

**Set Theory, Math 683, Fall 2015:  
Notes and homework assignments**

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**Class of August 18:**

Handed and discussed course syllabus.

Went over sections 1.1 and (part of) 1.2, up to **Empty Set Axiom**.

**Class of August 20:**

Restated some key points from the previous lecture.

Covered the rest of the section 1.2, leaving only (for the next class) a discussion of the **Infinity Axiom**.

Solved Ex 8 page 11.

**Written assignment for Tuesday, August 25:** Exercises 2 and 3, p. 11.

**Class of August 25:**

Go over three key points of the previous lecture: Introduced axioms allow definitions of all basic operations on sets, including the Cartesian product  $X \times Y$ .

Go over **Infinity Axiom**.

Go over section 2.1: **Relations and the axiom of choice**.

Covered up to the middle of page 14, that is, definition and examples of the equivalence relation.

**Class of August 27:**

Quick review of binary relations.

Finish section 2.1.

Go over section 2.2: Functions and the replacement scheme axiom.

Covered up to the replacement scheme axiom.

**Written assignment for September 1:** Exercise 3 page 15. (Solve only the problems from the first column.)

**Solutions of the assignment of August 20:** Exercises 2 and 3, p. 11.

*Ex. 2 page 11: Show that for every family  $\mathcal{F}$  and every set  $A$*

- (a) *if  $A \subset F$  for every  $F \in \mathcal{F}$ , then  $A \subset \bigcap \mathcal{F}$ , and*  
 (b) *if  $F \subset A$  for every  $F \in \mathcal{F}$ , then  $\bigcup \mathcal{F} \subset A$ .*

SOLUTION: Part (a): Actually, **the statement is false when  $\mathcal{F} = \emptyset$** , since, in this case, independently of  $A$ , we have  $\bigcup \mathcal{F} = \{x: \exists F \in \mathcal{F}(x \in F)\} = \emptyset$  (existence ensured by the Union Axiom) and so, by the definition of  $\bigcap \mathcal{F}$ ,

$$\bigcap \mathcal{F} = \left\{ z \in \bigcup \mathcal{F} : \forall F \in \mathcal{F} (z \in F) \right\} = \{z \in \emptyset : \mathcal{F} : \forall F \in \mathcal{F} (z \in F)\} = \emptyset.$$

Thus, for  $\mathcal{F} = \emptyset$  and any  $A \neq \emptyset$ , the property “ $A \subset F$  for every  $F \in \mathcal{F}$ ” holds, however  $A \subset \emptyset = \bigcap \mathcal{F}$  is false.

Nevertheless, *the statement (a) is true when  $\mathcal{F} \neq \emptyset$* . Indeed, in this case there exists an  $F_0 \in \mathcal{F}$  and (see e.g. Ex. 1)  $F_0 \subset \bigcup \mathcal{F}$ . Since, by (a),  $A \subset F_0$ , we also have  $A \subset \bigcup \mathcal{F}$ .

Now, to see that  $A \subset \bigcap \mathcal{F}$ , fix an  $a \in A$ . Then,  $a \in \bigcup \mathcal{F}$ , as  $A \subset \bigcup \mathcal{F}$ . Moreover, for every  $F \in \mathcal{F}$  we have  $a \in F$  (as  $a \in A \subset F$ ). Thus, by the definition of  $\bigcap \mathcal{F}$ ,  $a \in \bigcap \mathcal{F}$ , completing the proof.

Part (b): Take an arbitrary  $x \in \bigcup \mathcal{F}$ . We need to show that  $x \in A$ . Indeed,  $x \in \bigcup \mathcal{F}$  implies that there exists an  $F \in \mathcal{F}$  with  $x \in F$ . Since, by the assumption of (b),  $F \subset A$ , we have desired  $x \in A$ .

*Ex. 3 page 11: Prove that if  $\mathcal{F} \cap \mathcal{G} \neq \emptyset$  then  $\bigcap \mathcal{F} \cap \bigcap \mathcal{G} \subset \bigcap (\mathcal{F} \cap \mathcal{G})$ . Give examples showing that the inclusion cannot be replaced by equality and that the assumption  $\mathcal{F} \cap \mathcal{G} \neq \emptyset$  is essential.*

SOLUTION: Assume that  $H \in \mathcal{F} \cap \mathcal{G}$ . Then, by Ex. 1,  $H \subset \bigcup (\mathcal{F} \cap \mathcal{G})$  and  $\bigcap \mathcal{F} \subset H$ . Hence,  $\bigcap \mathcal{F} \subset \bigcap (\mathcal{F} \cap \mathcal{G})$ . Similarly,  $\bigcap \mathcal{G} \subset \bigcap (\mathcal{F} \cap \mathcal{G})$ . Combination of these two last inclusions gives  $\bigcap \mathcal{F} \cap \bigcap \mathcal{G} \subset \bigcap (\mathcal{F} \cap \mathcal{G})$ , as desired.

