ (g \circ f)(x) \in C \Leftrightarrow\ g(f(x)) \in C \Leftrightarrow$ $f(x) \in g^{-1}(C) \Leftrightarrow\ x \in f^{-1}(g^{-1}(C))$.

Proof of (b) is left as an exercise. (Not homework assignment.) ■

The next theorem gives a motivation of defining continuity of a functions via property (a) of Theorem 2. Note, that the proof is considerably easier than a standard ε-δ proof.

Theorem 4 *If functions $f, g\colon \mathbb{R} \to \mathbb{R}$ are continuous, then so is their composition $g \circ f\colon \mathbb{R} \to \mathbb{R}$.*

Proof. Let $U \in \mathcal{T}$. By Theorem 2 it is enough to prove that $(g \circ f)^{-1}(U) \in \mathcal{T}$. By Theorem 3(a), $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$. Now, $W = g^{-1}(U) \in \mathcal{T}$ by the continuity of g and Theorem 2. Therefore, by the continuity of f (and Theorem 2 used once again), $(g \circ f)^{-1}(U) = f^{-1}(W) \in \mathcal{T}$, as required. ■

The same proof will work for arbitrary continuous functions defined via a general notion of defined below. (See section 12 in the text.)

Proof of Theorem 2. (a)$\Longrightarrow$(b): Fix an $x \in \mathbb{R}$ and an $\varepsilon > 0$. Using (a), we need to find a δ satisfying (b).

Let $U = B(f(x), \varepsilon) = (f(x) - \varepsilon, f(x) + \varepsilon)$. Notice that $U \in \mathcal{T}$. (This requires checking, that U satisfies the definition of sets in $\mathcal{T}$.) So, by (a),

$f^{-1}(U) \in \mathcal{T}$. Note also, that $x \in f^{-1}(U)$, as $f(x) \in (f(x)-\varepsilon, f(x)+\varepsilon) = U$. Therefore, we have $x \in f^{-1}(U) \in \mathcal{T}$ and, by the definition of $\mathcal{T}$, there is a $\delta > 0$ such that $B(x,\delta) \subset f^{-1}(U)$. We show, that this δ satisfies (b).

Indeed, let $r \in \mathbb{R}$ be such that $|x-r| < \delta$. Then, $r \in (x-\delta, x+\delta) = B(x,\delta) \subset f^{-1}(U)$. Therefore, $f(r) \in U = (f(x)-\varepsilon, f(x)+\varepsilon)$ and so, $|f(x)-f(r)| < \varepsilon$, as required.

(b)$\Longrightarrow$(a): Fix a $U \in \mathcal{T}$. We need to show that $f^{-1}(U)$ is in $\mathcal{T}$. For this, take an $x \in f^{-1}(U)$. We need to find a $\delta > 0$ for which $B(x,\delta) \subset f^{-1}(U)$.

We have $f(x) \in U$, as $x \in f^{-1}(U)$. Since $U \in \mathcal{T}$, there exists an $\varepsilon > 0$ for which $B(f(x),\varepsilon) \subset U$. Using (b) for this x and ε, we can find a $\delta > 0$ such that $|f(x)-f(r)| < \varepsilon$ provided $|x-r| < \delta$. We will show that for this choice of δ we indeed have $B(x,\delta) \subset f^{-1}(U)$.

To see this, take an $r \in B(x,\delta) = (x-\delta, x+\delta)$. We need to show that $r \in f^{-1}(U)$. Since $r \in (x-\delta, x+\delta)$, we have $|x-r| < \delta$. So, by the choice of δ, $|f(x)-f(r)| < \varepsilon$. In particular, $f(r) \in (f(x)-\varepsilon, f(x)+\varepsilon) = B(f(x),\varepsilon) \subset U$. Thus, $r \in f^{-1}(U)$, as required. ■

Reading assignment: Read Sections 1-7.

It is assumed that you are familiar with the material presented there. Therefore, we will not cover this material in class. (If necessary, we will be reviewing these notion on "as needed" basis.)

Written assignment: Write for the next class:

Exercise 1 *Prove that* $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ *for every sets* A *and* B*, and a function* $f \colon X \to Y$*.*

Prove, for Tuesday, September 2, the following version of Theorem 2. Provide direct proof, that is, without using condition (b) of Theorem 2.
Note: Exercise 2 will be treated as ***bonus exercise***. (This is, really, a real analysis problem. So, do not feel bad, if you cannot solve this.)

Exercise 2 *(Motivational Theorem Part 2) Let* $f \colon \mathbb{R} \to \mathbb{R}$*. The following two definitions of continuity of* f *are equivalent:*

- *(a) (Topological definition)* $f^{-1}(U) \in \mathcal{T}$ *for every* $U \in \mathcal{T}$*.*
- *(c) (Sequential definition)* $\lim_{n\to\infty} f(x_n) = f(x)$ *for every sequence* $\langle x_n \rangle_{n=1}^{\infty}$ *in* $\mathbb{R}$ *converging to* $x \in \mathbb{R}$*.*

Class of August 21:

What we covered last class: For $x \in \mathbb{R}$ and $\varepsilon > 0$ we define an open ball

$$B(x,\varepsilon) = \{r \in \mathbb{R}: |x-r| < \varepsilon\} = (x-\varepsilon, x+\varepsilon).$$

Let $\mathcal{T}$ be the family of all subsets U of $\mathbb{R}$ such that for every $x \in U$ there is an $\varepsilon > 0$ such that $x \in B(x,\varepsilon) \subset U$:

$$\mathcal{T} = \{U \subset \mathbb{R}: \forall x \in U \exists \varepsilon > 0 (B(x,\varepsilon) \subset U)\}.$$

We will refer to $\mathcal{T}$ as the *standard topology* on $\mathbb{R}$ and its elements $U \in \mathcal{T}$ will be called *open sets*.

We proved

Let $f: \mathbb{R} \to \mathbb{R}$. The following two definitions of continuity of f are equivalent:

- *(a) (Topological definition) $f^{-1}(U) \in \mathcal{T}$ for every $U \in \mathcal{T}$.*
- *(b) (ε-δ definition) For every $x \in \mathbb{R}$ and every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $r \in \mathbb{R}$, if $|x-r| < \delta$, then $|f(x) - f(r)| < \varepsilon$.*

New material:

Definition 1 Let X be an arbitrary set having at least two elements. A *topology* on X is any family $\mathcal{T}$ of subsets of X having the following properties:

1. $\emptyset, X \in \mathcal{T}$.
2. The union of the any subfamily of $\mathcal{T}$ is in $\mathcal{T}$, that is, $\bigcup \mathcal{U} \in \mathcal{T}$ for every $\mathcal{U} \subset \mathcal{T}$.
3. The intersection of the any *finite* subfamily of $\mathcal{T}$ is in $\mathcal{T}$, that is, $\bigcap \mathcal{U} \in \mathcal{T}$ for every finite $\mathcal{U} \subset \mathcal{T}$.

The pair $\langle X, \mathcal{T} \rangle$ is called a *topological space*. For a fixed topological space $\langle X, \mathcal{T} \rangle$, the sets belonging to the family $\mathcal{T}$ will be refereed to as the *open sets* (with respect to this topology).

In the above definition, we used the following notation:

- Arbitrary unions and intersections of sets: Let $\mathcal{A}$ be a family of sets, say $\mathcal{A} = \{A_t: t \in T\}$. Then $\bigcup \mathcal{A} = \bigcup_{t \in T} A_t$ denotes the same set: $\{x: \exists A \in \mathcal{A}(x \in A)\}$, that is, $\{x: \exists t \in T(x \in A_t)\}$.

- Similarly, $\bigcap \mathcal{A} = \bigcap_{t\in T} A_t$ denotes the same set: $\{x\colon \forall A \in \mathcal{A}(x \in A)\}$, that is, $\{x\colon \forall t \in T(x \in A_t)\}$.