Example for  $\bigcap \mathcal{F} \cap \bigcap \mathcal{G} \subsetneq \bigcap (\mathcal{F} \cap \mathcal{G})$ : Let us take  $\mathcal{F} = \{\{0\}, \{1\}\}$  and  $\mathcal{G} = \{\{0\}, \{2\}\}$ . Then  $\bigcap \mathcal{F} = \{0\} \cap \{1\} = \emptyset$  and  $\bigcap \mathcal{G} = \{0\} \cap \{2\} = \emptyset$ , so that  $\bigcap \mathcal{F} \cap \bigcap \mathcal{G} = \emptyset$ . On the other hand  $\mathcal{F} \cap \mathcal{G} = \{\{0\}, \{1\}\} \cap \{\{0\}, \{2\}\} = \{\{0\}\}$  and  $\bigcap (\mathcal{F} \cap \mathcal{G}) = \bigcap \{\{0\}\} = \{0\}$ , so that  $\bigcap \mathcal{F} \cap \bigcap \mathcal{G} = \emptyset \subsetneq \{0\} = \bigcap (\mathcal{F} \cap \mathcal{G})$ .

Example for  $\mathcal{F} \cap \mathcal{G} = \emptyset$ : take families  $\mathcal{F}$  and  $\mathcal{G}$  as one element family each:  $\mathcal{F} = \{\{0, 1\}\}$  and  $\mathcal{G} = \{\{0, 2\}\}$ . Then  $\bigcap \mathcal{F} = \{0, 1\}$  and  $\bigcap \mathcal{G} = \{0, 2\}$ , so that  $\bigcap \mathcal{F} \cap \bigcap \mathcal{G} = \{0, 1\} \cap \{0, 2\} = \{0\}$  is non-empty, while  $\bigcap (\mathcal{F} \cap \mathcal{G}) = \bigcap \emptyset = \emptyset$  (as proved in Ex 2). Thus,  $\bigcap \mathcal{F} \cap \bigcap \mathcal{G} = \{0\} \not\subset \emptyset = \bigcap (\mathcal{F} \cap \mathcal{G})$ .

**Solution of the assignment of August 27:** Exercise 3 page 15, solve only the problems from the first column.

*Exercise 3 page 15, the first column: Prove the formulas*

$$(R \cup S) \circ T = (R \circ T) \cup (S \circ T), \quad (R \cap S) \circ T \subset (R \circ T) \cap (S \circ T).$$

*Find an example of relations  $R$ ,  $S$ , and  $T$  that show that the inclusion in the display cannot be replaced with the equation.*

**SOLUTION:** Assume that  $T \subset X \times Y$  and  $R, S \subset Y \times Z$ . Then

$$\begin{aligned} & (R \cup S) \circ T \\ &= \{ \langle x, z \rangle \in X \times Z : \exists y \in Y (\langle x, y \rangle \in T \ \& \ \langle y, z \rangle \in R \cup S) \} \\ &= \{ \langle x, z \rangle \in X \times Z : \exists y \in Y (\langle x, y \rangle \in T \ \& \ (\langle y, z \rangle \in R \vee \langle y, z \rangle \in S)) \} \\ &= \{ \langle x, z \rangle \in X \times Z : \exists y \in Y (\langle x, y \rangle \in T \ \& \ \langle y, z \rangle \in R) \vee (\langle x, y \rangle \in T \ \& \ \langle y, z \rangle \in S) \} \\ &= \{ \langle x, z \rangle \in X \times Z : \\ & \quad (\exists y \in Y (\langle x, y \rangle \in T \ \& \ \langle y, z \rangle \in R)) \vee (\exists y \in Y (\langle x, y \rangle \in T \ \& \ \langle y, z \rangle \in S)) \} \\ &= \{ \langle x, z \rangle \in X \times Z : \langle x, y \rangle \in R \circ T \vee \langle x, y \rangle \in S \circ T \} \\ &= \{ \langle x, z \rangle \in X \times Z : \langle x, y \rangle \in (R \circ T) \cup (S \circ T) \} \\ &= (R \circ T) \cup (S \circ T) \end{aligned}$$