Remark 5 In the definition, condition (3) can be replaced with

(3') The intersection of the any two sets in $\mathcal{T}$ is in $\mathcal{T}$, that is, if $U, V \in \mathcal{T}$, the also $U \cap V \in \mathcal{T}$.

PROOF. Easy induction. ∎

Example 6 *Here are some examples of topological spaces $\langle X, \mathcal{T}\rangle$, where X is an arbitrary set.*

- $\mathcal{T} = \mathcal{P}(X)$, *where* $\mathcal{P}(X)$ *is the* **power set of** X, *that is, the family of all subsets of* X. *This topology is called the* **discrete topology**.
- $\mathcal{T} = \{\emptyset, X\}$. *This topology is called* **trivial** *or* **indiscrete topology**.
- **The standard topology** $\mathcal{T}$ *on* $\mathbb{R}$, *defined for Theorem 2.*

More examples:

Example 7 *Examples of topologies on a set* X*:*

- *For a three elements set* $X = \{a, b, c\}$, *there are many different possible topologies. (Nine are indicated in Example 1, page76). E.g.* $\{\emptyset, \{a\}, \{a, b\}, X\}$. *Other examples from the text, section 12.*
- **Finite complement topology** $\mathcal{T}_f = \{\emptyset\} \cup \{X \setminus F\colon F \text{ is finite}\}$. *Notice that* $\langle X, \mathcal{T}_f\rangle$ *is discrete, for finite* X.
- **Countable complement topology** $\mathcal{T}_C = \{\emptyset\} \cup \{X \setminus F\colon F \text{ is countable}\}$. *Notice that* $\langle X, \mathcal{T}_C\rangle$ *is discrete, for countable* X.

Definition of *finer* and *coarser* topologies.
Proof that $f^{-1}(\bigcup_{t\in T} A_t) = \bigcup_{t\in T} f^{-1}(A_t)$:

$$
\begin{aligned}
x \in f^{-1}(\bigcup_{t\in T} A_t) &\Leftrightarrow f(x) \in \bigcup_{t\in T} A_t \ \text{(by the definition of preimage)}\\
&\Leftrightarrow \exists t \in T\ f(x) \in A_t \ \text{(by the definition of union)}\\
&\Leftrightarrow \exists t \in T\ x \in f^{-1}(x)A_t \ \text{(by the definition of preimage)}\\
&\Leftrightarrow x \in \bigcup_{t\in T} f^{-1}(A_t) \ \text{(by the definition of union).}
\end{aligned}
$$

Class of August 26:

Recall that a *topology* on X is a family $\mathcal{T}$ of subsets of X such that

(1) $\emptyset, X \in \mathcal{T}$;

(2) $\bigcup \mathcal{U} \in \mathcal{T}$ for every $\mathcal{U} \subset \mathcal{T}$;

(3) $\bigcap \mathcal{U} \in \mathcal{T}$ for every finite $\mathcal{U} \subset \mathcal{T}$.

Examples of topological spaces $\langle X, \mathcal{T} \rangle$:

- **Discrete topology** $\mathcal{T} = \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the **power set of** X.
- **Trivial** or **indiscrete topology** $\mathcal{T} = \{\emptyset, X\}$.
- **The standard topology** $\mathcal{T}$ on $\mathbb{R}$, defined for Theorem 2.
- **Finite complement topology** $\mathcal{T}_f = \{\emptyset\} \cup \{X \setminus F \colon F \text{ is finite}\}$. Notice that $\langle X, \mathcal{T}_f \rangle$ is discrete, for finite X.
- **Countable complement topology** $\mathcal{T}_C = \{\emptyset\} \cup \{X \setminus F \colon F \text{ is countable}\}$. Notice that $\langle X, \mathcal{T}_C \rangle$ is discrete, for countable X.

New material:

Section 13: Basis for a Topology

Definition 2 *Basis — Two related definitions*

FROM A BASIS TO TOPOLOGY — **Basis for a̲ topology:** A collection $\mathcal{B}$ of a subsets of a set X such that

(B1) For every $x \in X$ there is a $B \in \mathcal{B}$ with $x \in B$ (i.e., $\bigcup \mathcal{B} = X$).

(B2) For every $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ there is a $B \in \mathcal{B}$ with $x \in B \subset B_1 \cap B_2$.

[FROM A TOPOLOGY TO ITS BASIS — **Basis for a given topology** $\mathcal{T}$**:**

Let $\langle X, \mathcal{T} \rangle$ be a fixed topological space. A basis for $\mathcal{T}$ is any collection $\underline{\mathcal{B} \subset \mathcal{T}}$ such that for every $U \in \mathcal{T}$ and every $x \in U$ there exists a $B \in \mathcal{B}$ with $x \in B \subset U$.

The first of these notion is used to create new topologies. The second is used to easier deal with a given, fixed topology $\mathcal{T}$. This second notion is used considerably more often than the first one.

Fact 1 *If $\mathcal{B}$ satisfies (B1) and (B2), then the family*

$$\mathcal{T}(\mathcal{B}) = \{U \subset X \colon \forall x \in U \exists B \in \mathcal{B} (x \in B \subset U)\} = \left\{\bigcup \mathcal{U} \colon \mathcal{U} \subset \mathcal{B}\right\}$$

is a topology on X. The family $\mathcal{B}$ is a basis to the topology $\mathcal{T}(\mathcal{B})$.

Fact 2 *(Lemma 13.2) If $\mathcal{B}$ is a basis for a topology $\mathcal{T}$, then $\mathcal{T} = \mathcal{T}(\mathcal{B})$.*

Discuss examples 1–3.
Go over Lemma 13.3.

There may be more than one basis for a given topology: Example 4 (from Examples 1 and 2).

Example 8 *Two examples of topologies on $\mathbb{R}$:*

- **Standard topology**, *generated by basis $\mathcal{B}_{st} = \{(a, b) \colon a, b \in \mathbb{R}, a < b\}$, that is, the topology $\mathcal{T}_{st} = \mathcal{T}(\mathcal{B}_{st})$. We usually write just $\mathbb{R}$ for $\langle \mathbb{R}, \mathcal{T}_{st} \rangle$.*

  Notice, that this is the same topology that was used in Theorem 2.

- **Lower limit (or Sorgenfrey) topology** *$\mathcal{T}_\ell$ is generated by basis $\mathcal{B}_\ell = \{[a, b) \colon a, b \in \mathbb{R}, a < b\}$, that is, $\mathcal{T}_\ell = \mathcal{T}(\mathcal{B}_\ell)$. We usually write $\mathbb{R}_\ell$ for $\langle \mathbb{R}, \mathcal{T}_\ell \rangle$.*

Written assignment for Tuesday, September 2: Exercise 8, page 83. (In part (b), do not forget to prove, that $\mathcal{T}(\mathcal{C})$ is indeed a topology. Do you need to prove, in part (a), that $\mathcal{T}(\mathcal{B})$ is a topology?)

Be ready for a quiz next class time!

Class of September 2:

Recall that (rephrasing):

Basis for a <u>given</u> topology $\mathcal{T}$: Let $\langle X, \mathcal{T}\rangle$ be a fixed topological space. A basis for $\mathcal{T}$ is any collection $\underline{\mathcal{B} \subset \mathcal{T}}$ such that for every $U \in \mathcal{T}$ and every $x \in U$ there exists a $B \in \mathcal{B}$ with $x \in B \subset U$.

Fact 3 *For a collection $\mathcal{B}$ of subsets of X, let*

$$\mathcal{T}(\mathcal{B}) = \{U \subset X \colon \forall x \in U \exists B \in \mathcal{B}(x \in B \subset U)\}.$$

If $\mathcal{B}$ satisfies the following two conditions:

(B1) For every $x \in X$ there is a $B \in \mathcal{B}$ with $x \in B$ (i.e., $\bigcup \mathcal{B} = X$).

(B2) For every $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ there is a $B \in \mathcal{B}$ with $x \in B \subset B_1 \cap B_2$.

then $\mathcal{T}(\mathcal{B})$ is a topology on X and $\mathcal{B}$ is a basis for $\mathcal{T}(\mathcal{B})$.

Restate and prove Lemma 13.3.

New material:

Example 9 *Three examples of topologies on $\mathbb{R}$, defined via bases:*

- **Standard topology**, *generated by basis $\mathcal{B}_{st} = \{(a, b) \colon a, b \in \mathbb{R}, a < b\}$, that is, the topology $\mathcal{T}_{st} = \mathcal{T}(\mathcal{B}_{st})$. We usually write just $\mathbb{R}$ for $\langle \mathbb{R}, \mathcal{T}_{st}\rangle$.*

  Notice, that this is the same topology that was used in Theorem 2.

- **Lower limit (or Sorgenfrey) topology** *$\mathcal{T}_\ell$ is generated by basis $\mathcal{B}_\ell = \{[a, b) \colon a, b \in \mathbb{R}, a < b\}$, that is, $\mathcal{T}_\ell = \mathcal{T}(\mathcal{B}_\ell)$. We usually write $\mathbb{R}_\ell$ for $\langle \mathbb{R}, \mathcal{T}_\ell\rangle$.*

- **K-topology** *$\mathcal{T}_K$: Let $K = \{1/n \colon n = 1, 2, 3, \ldots\}$. Then $\mathcal{T}_K$ is generated by basis $\mathcal{B}_K = \mathcal{B}_{st} \cup \{(a, b) \setminus K \colon a, b \in \mathbb{R}, a < b\}$, that is, $\mathcal{T}_K = \mathcal{T}(\mathcal{B}_K)$. We usually write $\mathbb{R}_K$ for $\langle \mathbb{R}, \mathcal{T}_K\rangle$.*

Fact 4 *(Lemma 13.4) $\mathcal{T}_\ell$ and $\mathcal{T}_K$ are strictly finer than $\mathcal{T}_{st}$.*

Definition of *subbasis* for a topology.

Note that $\mathcal{S} = \{(a, \infty) \colon a \in \mathbb{R}\} \cup \{(-\infty, b) \colon b \in \mathbb{R}\}$ is a subbasis for $\mathbb{R}$ (with the standard topology).

Go over exercises 1, 3, 6.

Class of September 3:

Go briefly over:

Section 14: Order Topology: For linearly ordered set $\langle X, \leq \rangle$, order topology is generated by subbasis $\mathcal{S} = \{(a, \infty) \colon a \in X\} \cup \{(-\infty, b) \colon b \in X\}$.

Describe basis for X. (Definition, page 84.)

Go over examples 1-4.

Section 15: Product Topology on $X \times Y$

Definition 3 For topological spaces $\langle X, \mathcal{T}_1 \rangle$ and $\langle Y, \mathcal{T}_2 \rangle$ let $\mathcal{B}_{\mathcal{T}_1,\mathcal{T}_2}$ be the family of all open rectangles, that is,

$$\mathcal{B}_{\mathcal{T}_1,\mathcal{T}_2} = \{U \times V \colon U \in \mathcal{T}_1 \ \& \ V \in \mathcal{T}_2\}.$$

Note that $\mathcal{B}_{\mathcal{T}_1,\mathcal{T}_2}$ satisfies conditions (B1) and (B2) for a topology on $X \times Y$. So, the family $\mathcal{T}(\mathcal{B}_{\mathcal{T}_1,\mathcal{T}_2})$ is a topology on $X \times Y$.

The topology $\mathcal{T}(\mathcal{B}_{\mathcal{T}_1,\mathcal{T}_2})$ is called the *product topology* on $X \times Y$.

Note that, in general,

$$\mathcal{T}(\mathcal{B}_{\mathcal{T}_1,\mathcal{T}_2}) \neq \mathcal{B}_{\mathcal{T}_1,\mathcal{T}_2},$$

since, $\mathcal{B}_{\mathcal{T}_1,\mathcal{T}_2}$ is not closed under unions, as, usually, $(U_1 \times V_1) \cup (U_2 \times V_2)$ is not a rectangle (so, itdoes not belong to $\mathcal{B}_{\mathcal{T}_1,\mathcal{T}_2}$).

Theorem 10 *If $\mathcal{B}_1$ is a basis for $\langle X, \mathcal{T}_1 \rangle$ and $\mathcal{B}_2$ is a basis for $\langle Y, \mathcal{T}_2 \rangle$, then the family*

$$\mathcal{B}_{\mathcal{B}_1,\mathcal{B}_2} = \{U \times V \colon U \in \mathcal{B}_1 \ \& \ V \in \mathcal{B}_2\}$$

is a basis for the product topology on $X \times Y$.

Corollary 11 *(Example 1) The family $\mathcal{B} = \{(a, b) \times (c, d) \colon a, b, c, d \in \mathbb{R}\}$ is a basis for the product topology on $\mathbb{R} \times \mathbb{R}$, where $\mathbb{R}$ is considered with the standard topology. Thus, the product topology on $\mathbb{R} \times \mathbb{R}$ coincides with the standard topology $\mathcal{T}(\mathcal{B})$ on $\mathbb{R} \times \mathbb{R}$.*

Definition 4 For the Cartesian product $X_1 \times X_2$ define the *projection function* $\pi_1 \colon X_1 \times X_2 \to X_1$ onto the first coordinate as $\pi_1(x_1, x_2) = x_1$. Similarly, the projection onto the second coordinate is the function $\pi_2 \colon X_1 \times X_2 \to X_2$ defined as $\pi_2(x_1, x_2) = x_2$.

Notice that for $U \subset X_1$ and $V \subset X_2$ we have

$$\pi_1^{-1}(U) = U \times X_2 \quad \text{and} \quad \pi_2^{-1}(V) = X_1 \times V.$$

In particular, for topological spaces $\langle X_1, \mathcal{T}_1 \rangle$ and $\langle X_2, \mathcal{T}_2 \rangle$, the family

$$\mathcal{S} = \{\pi_i^{-1}(W) \colon i \in \{1,2\} \ \&\ W \in \mathcal{T}_i\}$$

forms a subbasis for the product topology on $X_1 \times X_2$, since we have the identity $\pi_1^{-1}(U) \cap \pi_2^{-1}(V) = U \times V$.

Class of September 9:

Recall that:

- For the topological spaces $\langle X, \mathcal{T}_1\rangle$ and $\langle Y, \mathcal{T}_2\rangle$, the *product topology* on $X \times Y$ is generated by a basis: $\mathcal{B}_{\mathcal{T}_1,\mathcal{T}_2} = \{U \times V : U \in \mathcal{T}_1 \ \& \ V \in \mathcal{T}_2\}$.
- If $\mathcal{B}_1$ is a basis for $\langle X, \mathcal{T}_1\rangle$ and $\mathcal{B}_2$ is a basis for $\langle Y, \mathcal{T}_2\rangle$, then the family $\mathcal{B}_{\mathcal{B}_1,\mathcal{B}_2} = \{U \times V : U \in \mathcal{B}_1 \ \& \ V \in \mathcal{B}_2\}$ is a basis for the product topology on $X \times Y$.

Section 16: Subspace Topology

Definition 5 Let $\langle X, \mathcal{T}\rangle$ be a topological space and Y be any subset of X (containing at least two points). Then the family

$$\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\}$$

forms a topology on Y called the *subspace topology.*

Lemma 12 *If $\mathcal{B}$ is a basis for a topological space $\langle X, \mathcal{T}\rangle$ and $Y \subset X$, then the family*

$$\mathcal{B}_Y = \{Y \cap B : B \in \mathcal{B}\}$$

is a basis for $\langle Y, \mathcal{T}_Y\rangle$.

Go over Lemma 16.2 and Example 1.

Discuss briefly Theorem 16.4.