and

$$\begin{aligned} & (R \cap S) \circ T \\ &= \{ \langle x, z \rangle \in X \times Z : \exists y \in Y (\langle x, y \rangle \in T \ \& \ \langle y, z \rangle \in R \cap S) \} \\ &= \{ \langle x, z \rangle \in X \times Z : \exists y \in Y (\langle x, y \rangle \in T \ \& \ (\langle y, z \rangle \in R \ \& \ \langle y, z \rangle \in S)) \} \\ &= \{ \langle x, z \rangle \in X \times Z : \exists y \in Y (\langle x, y \rangle \in T \ \& \ \langle y, z \rangle \in R) \ \& \ (\langle x, y \rangle \in T \ \& \ \langle y, z \rangle \in S) \} \\ &\subset \{ \langle x, z \rangle \in X \times Z : \\ & \quad (\exists y \in Y (\langle x, y \rangle \in T \ \& \ \langle y, z \rangle \in R)) \ \& \ (\exists y \in Y (\langle x, y \rangle \in T \ \& \ \langle y, z \rangle \in S)) \} \\ &= \{ \langle x, z \rangle \in X \times Z : \langle x, y \rangle \in R \circ T \ \& \ \langle x, y \rangle \in S \circ T \} \\ &= \{ \langle x, z \rangle \in X \times Z : \langle x, y \rangle \in (R \circ T) \cap (S \circ T) \} \\ &= (R \circ T) \cap (S \circ T). \end{aligned}$$

For an example, take  $X = Y = Z = \mathbb{R}$ ,  $R = \{ \langle 1, 2 \rangle \}$ ,  $S = \{ \langle 1, 3 \rangle \}$ , and  $T = \{ \langle 2, 1 \rangle, \langle 3, 1 \rangle \}$ . Then  $(R \cap S) \circ T = \emptyset \circ T = \emptyset$ , while  $R \circ T = S \circ T = \{ \langle 1, 1 \rangle \}$ , that is,  $(R \cap S) \circ T = \emptyset \subsetneq \{ \langle 1, 1 \rangle \} = R \circ T \cap (S \circ T)$ .

**Class of September 1:**

Quick review of functions.

Finish section 2.2, the replacement scheme axiom and beyond.

Go over sec. 2.3: *Generalized union, intersection, and Cartesian product*

Go over sec. 2.4: *Partial- and linear-order relations*

**Written assignment for September 8:** Exercises 5 and 7, page 21.

**Class of September 3:**

Go, relatively briefly, over Chapter 4: *Natural numbers, integers, and real numbers*

**Class of September 8:**

PROOF OF THEOREM 3.1.2(f) that  $m \notin m$  for every  $m \in \omega$ . The proof follows by induction. Let

$$K = \{m \in \omega : m \notin m\}.$$

Notice that  $0 = \emptyset \in K$ , since there is no  $x \in 0 = \emptyset$ . Now assume that  $m \in K$ . We need to show that  $m+1 \in m+1 = m \cup \{m\}$  leads to a contradiction with this assumption. Indeed,  $m+1 \in m$  implies that  $m \in m \cup \{m\} = m+1 \in m$ , while, by (d),  $m \in m+1 \in m$  implies  $m \in m$ , contradicting  $m \in K$ . Similarly,  $m+1 \in \{m\}$  implies that  $m \in m \cup \{m\} = m+1 = m$ , once again contradicting  $m \in K$ . So,  $K = \omega$ . ■

Go over Section 4.1: *Well-ordered sets and the axiom of foundation*.

Covered up to, including, Corollary 4.1.7.

**Solution of the assignment of September 1:**

*Exercise 5 page 21: If  $f \in Y^X$ ,  $\{F_t\}_{t \in T}$  is an indexed family of subsets of  $X$ ,  $\{G_t\}_{t \in T}$  is an indexed family of subsets of  $Y$ ,  $A, B \subset X$ , and  $C, D \subset Y$  then*

$$(a) f \left[ \bigcup_{t \in T} F_t \right] = \bigcup_{t \in T} f[F_t];$$

$$(b) f \left[ \bigcap_{t \in T} F_t \right] \subset \bigcap_{t \in T} f[F_t];$$

$$(c) f[A] \setminus f[B] \subset f[A \setminus B].$$

*Show, by giving examples, that the inclusions in parts (b) and (c) cannot be replaced by equality.*

SOLUTION. (a):

$$\begin{aligned} f \left[ \bigcup_{t \in T} F_t \right] &= \left\{ f(x) : x \in \bigcup_{t \in T} F_t \right\} \\ &= \{ f(x) : x \in \exists t \in T (x \in F_t) \} \\ &= \bigcup_{t \in T} \{ f(x) : x \in F_t \} = \bigcup_{t \in T} f[F_t]. \end{aligned}$$

(b):  $y \in f \left[ \bigcap_{t \in T} F_t \right]$  if, and only if,  $\exists x \in \bigcap_{t \in T} F_t (y = f(x))$  if, and only if,  $\exists x (y = f(x) \ \& \ \forall t \in T (x \in F_t))$ . This last condition implies that  $\forall t \in T \exists x \in (y = f(x) \ \& \ x \in F_t)$ , what is equivalent to  $\forall t \in T (y \in f[F_t])$ , that is, to the property that  $y \in \bigcap_{t \in T} f[F_t]$ . The inclusion is proved.