Theorem 13 *(Theorem 16.3) Let $\langle A, \mathcal{T}_A\rangle$ be a subspace of $\langle X, \mathcal{T}_1\rangle$ and $\langle B, \mathcal{T}_B\rangle$ be a subspace of $\langle Y, \mathcal{T}_2\rangle$. Then the following two topologies on $A \times B$ coincide:*

- *$\mathcal{T}_{A\times B}$, the subspace topology of the product topology on $X \times Y$;*
- *$\mathcal{T}(\mathcal{B}_{\mathcal{T}_A,\mathcal{T}_B})$, the product topology for the spaces $\langle A, \mathcal{T}_A\rangle$ and $\langle B, \mathcal{T}_B\rangle$.*

Go over Examples 2 & 3 and discuss Theorem 13.
Go over Exercise 1.

Written assignment for Tuesday, September 16: Exercises 8 and 9, page 92.

Class of September 11:

Recall that:

- If $\langle X, \mathcal{T}\rangle$ is a topological space and $Y \subset X$, then $\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\}$ is the *subspace topology* on Y.
- If $\mathcal{B}$ is a basis for a topological space $\langle X, \mathcal{T}\rangle$ and $Y \subset X$, then the family $\mathcal{B}_Y = \{Y \cap B : B \in \mathcal{B}\}$ is a basis for $\langle Y, \mathcal{T}_Y\rangle$.

Go over Exercises 4 and:

Ex. 10. p. 92: *Let* $I = [0, 1]$. *Compare the following topologies on* I^2*: the standard product topology* τ_{st}*, the dictionary order topology* $\tau_{\preceq}$*, and the subspace topology* $\tau^*_{\preceq}$ *of* $\mathcal{T}_{\preceq}$*.*

Proof. In the proof, we will use the following two facts, mentioned many times in class. (For notation, see lecture for Section 13.)

(i) If $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{P}(X)$, then $\mathcal{T}(\mathcal{B}_0) \subset \mathcal{T}(\mathcal{B}_1)$.

(ii) If $\mathcal{T}_0$ is a topology (on X), then $\mathcal{T}(\mathcal{T}_0) = \mathcal{T}_0$.

Property (i) holds, as $\mathcal{T}(\mathcal{B}_0) = \{\bigcup \mathcal{B} : \mathcal{B} \subset \mathcal{B}_0\} \subset \{\bigcup \mathcal{B} : \mathcal{B} \subset \mathcal{B}_1\} = \mathcal{T}(\mathcal{B}_1)$. Property (ii) holds, since the family $\mathcal{B}_0 = \mathcal{T}_0$ is a basis for $\mathcal{T}_0$, and so $\mathcal{T}(\mathcal{T}_0) = \mathcal{T}(\mathcal{B}_0) = \mathcal{T}_0$.

We will prove, that the only inclusions between the topologies are $\tau_{st} \subset \tau^*_{\preceq}$ and $\tau_{\preceq} \subset \tau^*_{\preceq}$.

$\tau_{st} \subset \tau^*_{\preceq}$: By Theorem 15.1 the family $\mathcal{B}_{st} = \{(a, b) \times (c, d) : a, b, c, d \in \mathbb{R}\}$ is a basis for $\mathbb{R}^2$ with the standard topology $\mathcal{T}_{st}$. Hence, by Lemma 16.1, the family $\mathcal{D}_{st} = \{B \cap I^2 : B \in \mathcal{B}_{st}\}$ forms a basis for τ_{st}.

Next, notice that $\mathcal{D}_{st} \subset \tau^*_{\preceq}$. Indeed, if $[(a, b) \times (c, d)] \cap I^2 \in \mathcal{D}_{st}$ and $x \in (a, b)$, then $\{x\} \times (c, d) = (\langle x, c\rangle, \langle x, d\rangle)$ is a basic open set for the dictionary order topology on $\mathbb{R}^2$ so $[(a, b) \times (c, d)] \cap I^2 = \bigcup_{x\in(a,b)}[\{x\} \times (c, d)] \cap I^2 \in \tau^*_{\preceq}$.

Hence, by (i) and (ii), $\tau_{st} = \mathcal{T}(\mathcal{D}_{st}) \subset \mathcal{T}(\tau^*_{\preceq}) = \tau^*_{\preceq}$.

$\tau_{\preceq} \subset \tau^*_{\preceq}$: Notice that $\mathcal{D}_{\preceq} = \{(\langle a, b\rangle, \langle c, d\rangle) \cap I^2 : a, b, c, d \in I\}$ is a basis for $\tau_{\preceq}$ (straight from the definition of order topology) while, by Lemma 16.1, $\mathcal{D}^*_{\preceq} = \{(\langle a, b\rangle, \langle c, d\rangle) \cap I^2 : a, b, c, d \in \mathbb{R}\}$ is a basis for $\tau^*_{\preceq}$. Clearly, $\mathcal{D}_{\preceq} \subset \mathcal{D}^*_{\preceq}$. Therefore, by (i), $\tau_{\preceq} = \mathcal{T}(\mathcal{D}_{\preceq}) \subset \mathcal{T}(\mathcal{D}^*_{\preceq}) = \tau^*_{\preceq}$, as desired.

To finish the argument, we need to show that the topologies τ_{st} and $\tau_{\preceq}$ and not comparable. Indeed, $\tau_{st} \not\subset \tau_{\preceq}$ since a set $[0,1)^2 = (-1,1)^2 \cap I^2 \in \tau_{st}$ but it does not belong to $\tau_{\preceq}$ since there is no $J \in \mathcal{D}_{\preceq}$ with $\langle .5, 0\rangle \in J \subset [0,1)^2$ (as any $J \in \mathcal{D}_{\preceq}$ containing $\langle .5, 0\rangle$ must contain also $\langle x, 1\rangle$ for some $x \in (0, .5)$).

Similarly, $\tau_{\preceq} \not\subset \tau_{st}$, as $\{0\} \times (0,1) = (\langle 0,0\rangle, \langle 0,1\rangle) \in \tau_{\preceq}$ does not belong to τ_{st}. ■

Class of September 16:

Section 17: Closed sets; Closure and Interior of a Set

Definition 6 A set $A \subset X$ is *closed* in the topological space $\langle X, \mathcal{T} \rangle$ if its complement $X \setminus A$ is open.

Go over Examples 1-5.
Go over Theorem 17.1.
Go over Exercise 1.

Theorem 14 *(Theorem 17.2) Let Y be a subspace of X. Then, $A \subset Y$ is closed in Y iff $A = Y \cap F$ for some closed subset F of X.*

Go over Theorem 17.3.
Go over Exercises 2, 3, and 4.

Definition 7 Let $A \subset X$ be a subset of a topological space $\langle X, \mathcal{T} \rangle$.

- The *interior* of A, denoted as $\mathrm{int}(A)$, is defined as a union of all open subsets contained in A, that is, $\mathrm{int}(A) = \bigcup\{U \in \mathcal{T} \colon U \subset A\}$.

  Notice that $\mathrm{int}(A)$ is open and that it is the largest open subset of A.

- The *closure* of A, denoted either as $\mathrm{cl}(A)$ or as $\bar{A}$, is defined as an intersection of all closed subsets containing in A, that is,
  $\mathrm{cl}(A) = \bigcap\{F \supset A \colon F \text{ is closed in } X\}$.

  Notice that $\mathrm{cl}(A)$ is closed and that it is the smallest closed set containing A.

We will sometimes use symbols $\mathrm{int}_X(A)$ and $\mathrm{cl}_X(A)$ in place of $\mathrm{int}(A)$ and $\mathrm{cl}(A)$ to stress that the operation is with respect to the given topology on X.

Go over Exercise 6(a) and (b).

Class of September 18:

Recall, from the last class:

- A set $A \subset X$ is *closed* in the topological space $\langle X, \mathcal{T} \rangle$ if its complement $X \setminus A$ is open.
- (Theorem 17.2) Let Y be a subspace of X. Then, $A \subset Y$ is closed in Y iff $A = Y \cap F$ for some closed subset F of X.
- The *interior* of A is $\operatorname{int}(A) = \bigcup\{U \in \mathcal{T} \colon U \subset A\}$.
- The *closure* of A is $\operatorname{cl}(A) = \bigcap\{F \supset A \colon F \text{ is closed in } X\}$.

New material:

Theorem 15 *(Theorem 17.4) Let Y be a subspace of X and $A \subset Y$. Then* $\operatorname{cl}_Y(A) = Y \cap \operatorname{cl}_X(A)$.

Theorem 16 *(Theorem 17.5) Let $A \subset X$ be a subset of a topological space $\langle X, \mathcal{T} \rangle$ and $\mathcal{B}$ be a basis for X. Then*

$$x \in \operatorname{cl}(A) \textit{ if, and only if, } A \cap B \neq \emptyset \textit{ for every } B \in \mathcal{B} \textit{ with } x \in B.$$

In particular, the result is true with $\mathcal{B} = \mathcal{T}$.

Go over Exercises 8 and 9; also Examples 6 and 7.

Let $A = K \cup (2,3)$, where $K = \{1/n \colon n \in \{1,2,3,\ldots\}\}$. Find the closures of A in: $\mathbb{R}$ (i.e., $\mathbb{R}$ with the standard topology), $\mathbb{R}_\ell$, $\mathbb{R}_d$ (i.e., $\mathbb{R}$ with the discrete topology), and $\mathbb{R}_K$.

Answer: $\operatorname{cl}_{\mathbb{R}}(A) = \{0\} \cup K \cup [2,3]$; $\operatorname{cl}_{\mathbb{R}_\ell}(A) = \{0\} \cup K \cup [2,3)$; $\operatorname{cl}_{\mathbb{R}_d}(A) = A$; $\operatorname{cl}_{\mathbb{R}_K}(A) = K \cup [2,3]$;

Section 17: Limit Points

Definition 8 Let $A \subset X$ be a subset of a topological space $\langle X, \mathcal{T} \rangle$. A point $x \in X$ is a *limit point* (or *accumulation point*) of A provided $x \in \operatorname{cl}(A \setminus \{x\})$. The set of all limit points of A is denoted as A'.

Go over Example 8. State the following theorems:

Theorem 17 *(Theorem 17.6) Let A be a subset of a topological space $\langle X, \mathcal{T} \rangle$. Then $\operatorname{cl}(A) = A \cup A'$.*

Theorem 18 *(Theorem 17.7) Let A be a subset of a topological space $\langle X, \mathcal{T} \rangle$. Then A is closed in X if, and only if, $A' \subset A$.*

Written assignment due Tuesday, Sept. 23: Exercise 6(c), page 101.

Class of September 23, 2014:

Recall, from the last class:

- If Y be is subspace of X and $A \subset Y$, then $\mathrm{cl}_Y(A) = Y \cap \mathrm{cl}_X(A)$.
- If $A \subset X$ and $\mathcal{B}$ is a basis for X, then

  $x \in \mathrm{cl}(A)$ if, and only if, $A \cap B \neq \emptyset$ for every $B \in \mathcal{B}$ with $x \in B$.
- $A' = \{x \in X\colon x \in \mathrm{cl}(A \setminus \{x\})\}$.

New material: Prove the theorems:

(Theorem 17.6) Let A be a subset of $\langle X, \mathcal{T}\rangle$. Then $\mathrm{cl}(A) = A \cup A'$.

(Theorem 17.7) Let A be a subset of a topological space $\langle X, \mathcal{T}\rangle$. Then A is closed in X if, and only if, $A' \subset A$.

Section 17: Hausdorff spaces

Definition 9 Let $\langle X, \mathcal{T}\rangle$ be a topological space. We say that:

- X is *Hausdorff* (or a T_2 *space*) provided for every distinct $x, y \in X$ there exists disjoint open sets $U, V \subset X$ such that $x \in U$ and $y \in V$.
- X is a T_1 *space* provided for every distinct $x, y \in X$ there exists an open set $U \subset X$ such that $x \in U$ and $y \notin U$.
- X is a T_0 *space* provided for every distinct $x, y \in X$ there exists an open set $U \subset X$ such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$ (i.e., such that U contains precisely one of the points x and y).

Notice that if X is T_2 then it is also T_1, and if X is T_1 then it is also T_0. Examples:

- A space X with a trivial topology $\mathcal{T} = \{\emptyset, X\}$ is not T_0.
- $X = \{0, 1\}$ with a topology $\mathcal{T} = \{\emptyset, \{0\}, X\}$ is T_0 but not T_1.
- $X = \mathbb{R}$ with a cofinite topology $\mathcal{T} = \{\emptyset\} \cup \{X \setminus F\colon F \text{ is finite}\}$ is T_1 but not T_2.

- The following spaces are T_2: any space with the discrete topology, $\mathbb{R}$ with the standard topology, $\mathbb{R}_\ell$, $\mathbb{R}_K$.

Theorem 19 *(Exercise 15) A space X is T_1 if, and only if, every finite subset of X is closed.*

Corollary 20 *(Theorem 17.8) Every finite subset in a Hausdorff space is closed.*

Theorem 21 *(Theorem 17.9) Let X be a T_1 topological space and $A \subset X$. Then $x \in A'$ if, and only if, $U \cap A$ is infinite for every open U containing x.*

Definition 10 Let X be a topological. We say that $x \in X$ is an *isolated point* provided $\{x\}$ is open in X.

Remark 22 If X is T_1 and an open set U is finite, then every $x \in U$ is isolated.

Definition 11 Let X be a topological. We say that a sequence $\langle x_n \rangle_{n=1}^{\infty}$ of points of X *converges* to an $x \in X$ provided for every open set U containing x there exists an N such that $x_n \in U$ for every $n \geq N$.

If this is the case, we say also, that x is a *limit* of a sequence $\langle x_n \rangle_{n=1}^{\infty}$.

Theorem 23 *(Theorem 17.10) If X is a Hausdorff topological space, then any sequence $\langle x_n \rangle_{n=1}^{\infty}$ of points of X* converges *to at most one point in X.*

Proof: next class.

Example: (Exercise 14) Theorem 21 is false for T_1 spaces. For example, if $X = \mathbb{R}$ is considered with the cofinite topology (which is T_1) and $x_n = 1/n$ for every n, then every real number is a limit of $\langle x_n \rangle_{n=1}^{\infty}$.

Theorem 24 *(Theorem 17.11) The product of two Hausdorff topological spaces is a Hausdorff space. A subspace of a Hausdorff topological space is a Hausdorff space.*

Proof: next class.

Class of September 25:

Recall that for a topological space $\langle X, \mathcal{T} \rangle$:

- X is *Hausdorff* (or a T_2 *space*) provided for every distinct $x, y \in X$ there exists disjoint open sets $U, V \subset X$ such that $x \in U$ and $y \in V$.
- X is a T_1 *space* provided for every distinct $x, y \in X$ there exists an open set $U \subset X$ such that $x \in U$ and $y \notin U$. Equivalently, X is T_1 if, and only if, every singleton is closed in X.
- If X is a T_1 space and $A \subset X$, then

  $x \in A'$ if, and only if, $U \cap A$ is infinite for every open U containing x.

New material

Theorem 23 *(Theorem 17.10) If X is a Hausdorff topological space, then any sequence $\langle x_n \rangle_{n=1}^{\infty}$ of points of X* converges *to at most one point in X.*

Theorem 24 *(Theorem 17.11) The product of two Hausdorff topological spaces is a Hausdorff space. A subspace of a Hausdorff topological space is a Hausdorff space.*

Go over Exercises 10, 11, and 12.

Written assignment due Tuesday, Sept. 30: Exercise 13, page 101.

Explain policy on the mid term test.

Section 18: Continuous functions

Definition 12 Let X and Y be the topological spaces. A function $f\colon X \to Y$ is *continuous* provided $f^{-1}(V)$ is open in X for every open subset V of Y.

Notice, that the definition agrees with (a) from Theorem 2.