For an example, take  $T = \{0, 1\}$ ,  $f: \mathbb{R} \rightarrow \mathbb{R}$  given as  $f(x) = x^2$ ,  $F_0 = (-1, 0)$  and  $F_1 = (0, 1)$ . Then  $f \left[ \bigcap_{t \in T} F_t \right] = f[(-1, 0) \cap (0, 1)] = f[\emptyset] = \emptyset$ , while  $\bigcap_{t \in T} f[F_t] = f[(-1, 0)] \cap f[(0, 1)] = (0, 1) \cap (0, 1) = (0, 1)$ . In particular,  $f \left[ \bigcap_{t \in T} F_t \right] = \emptyset \subsetneq (0, 1) = \bigcap_{t \in T} f[F_t]$ .

(c):  $y \in f[A] \setminus f[B]$  if, and only if,  $\exists x \in A (y = f(x) \ \& \ y \notin f[B])$ . As  $f(x) = y \notin f[B]$  ensures that  $x \notin B$ ,  $\exists x \in A (y = f(x) \ \& \ y \notin f[B])$  implies that  $\exists x \in A \setminus B (y = f(x))$ , that is, that  $y \in f[A \setminus B]$ . The inclusion is proved.

For an example, let  $T$  and  $F$  be as in (b), put  $A = (-1, 0)$  and  $B = (0, 1)$ . Then  $f[A] \setminus f[B] = (0, 1) \setminus (0, 1) = \emptyset$ , while  $f[A \setminus B] = f[A] = f[A \setminus B]$ . In particular,  $f[A] \setminus f[B] = \emptyset \subsetneq (0, 1) = f[A \setminus B]$ .

*Exercise 7 page 21: For  $r, s \in \mathbb{R}$  let  $A_r = [r, r + 1]$  and  $B_{rs} = [r, s)$ . Calculate  $\bigcup_{s \leq 0} \bigcap_{r \geq s} A_r$ ,  $\bigcap_{s \leq 0} \bigcup_{r \geq s} A_r$ ,  $\bigcup_{r \leq 0} \bigcap_{s > r} B_{rs}$ , and  $\bigcap_{r \leq 0} \bigcup_{s > r} B_{rs}$ .*

SOLUTION. (1)  $\bigcap_{r \geq s} A_r = \bigcap_{r \geq s} [r, r + 1] \subset [s, s + 1] \cap [s + 3, s + 5] = \emptyset$ . So,

$$\bigcup_{s \leq 0} \bigcap_{r \geq s} A_r = \bigcup_{s \leq 0} \emptyset = \emptyset.$$

(2)  $\bigcup_{r \geq s} A_r = \bigcup_{r \geq s} [r, r + 1] = [s, \infty)$ . So,

$$\bigcap_{s \leq 0} \bigcup_{r \geq s} A_r = \bigcap_{s \leq 0} [s, \infty) = [0, \infty).$$

(3)  $\bigcap_{s > r} B_{rs} = \bigcap_{s > r} [r, s) = \{r\}$ . So,

$$\bigcup_{r \leq 0} \bigcap_{s > r} B_{rs} = \bigcup_{r \leq 0} \{r\} = (-\infty, 0].$$

(4)  $\bigcup_{s > r} B_{rs} = \bigcup_{s > r} [r, s) = [r, \infty)$ . So,

$$\bigcap_{r \leq 0} \bigcup_{s > r} B_{rs} = \bigcap_{r \leq 0} [r, \infty) = [0, \infty).$$

### Class of September 10:

Recall:

- Definitions of: *well-ordered set*, *order isomorphism*, *initial segment*, and of  $O(x_0)$ .
- Theorems 4.1.5 and 4.1.6.

Finish Section 4.1: *Well-ordered sets and the axiom of foundation*.

### Written assignment for September 15: Exercise 5, page 43.

### Class of September 15:

Section 4.2, up to (including) Theorem 4.2.4.

### Class of September 17:

NOTE: Part “ $\implies$ ” of the second part of the homework exercise 5, page 43, will be considered as *bonus exercise*.

Finish section 4.2 and Recursion Theorem from section 4.3.

**Solution of the assignment of September 10:**

*Exercise 5 page 43: Let  $\{F_t\}_{t \in T}$  be an indexed family of nonempty linearly ordered sets  $\langle F_t, \leq_t \rangle$  and let  $T$  be well ordered by  $\leq$ . If  $\preceq$  is a binary relation on the Cartesian product  $P = \prod_{t \in T} F_t$  defined by the formula*

$$(*) \quad f \preceq g \Leftrightarrow f = g \vee \exists s (s = \min\{t \in T: f(t) \neq g(t)\} \ \& \ f(s) \leq_s g(s))$$

for every  $f, g \in P$ , then  $\preceq$  is a linear order on  $P$ .

Moreover, if  $T$  is finite then  $P$  is well ordered by  $\preceq$  if and only if all sets  $F_t$  are well ordered.

SOLUTION OF THE MAIN PART, HANDED SEPTEMBER 17. We need to show that  $\preceq$  is reflexive, antisymmetric, transitive and that any two elements of  $P$  are  $\preceq$ -comparable.

Clearly,  $\preceq$  is reflexive, as, by  $(*)$ ,  $f = g$  implies  $f \preceq g$ .

To see that  $\preceq$  is antisymmetric assume that  $f \preceq g$  and  $g \preceq f$ . By  $(*)$   $f \preceq g$  implies that either  $f = g$ , or there exists an  $s$  with the property that  $s = \min\{t \in T: f(t) \neq g(t)\}$  and  $f(s) \leq_s g(s)$ . However, existence of such  $s$  contradicts  $g \preceq f$ .

Indeed, if there exists such an  $s$ , then  $f(s) \leq_s g(s)$  and  $f(s) \neq g(s)$ . In particular, since  $\leq_s$  is antisymmetric, we cannot have at the same time  $g(s) \leq_s f(s)$ . Thus, since the same  $s$  would be applicable for  $g \preceq f$ , this ensures that, in this case,  $g \preceq f$  is impossible. Thus,  $s$  cannot exist and we must have  $f = g$ , as desired.

To see that  $\preceq$  is transitive assume that  $f \preceq g$  and  $g \preceq h$ . We must show that this implies  $f \preceq h$ . If either  $f = g$  or  $g = h$ , then this conclusion is obvious. So, assume that  $f \neq g$  and  $g \neq h$ . Then there exist the numbers  $s_{fg} = \min\{t \in T: f(t) \neq g(t)\}$  and  $s_{gh} = \min\{t \in T: g(t) \neq h(t)\}$  for which we have  $f(s_{fg}) \leq_{s_{fg}} g(s_{fg})$  and  $f(s_{gh}) \leq_{s_{gh}} g(s_{gh})$ . Let  $s = \min\{s_{fg}, s_{gh}\}$  and notice that, by the definition of numbers  $s_{fg}$  and  $s_{gh}$ , we have  $f(s) \leq_s g(s)$  and  $g(s) \leq_s h(s)$ , with at least one of these inequalities being strict. Thus,  $f(s) <_s g(s)$ . Moreover, by  $s = \min\{s_{fg}, s_{gh}\}$  and the definitions of  $s_{fg}$  and  $s_{gh}$ , we can conclude that  $f(t) = g(t)$  and  $g(t) = h(t)$  for every  $t < s$ . In particular,  $s = \min\{t \in T: f(t) \neq h(t)\}$ . Combining this with  $f(s) <_s g(s)$ , we obtain, by  $(*)$ , the desired  $f \preceq h$ .

Clearly, any two elements of  $f, g \in P$  are  $\preceq$ -comparable, since  $f \neq g$  implies that the set  $\{t \in T: f(t) \neq g(t)\}$  is not empty and the existence of  $s$  is ensured by well ordering of  $T$ . Then, since  $\leq_s$  is linear order, either  $f(s) <_s g(s)$  or  $g(s) <_s f(s)$ , making  $f$  and  $g$   $\preceq$ -comparable.

**Class of September 22:**

Restate Recursion Theorem. Finish section 4.3. (Covered only up to Theorem 4.3.3.) Mention the content of section 4.4.

**Class of September 29:**

Restate Zermello's well-ordering theorem. Finish section 4.3 by proving Zorn's Lemma and Hausdorff Maximal Principle.

Prove Theorem 4.4.1, on existence of basis for every vector space.

Sketch the proof of Theorem 4.4.3, Tychonoff Theorem.

**Written assignment for October 6:**

**Students with an odd number on the class roster:** Exercise 1, p. 60.

**Students with an even number on the class roster:** Exercise 2, p. 60.

**Solution of the additional part of the assignment of September 10:**

*Exercise 5 page 43: Let  $\{F_t\}_{t \in T}$  be an indexed family of nonempty linearly ordered sets  $\langle F_t, \leq_t \rangle$  and let  $T$  be well ordered by  $\leq$ . If  $\preceq$  is a binary relation on the Cartesian product  $P = \prod_{t \in T} F_t$  defined by the formula*

$$(*) \quad f \preceq g \Leftrightarrow f = g \vee \exists s (s = \min\{t \in T: f(t) \neq g(t)\} \ \& \ f(s) \leq_s g(s))$$

for every  $f, g \in P$ , then  $\preceq$  is a linear order on  $P$ .

Moreover, if  $T$  is finite then  $P$  is well ordered by  $\preceq$  if and only if all sets  $F_t$  are well ordered.