Theorem 25 *Let X and Y be the topological spaces and $\mathcal{B}$ a basis for Y. Then $f\colon X \to Y$ is continuous if, and only if, $f^{-1}(B)$ is open in X for every $B \in \mathcal{B}$.*

Similarly, if $\mathcal{S}$ is a subbasis for Y, then $f\colon X \to Y$ is continuous if, and only if, $f^{-1}(S)$ is open in X for every $S \in \mathcal{S}$.

Example 3:

- $f\colon \mathbb{R} \to \mathbb{R}_\ell$, $f(x) = x$, is discontinuous, as $f^{-1}([0,1)) = [0,1)$ is not open in $\mathbb{R}$.
- $f\colon \mathbb{R}_\ell \to \mathbb{R}$ is continuous, as $f^{-1}(U) = U \in \mathcal{T}_{st} \subset \mathcal{T}_\ell$ for every $U \in \mathcal{T}_{st}$.

Go over Exercise 3(a).

Class of September 30:

Recall that:

- If X is a Hausdorff topological space, then any sequence $\langle x_n \rangle_{n=1}^{\infty}$ of points of X *converges* to at most one point in X.
- The product of two Hausdorff topological spaces is a Hausdorff space. A subspace of a Hausdorff topological space is a Hausdorff space.
- A function $f\colon X \to Y$ is *continuous* provided $f^{-1}(V)$ is open in X for every open subset V of Y.
- If $\mathcal{B}$ a basis for Y, then $f\colon X \to Y$ is continuous if, and only if, $f^{-1}(B)$ is open in X for every $B \in \mathcal{B}$.

New material

Go over Theorem 18.1. (Very important!)
Stress continuity at a point.
Go over Exercise 2.

Section 18: Homeomorphisms

Definition 13 Let X and Y be the topological spaces and let $f\colon X \to Y$ be a bijection (i.e., one-to-one and onto). Then f is a *homeomorphism* (from X onto Y) provided both f and $f^{-1}\colon Y \to X$ are continuous.

Topological spaces X and Y are *homeomorphic* provided there is a homeomorphism from X onto Y.

Fact. *If $f\colon X \to Y$ is homeomorphism, then $U \subset X$ is open in X, if, and only if, $f[U]$ is open in Y. In particular, if τ is a topology on X and $\mathcal{T}$ is a topology on Y, then $\mathcal{T} = \{f[U]\colon U \in \tau\}$ and $\tau = \{f^{-1}[V]\colon V \in \mathcal{T}\}$.*

Proof. Notice that $(f^{-1})^{-1} = f$.

If $U \in \tau$, then, since $f^{-1}\colon Y \to X$ is continuous, $f[U] = (f^{-1})^{-1}(U) \in \mathcal{T}$.
If $f[U] \in \tau$, then, since $f\colon X \to Y$ is continuous, $U = f^{-1}(f[U]) \in \mathcal{T}$. ∎

Go over Examples 4-6.

Class of October 2:

Recall that:

- Spaces X and Y are *homeomorphic* provided there exists a *homeomorphism* $f\colon X \to Y$, that is, a bijection such that both f and $f^{-1}\colon Y \to X$ are continuous.
- **Fact.** *If $f\colon \langle X, \tau\rangle \to \langle Y, \mathcal{T}\rangle$ is a homeomorphism, then $U \in \tau$, if, and only if, $f[U] \in \mathcal{T}$. In particular, $\mathcal{T} = \{f[U]\colon U \in \tau\}$ and $\tau = \{f^{-1}[V]\colon V \in \mathcal{T}\}$.*

PROOF OF **Fact.** Notice that $(f^{-1})^{-1} = f$.

If $U \in \tau$, then, since $f^{-1}\colon Y \to X$ is continuous, $f[U] = (f^{-1})^{-1}(U) \in \mathcal{T}$.

If $f[U] \in \tau$, then, since $f\colon X \to Y$ is continuous, $U = f^{-1}(f[U]) \in \mathcal{T}$. ■

Go over Exercises 5 and 6.

A mapping $f\colon X \to Y$ is an *imbedding* provided f is injective (i.e., one-to-one), continuous, and $f^{-1}\colon f[X] \to X$ is also continuous. In such a case a mapping $f'\colon X \to f[X]$, $f'(x) = f(x)$, is a homeomorphism (from X onto $f[X]$.

Go over Exercise 4.

Section 18: Constructing Continuous Functions

Go over Theorem 18.2.

Go over Theorem 18.3 (The pasting Lemma).

Go over Example 8.

Class of October 7:

Solutions for the remaining homework will be given next class, October 9. No further solutions will be accepted.

In class mid term test will be on Thursday, October 16

We will start test 15 minutes earlier, that is, 6:45 pm.

Go over Theorem 18.4.

Go over Exercise 11.

Variant of Exercise 12, with $f(x,y) = \frac{xy^2}{x^2+y^4}$ for $\langle x,y\rangle \neq \langle 0,0\rangle$ and $f(0,0)=0$. Show that it is discontinuous (on curve $y^2 = x$), but $f \restriction L$ is continuous for every straight line L.

Section 19: The product topology

Definition 14 For sets J and X let X^J denotes the family of all functions $f\colon J \to X$.

Let $\{A_\alpha\}_{\alpha\in J}$ be an arbitrary indexed family of sets and let $X = \bigcup_{\alpha\in J} A_\alpha$. (Notice that the index set J may be uncountable!) The *cartesian product* of the family $\{A_\alpha\}_{\alpha\in J}$, denoted by $\prod_{\alpha\in J} A_\alpha$, is defined as

$$\prod_{\alpha\in J} A_\alpha = \{f \in X^J\colon f(j) \in A_j \text{ for all } j \in J\}.$$

Elements of $\{A_\alpha\}_{\alpha\in J}$ will be also sometimes denotes as $\langle a_\alpha\rangle_{\alpha\in J}$ and referred to as *J-tuples*.

Notice that $X^J = \prod_{\alpha\in J} A_\alpha$, where $A_\alpha = X$ for every $\alpha \in J$.

Notice, that this definition agrees the definition of the finite cartesian product (over the set $J = \{1,\ldots,n\}$) $\prod_{i=1}^n A_i = A_1 \times\cdots\times A_n$ as the set of all sequences $\langle a(1),\ldots,a(n)\rangle$ with $a(i) \in A_i$, since any such sequence can be considerd as a function $a\colon\{1,\ldots,n\} \to X$. Similar agreement is also for $J = \{1,2,3,\ldots\}$.

Definition 15 Let $\{X_\alpha\}_{\alpha\in J}$ be an indexed family of topological spaces. Then, on the product space $X = \prod_{\alpha\in J} X_\alpha$, we define the following two kinds of topologies.

box topology $\mathcal{T}_{box}$: Generated by a basis $\mathcal{B}_{box}$ formed by all sets of the form

$$\prod_{\alpha\in J} U_\alpha \quad \text{where each } U_\alpha \text{ is open in } X_\alpha.$$

product topology $\mathcal{T}_{prod}$: Generated by a subbasis $\mathcal{S}$ formed by all sets of the form

$$\pi_\beta^{-1}(U_\beta) \text{ for all } \beta \in J \text{ and open subsets } U_\beta \text{ of } X_\beta,$$

where $\pi_\beta\colon X \to X_\beta$ is the *projection* onto βth coordinate, that is, defined as $\pi_\beta(x) = x(\beta)$.

Notice that $\pi_\beta^{-1}(U_\beta) = \prod_{\alpha\in J} U_\alpha$, where $U_\alpha = X_\alpha$ for all $\alpha \neq \beta$.

A natural basis, $\mathcal{B}_{prod}$ associated with $\mathcal{S}$ is formed by finite intersections of sets from $\mathcal{S}$, that is, all sets of the form $\prod_{\alpha\in J} U_\alpha$ where each U_α is open in X_α and *the set* $\{\alpha \in J\colon U_\alpha \neq X_\alpha\}$ *is finite.*

Go over Theorem 19.6 and Example 2.