**SOLUTION OF THE ADDITIONAL PART, HANDED SEPTEMBER 29.** Assume that  $T$  is finite. To see the implication “ $\implies$ ” assume that  $P$  is well ordered and fix an  $s \in T$ . We need to show that  $F_s$  is well ordered.

To see this, for every  $t \in T$  fix an  $m_t \in F_t$  (for example, the minimal element of  $F_t$ ). Let  $F = \{f \in P: f(t) = m_t \text{ for every } t \neq s\}$ . Then  $F$  is well ordered and the map  $\pi: F \rightarrow F_s$  defined by  $\pi(f) = f(s)$  is an order isomorphism. (To see that it is strictly increasing, see (\*).) This,  $F_t$  well ordered, as an isomorphic copy of a well ordered set  $F$ .

To see the other inclusion, let  $B \subset P$  be non-empty. We need to find its smallest element  $f = \min B$ . For this, let  $\{t_1, \dots, t_n\}$  be an increasing enumeration of  $T$ . We will define values  $f(t_i)$  by induction on  $i \in \{1, \dots, n\}$ . For this, we put  $B_0 = B$  and define  $f(t_1) = \min\{g(t_1): g \in B_0\}$ . Also, we define  $B_1 = \{g \in B_0: g(t_1) = f(t_1)\} \neq \emptyset$ . More generally, if for some  $i \in \{1, \dots, n-1\}$  the set  $B_i$  is defined, we put  $f(t_{i+1}) = \min\{g(t_{i+1}): g \in B_i\}$  and define  $B_{i+1} = \{g \in B_i: g(t_{i+1}) = f(t_{i+1})\} \neq \emptyset$ .

Thus, by induction, we constructed the function (sequence)  $f \in P$ . To see that  $f$  is in  $B$ , notice that  $B = B_0 \supset \dots \supset B_n = \{f\}$ . This can be proved by noticing that  $B_i = \{g \in B: g(t_k) = f(t_k) \text{ for all } k \in \{1, \dots, i\}\}$  for every  $i \in \{0, \dots, n\}$ . To see that  $f$  is minimal, fix an  $h \in B$ . We need to show that  $f \preceq h$ . This is obvious when  $h = f$ . So, assume that  $h \neq f$ . Then there exists an  $s = \min\{t \in T: f(t) \neq h(t)\}$ . Let  $i \in \{0, \dots, n-1\}$  be such that  $s = t_{i+1}$ . Then  $h(t_k) = f(t_k)$  for all  $k \in \{1, \dots, i\}$ , that is,  $h \in B_i$ . But then  $f(s) = f(t_{i+1}) = \min\{g(t_{i+1}): g \in B_i\} <_{t_{i+1}} h(t_{i+1}) = h(s)$ , that is, by (\*), we have the desired  $f \preceq h$ .

**Class of October 1:**

Start section 5.1, on cardinal numbers – covered up to, excluding, Cantor’s theorem.

**Class of October 6:**

Finish section 5.1.

**Class of October 8:**

Shortly discuss the handed solutions of homework.

Go over section 5.2, Cardinal arithmetic. (Covered only part.)

**Written assignment for October 15:**

**Regular Exercise:** Solve Exercise 4 page 73.

**Replacement Exercise:** Exercise 3, page 60.

*If your score for this exercise is larger than the one you received for Exercise 1 or 2, page 60, I will substitute it for the former score.*

**Information on the mid term test:** Planed for October 22, from 7pm to around 8:45pm. It will be administered in class. No written or electronic material can be used during the test. It will consists of:

- Statement of 2-3 theorems or definition covered so far.
- Sketch of a proof of one of the theorems proved in class.
- Solution of 2-3 exercises (or their parts) from:
  - Homework exercises assigned (or suggested) this semester;
  - The list of 3-5 exercises, I plan to give you on October 15.

**Solutions of the assignment of September 29:**

*Exercise 1 page 60: Prove that every partial order relation  $\preceq$  on a set  $X$  can be extended to a linear order relation  $\leq$  on  $X$ . (Here  $\leq$  extends  $\preceq$  if  $\preceq \subset \leq$ .)*

SOLUTION. Let  $\mathcal{F}$  be the family of all partial orderings on  $X$  extending  $\preceq$ . Then,  $\mathcal{F}$  is non-empty and is partially ordered by  $\subset$ .

I will leave without a proof the fact that  $\mathcal{F}$  satisfies the assumptions of Hausdorff Maximal Principle.

Let  $\leq$  be a maximal element of  $\mathcal{F}$ . Then  $\leq$  is a partial order relation on  $X$  extending  $\preceq$ . It is enough to prove that  $\leq$  is a linear order relation. By way of contradiction assume that it is not. Then, there exist  $a, b \in X$  such that neither  $a \leq b$  nor  $b \leq a$ . Define

$$\ll = \leq \cup \{\langle x, y \rangle : x \leq a \text{ \& } b \leq y\}.$$

Notice that  $\ll$  is a proper extension of  $\leq$  since  $\langle a, b \rangle \in \ll$  while  $\langle a, b \rangle \notin \leq$ . We will obtain a contradiction with the maximality of  $\leq$  by proving that  $\ll$  is a partial order relation.

Clearly  $\ll$  is reflexive, since  $\langle x, x \rangle \in \leq \subset \ll$  for every  $x \in X$ .

To prove that  $\ll$  is antisymmetric assume that  $x \ll y$  and  $y \ll x$ . If  $\langle x, y \rangle, \langle y, x \rangle \notin \leq$  then  $x \leq a$  and  $b \leq y$  by  $x \ll y$ , and also  $y \leq a$  and  $b \leq x$  by  $y \ll x$ . In particular,  $b \leq y$  and  $y \leq a$ , so that  $b \leq a$ , contradicting our assumption that  $a$  and  $b$  are not  $\leq$ -comparable. Thus, at least one of  $\langle x, y \rangle$  and  $\langle y, x \rangle$  belongs to  $\leq$ . By symmetry we can assume that  $x \leq y$ . Now, if  $\langle y, x \rangle \notin \leq$  then again  $y \leq a$  and  $b \leq x$ . Hence,  $b \leq x$ ,  $x \leq y$ ,  $y \leq a$ , and so,  $b \leq a$ , a contradiction. Thus, we have  $x \leq y$  and  $y \leq x$ . Since  $\leq$  is antisymmetric,  $x = y$ .

To prove that  $\ll$  is transitive, assume that  $x \ll y$  and  $y \ll z$ . We have four cases.

$x \leq y$  and  $y \leq z$ . Then  $x \leq z$  and  $x \ll z$ .

$x \leq y$ ,  $y \leq a$  and  $b \leq z$ . Then  $x \leq a$ ,  $b \leq z$  and so  $x \ll z$ .

$x \leq a$ ,  $b \leq y$  and  $y \leq z$ . Then  $x \leq a$ ,  $b \leq z$  and so  $x \ll z$ .

$x \leq a$ ,  $b \leq y$ ,  $y \leq a$  and  $b \leq z$ . Then  $b \leq a$ , contradicting our assumption that  $\langle b, a \rangle \notin \leq$ .

*Exercise 2 page 60: We say that a subset  $A \subset \mathbb{R}$  is algebraically independent if for every non-zero polynomial  $p(x_1, \dots, x_n)$  of  $n$  variables with rational coefficients and any sequence  $a_1, \dots, a_n$  of different elements from  $A$ ,  $p(a_1, \dots, a_n) \neq 0$ . Show that there exists an algebraically independent subset*

A of  $\mathbb{R}$  such that if  $\mathbb{Q}(\mathcal{A})$  is a field generated by  $\mathbb{Q}$  and  $\mathcal{A}$  then for every  $b \in \mathbb{R} \setminus \mathbb{Q}(\mathcal{A})$  there exists a non-zero polynomial  $p(x)$  with coefficients from  $\mathbb{Q}(\mathcal{A})$  such that  $p(b) = 0$ . (The family  $\mathcal{A}$  with this property is called a transcendental base of  $\mathbb{R}$  over  $\mathbb{Q}$ .)

SOLUTION. Let

$$\mathcal{F} = \{A \subset \mathbb{R}: A \text{ is algebraically independent}\}.$$

Notice first that  $\mathcal{F}$  satisfies the assumption of Hausdorff Maximal Principle.

To see this, let  $\mathcal{G} \subset \mathcal{F}$  be a chain in  $\mathcal{F}$  with respect to  $\subset$ . We will show that  $\bigcup \mathcal{G}$  is algebraically independent. So, let  $a_1, \dots, a_n \in \bigcup \mathcal{G}$  be different and let  $p(x_1, \dots, x_n)$  be a polynomial of  $n$  variables with rational coefficients such that  $p(a_1, \dots, a_n) = 0$ . We will show that  $p$  is a zero polynomial.

For every  $i \in \{1, \dots, n\}$  let  $G_i \in \mathcal{G}$  be such that  $a_i \in G_i$ . Since  $\{G_1, \dots, G_n\}$  is a finite subset of a linearly ordered set  $\mathcal{G}$ , we can find the largest element  $G_j$  in this set. Then,  $a_i \in G_i \subset G_j$  for all  $i \in \{1, \dots, n\}$ . Hence, all  $a_i$ 's are in an algebraically independent set  $G_j$ . Thus,  $p$  must be a zero polynomial.

Now, we can use Hausdorff Maximal Principle to  $\mathcal{F}$ .