Class of October 9:

Recall that for $X = \prod_{\alpha \in J} X_\alpha$, each X_α being a topological space,

- *Box topology* $\mathcal{T}_{box}$ on X is generated by basis
  $\mathcal{B}_{box} = \{\prod_{\alpha \in J} U_\alpha\colon$ each U_α is open in $X_\alpha\}$.
- *Product topology* $\mathcal{T}_{prod}$ on X is generated by subbasis
  $\mathcal{S}_{box} = \{\pi_\beta^{-1}(U_\beta)$ for all $\beta \in J$ and open subsets U_β of $X_\beta\}$
  or, equivalently, by a basis
  $\mathcal{B}_{box} = \{\prod_{\alpha \in J} U_\alpha \in \mathcal{B}_{box}\colon U_\alpha = X_\alpha$ for all but finitely many $\alpha\}$.
- $\mathcal{T}_{prod} \subset \mathcal{T}_{box}$; equation holds when J is finite (or all but finitely many spaces X_α have trivial topology $\{\emptyset, X_\alpha\}$);
- *If $f_\alpha\colon A \to X_\alpha$ and $f\colon A \to X$ is given by $f(a)(\alpha) = f_\alpha(a)$, then*
  continuity of f implies the continuity of each f_α;
  continuity of all f_α's implies the continuity of $f\colon A \to \langle X, \mathcal{T}_{prod}\rangle$;
  continuity of all f_α's does not imply continuity of $f\colon A \to \langle X, \mathcal{T}_{box}\rangle$
  (as $f\colon \mathbb{R} \to \langle \mathbb{R}^\omega, \mathcal{T}_{box}\rangle$, $f(x) = \langle x, x, x, \ldots\rangle$ is discontinuous).

New material

Go over Theorem 19.2: bases for $\mathcal{T}_{box}$ and $\mathcal{T}_{prod}$ in term of basis for X_α's.

Go over Theorem 19.3: subspace topology on $A = \prod_{\alpha \in J} A_\alpha \subset X$.

Theorem 19.4: product of Hausdorff spaces is Hausdorff ($\mathcal{T}_{box}$ and $\mathcal{T}_{prod}$).

State Theorem 19.5: $\prod_{\alpha \in J} \operatorname{cl}(A_\alpha) = \operatorname{cl}\left(\prod_{\alpha \in J} A_\alpha\right)$ (in $\mathcal{T}_{box}$ and $\mathcal{T}_{prod}$).

Solve Exercise 7.

No class on Tuesday October 11: fall break.

Class of October 16:

Mid Term Test

Class of October 21:

Discuss the Mid Term Test.

Go over Exercises 13, page 112.
Go over Exercise 8, pages 111 and 112.

Class of October 23:

Prove Theorem 19.5: $\prod_{\alpha\in J} \mathrm{cl}(A_\alpha) = \mathrm{cl}\left(\prod_{\alpha\in J} A_\alpha\right)$ (in $\mathcal{T}_{box}$ and $\mathcal{T}_{prod}$).

Go over Exercises 10, page 112.

Written assignment for Tuesday, October 28: Exercise 8, page 118.

Section 20: The Metric Topology

Define a *metric* (*distance*) on X as a function $d\colon X\times X \to [0,\infty)$.

A *metric space* is a pair $\langle X, d\rangle$, where d is a metric on X.

In a metric space $\langle X, d\rangle$, define an *open ball* (centered at $x\in X$ with radius $\varepsilon > 0$) as $B_d(x,\varepsilon) = \{y \in X\colon d(x,y) < \varepsilon\}$.

Prove that a family $\mathcal{B}_d = \{B(x,\varepsilon)\colon x\in X \ \&\ \varepsilon > 0\}$ is a basis for a topology on X.

Define a metric topology for a metric space $\langle X, d\rangle$ as $\mathcal{T}(\mathcal{B}_d)$, that is, as a topology generated by the family of all open balls in $\langle X, d\rangle$.

Go over Example 1 (discrete metric) and 2 (standard metric on $\mathbb{R}$).

Go over Exercise 3(a): $d\colon X\times X\to\mathbb{R}$ is continuous in X^2, where X is considered with the metric topology.

PROOF. Let $B=(a,b)$ be basic open set in $\mathbb{R}$. Need to prove that $d^{-1}(B)$ is open in X^2.

Fix $\langle x,y\rangle \in d^{-1}(B)$. So, $d(x,y)\in B$. We need to find an open set U in X^2 with $\langle x,y\rangle\in U\subset d^{-1}(B)$. Let $\varepsilon>0$ be such that $(d(x,y)-\varepsilon, d(x,y)+\varepsilon)\subset B$. Define $U = B(x,\varepsilon/2)\times B(y,\varepsilon/2)$. It is open in X^2 and contains $\langle x,y\rangle$.

So, fix $\langle z,t\rangle\in U$. Then $d(x,z)<\varepsilon/2$ and $d(y,t)<\varepsilon/2$. By the triangle inequality we get $d(z,x)+d(x,y)+d(y,t)\geq d(z,t)$, so

$$d(z,x)+d(y,t)\geq d(z,t)-d(x,y).$$

Similarly, $d(x,z)+d(z,t)+d(t,y)\geq d(x,y)$, so

$$d(x,z)+d(t,y)\geq d(x,y)-d(z,t).$$

Hence, $|d(z,t)-d(x,y)|\leq d(x,z)+d(t,y) < \varepsilon/2+\varepsilon/2$ and so we have $d(z,t)\in(d(x,y)-\varepsilon, d(x,y)+\varepsilon)\subset B$, as required.

Definition 16 A topological space $\langle X,\tau\rangle$ is *metrizable* provided there exists a metric d on X such that $\tau=\mathcal{T}(\mathcal{B}_d)$.

Class of October 28:

Recall that:

- A *metric space* is a pair $\langle X, d\rangle$, where d is a metric on X.
- $B_d(x,\varepsilon) = \{y \in X \colon d(x,y) < \varepsilon\}$ is an *open ball* in $\langle X, d\rangle$.
- $\mathcal{B}_d = \{B(x,\varepsilon)\colon x \in X \ \&\ \varepsilon > 0\}$ is a basis for a topology on X.
- $\mathcal{T}(\mathcal{B}_d)$ is the metric topology on X (for metric d).

New material

Define: bounded set and its diameter.

Go over Theorem 20.1. (So, boundedness is not a topological property!)

Define Euclidean metric and square metric on $\mathbb{R}^n$.

Go over Theorem 20.3, using Lemma 20.2.

Define uniform metric on $\mathbb{R}^J$.

State and prove Theorem 20.4.

Class of October 30:

Recall

- uniform metric on $\mathbb{R}^J$ is defined as: $\bar{\rho}(x, y) = \sup\{\bar{d}(x_\alpha, y_\alpha) \colon \alpha \in J\}$, where $\bar{d}(x, y) = \min\{|x - y|, 1\}$
- uniform topology on $\mathbb{R}^J$: generated by $\bar{\rho}$.

New material

Recall Theorem 20.4 (relations between box, uniform, and product topologies on $\mathbb{R}^J$).

Go over Exercise 6.

Go over Exercise 5, page 127. Note, that this implies that, on $\mathbb{R}^\omega$, box, uniform, and product topologies are distinct.

Suggested to solve at home (no homework): Exercise 4 page 127.

Stated and briefly discussed Theorem 20.5 (countable product of metric spaces is metrizable).

Class of November 6:

Go over Theorem 20.5 (countable product of metric spaces is metrizable).

Section 21: The Metric Topology continued

- Subspace of a metric space is metric.
- No relation between ordered topologies and metric topologies.
- Every metrizable space is Hausdorff.
- Finite and countable product of metric spaces is metrizable.

State Theorem 21.1: for metric spaces, ε-δ definition of continuity is equivalent to topological definition of continuity. (This is an obvious generalization of Theorem 2.)

Definition 17 Let $\langle X, \tau\rangle$ be a topological space.

- A family $\mathcal{B}_x \subset \tau$ is a *basis (for X) at x* provided for every open set $U \ni x$ there is a $B \in \mathcal{B}_x$ with $x \in B \subset U$.
- A topological space X is *first countable* (or *satisfies the first countability axiom*) provided for every $x \in X$ there exists a countable basis $\mathcal{B}_x$ of X at x.

Proposition 26 *Every metrizable space is first countable.*

Note that for first countable spaces, a countable basis $\{B_n : n = 1, 2, 3, \ldots\}$ can be chosen monotone: $B_1 \supset B_2 \supset B_3 \supset \cdots$.

Go over Lemma 21.2, version for first countable spaces:

Lemma 27 *Let X be a first countable topological space and let $A \subset X$. Then $x \in \operatorname{cl}(A)$ if, and only if, there is a sequence of points of A converging to x. Moreover, the implication "$\Longleftarrow$" does not require the assumption of first countability.*

Stated Theorem 21.3, version for first countable spaces.

Class of November 11:

Recall that:

- A topological space $\langle X, \mathcal{T} \rangle$ is *first countable* (or *satisfies the first countability axiom*) provided for every $x \in X$ there exists a countable basis $\mathcal{B}_x$ of X at x, that is, $\mathcal{B}_x \subset \mathcal{T}$ and for every open set $U \ni x$ there is a $B \in \mathcal{B}_x$ with $x \in B \subset U$.
- **Lemma** *Let X be a first countable topological space and let $A \subset X$. Then $x \in \mathrm{cl}(A)$ if, and only if, there is a sequence of points of A converging to x. Moreover, the implication "$\Longleftarrow$" does not require the assumption of first countability.*

New material

Go over Theorem 21.3, version for first countable spaces:

Theorem 28 *Let X and Y topological spaces and let $f\colon X \to Y$. Assume also that X is first countable. Then f is continuous if, and only if, for every sequence $\langle x_n \rangle_n$ in X converging to an $x \in X$, $\langle f(x_n) \rangle_n$ converges to $f(x)$.*

Moreover, the implication "$\Longrightarrow$" does not require the assumption of first countability.

Go over Lemma 21.4 (no proof).

Go over Theorem 21.5.

Definition 18 Let $\langle Y, d \rangle$ be a metric space, X any set, and $f_n\colon X \to Y$ be a sequence of functions. We say that the sequence $\langle f_n \rangle_n$ *converges uniformly* to an $f\colon X \to Y$ provided for every $\varepsilon > 0$ there exists an N (independent of x) such that for every $x \in X$

$$d(f_n(x), f(x)) < \varepsilon \text{ for all } n > N.$$

State Theorem 21.6: uniform limit of continuous functions is continuous.

Go over Exercise 6: *uniform* convergence assumption in Theorem 21.6 is essential.

Prove Theorem 21.6.

Discuss Exercise 9: the implication in Theorem 21.6 cannot be reversed.

Go (briefly) over Example 1: $\mathbb{R}^\omega$ with the box topology is not first countable. In particular, it is not metrizable.

Go (briefly) over Example 2: uncountable product $\mathbb{R}^J$, considered with the product topology, is not first countable. In particular, it is not metrizable.

Written assignment for Tuesday, November 15: Exercise 7, p. 134.

Skip the rest of Chapter 2, that is, section 22.

Class of November 13:

Written assignment for Tuesday, November 15: Exercise 7, p. 134.

Chapter 3: Connectedness and Compactness

Stress usability of these notions to the proofs of three classical calculus theorems: *Intermediate Value Theorem*, *Maximum Value Theorem*, and *Uniform Continuity Theorem.*

Intermediate Value Theorem is a consequence of *connectedness* property.

The other two theorems are the consequences of *compactess* property.

Section 23: Connected spaces

Definition 19 Let X be a topological space. A *separation* of X is any pair U, V, of open, non-empty disjoint sets with $X = U \cup V$. A topological space X is *connected* provided it **does not** exist a separation of X.

Example 1: Any X with indiscreet topology is connected.

Any X with discreet topology is *disconnected*, that is, not connected.

Fact: A space is connected, when $\emptyset$ and X are its only subsets that are simultaneously closed and open.

Definition 20 Let Y be a subspace of X. A *separation* of Y is any pair $A, B \subset Y$ non-empty sets such that $Y = A \cup B$ and $\mathrm{cl}(A) \cap B = A \cap \mathrm{cl}(B) = \emptyset$.

Lemma 29 *A subspace Y of X is connected is, and only if, there is no separation of Y.*

Go over Examples 2, 3, 4, and 5.

Lemma 30 *Assume that sets C and D forms separation of X. If a subspace Y of X is connected, then either $Y \subset C$ or $Y \subset D$.*

Theorem 31 (Star Lemma) *Let $\{A_\alpha\}_{\alpha \in J}$ be a family of connected subspaces of X. If $\bigcap_{\alpha \in J} A_\alpha \neq \emptyset$, then $\bigcup_{\alpha \in J} A_\alpha$ is connected.*

Theorem 32 (Theorem 23.4) *Let A be a connected subspace of X. If $A \subset B \subset \mathrm{cl}(A)$, then B is connected.*

Theorem 33 (Theorem 23.5) *Continuous image of connected space is connected.*

This, together with the fact that intervals are connected, is the Intermediate Value Theorem.

Theorem 34 (Theorem 23.6) *Finite product of connected spaces is connected.*

Actually, arbitrary product of connected spaces, considered with the product topology, is connected. We show this only for $\mathbb{R}^\omega$, Example 7. (In general, this is Exercise 10.)

Example 6: $\mathbb{R}^\omega$ with the box topology is disconnected.

Class of November 18:

Recall

- A topological space X is *connected* provided it **does not** exist a separation of X, where a *separation* of X is any pair U, V of open, non-empty disjoint sets with $X = U \cup V$.
- **(Star Lemma)** Let $\{A_\alpha\}_{\alpha \in J}$ be a family of connected subspaces of X. If $\bigcap_{\alpha \in J} A_\alpha \neq \emptyset$, then $\bigcup_{\alpha \in J} A_\alpha$ is connected.
- A closure of a connected space is connected.
- Continuous image of connected space is connected.
- Finite product of connected spaces is connected.
- $\mathbb{R}^\omega$ with the box topology is disconnected, while with the product topology is connected.

New material

Go over Exercises 2, 7, and 8, page 152.

Suggestion to students: Look over Exercises 3,4, and 9, page 152.

Section 24: Connected spaces of the Real Line

Recall that $\mathbb{R}$ has the *least upper bound property* provided every non-empty bounded above subset A of $\mathbb{R}$ has an upper bound $\sup(A) \in \mathbb{R}$.

Theorem 35 (Theorem 24.1, for $\mathbb{R}$ only) *A subset A of $\mathbb{R}$ (considered with the standard topology) is connected if, and only if, A is an interval (possible degenerated).*

Go over the Intermediate Value Theorem, Theorem 24.3.

Go over Exercise 2.

Class of November 20:

Recall

- Continuous image of connected space is connected.
- $A \subset \mathbb{R}$ is connected if, and only if, A is convex (an interval).
- Intermediate Value Theorem.

New material

Go over Exercise 1.

Define *path connectedness*.

Note that every path connected space is connected.

Go over Examples 3, 4, and 5.

Go over Examples 7, *topologists sine curve*: it is connected but not path connected.

Go over Exercises 2 (again), 3, and 8.

Class of December 2:

Go over Exercises 9, **10**, and 11 page 158.

Go over Section 25

Define components and path components. Go over thms 25.1 and 25.2.

Go over Examples 1 and 2.

Define locally connected spaces and locally path connected spaces.

Go over Example 3.

Briefly discuss Exercise 10: quasi components.

Go over Theorems 25.3, 25.4, and 25.5.

Class of December 4:

Review for Final Test.

Class of December 9:

Administration of Final Test.