Let  $\mathcal{A}$  be a maximal element in  $\mathcal{F}$ . We will show that it is a transcendental base of  $\mathbb{R}$  over  $\mathbb{Q}$ .

So, let  $b \in \mathbb{R} \setminus \mathbb{Q}(\mathcal{A})$ . By maximality of  $\mathcal{F}$  there exists a non-zero polynomial  $P(x_1, \dots, x_n, y)$  with rational coefficients and different elements  $a_1, \dots, a_n \in \mathcal{A}$  such that  $P(a_1, \dots, a_n, b) = 0$ . But polynomial  $P$  can be expressed as

$$P(x_1, \dots, x_n, y) = \sum_{i=0}^m P_i(x_1, \dots, x_n) y^i$$

for some polynomials  $P_i(x_1, \dots, x_n)$  with rational coefficients. Let

$$p(y) = \sum_{i=0}^m P_i(a_1, \dots, a_n) y^i.$$

Then,  $p(b) = 0$  and  $p$  has coefficients from  $\mathbb{Q}(\mathcal{A})$ . To finish the proof it is enough to show that  $p$  is non-zero. But  $P$  has at least one non-zero coefficient  $q$  and  $q$  is also a coefficient of  $P_i$  for some  $i$ . Thus,  $P_i$  is non-zero polynomial with rational coefficients so  $P_i(a_1, \dots, a_n) \neq 0$ , since  $\mathcal{A}$  is algebraically independent. Therefore  $p$  is a non-zero polynomial.

**Solutions of the assignment of October 8:**

*Exercise 3 page 60: A filter on a set  $X$  is a nonempty family  $\mathcal{F}$  of subsets of  $X$  such that (1)  $A \cap B \in \mathcal{F}$  provided  $A, B \in \mathcal{F}$ ; and (2) if  $A \subset B \subset X$  and  $A \in \mathcal{F}$  then  $B \in \mathcal{F}$ . A filter  $\mathcal{F}$  on  $X$  is proper if  $\mathcal{F} \neq \mathcal{P}(X)$ , and it is prime if for every  $A \subset X$  either  $A \in \mathcal{F}$  or  $X \setminus A \in \mathcal{F}$ . Show that every proper filter on a set  $X$  can be extended to a proper prime filter.*

SOLUTION, HANDED OCTOBER 15. Let  $\mathcal{F}_0$  be a proper filter on  $X$ . We need to find a proper prime filter  $\mathcal{F}$  on  $X$  with  $\mathcal{F}_0 \subset \mathcal{F}$ . So, let

$$\mathcal{G} = \{\mathcal{F} \subset \mathcal{P}(X): \mathcal{F}_0 \subset \mathcal{F} \text{ \& } \mathcal{F} \text{ is a proper filter on } X\}.$$

Notice first that  $\mathcal{G}$  satisfies the assumption of Hausdorff Maximal Principle.

To see this, let  $\mathcal{H} \subset \mathcal{G}$  be a chain in  $\mathcal{G}$  with respect to  $\subset$ . We will show that  $\bigcup \mathcal{H}$  is a proper filter on  $X$ .

To see (1) for  $\bigcup \mathcal{H}$ , choose  $A, B \in \bigcup \mathcal{H}$ . Then, there exist  $\mathcal{F}_A \in \mathcal{H}$  with  $A \in \mathcal{F}_A$  and  $\mathcal{F}_B \in \mathcal{H}$  with  $B \in \mathcal{F}_B$ . Since  $\mathcal{F}_A, \mathcal{F}_B \in \mathcal{H}$  are  $\subset$ -comparable, there is  $\mathcal{F} \in \{\mathcal{F}_A, \mathcal{F}_B\}$  containing both  $\mathcal{F}_A$  and  $\mathcal{F}_B$ . So, since  $\mathcal{F}$  is a filter,  $A \cap B \in \mathcal{F} \subset \bigcup \mathcal{H}$ , giving (1) for  $\bigcup \mathcal{H}$ .

To see (2) for  $\bigcup \mathcal{H}$ , choose  $A \subset B \subset X$  with  $A \in \bigcup \mathcal{H}$ . Then, there exists an  $\mathcal{F} \in \mathcal{H}$  with  $A \in \mathcal{F}$  and, since  $\mathcal{F}$  satisfies (2),  $B \in \mathcal{F} \subset \bigcup \mathcal{H}$ , giving us (2) for  $\bigcup \mathcal{H}$ .

To see that  $\bigcup \mathcal{H}$  is a proper filter, notice that, by (2), no proper filter contains  $\emptyset$ . So,  $\emptyset \notin \mathcal{F}$  for any  $\mathcal{F} \in \mathcal{H}$ . Thus,  $\emptyset \notin \bigcup \mathcal{H}$ , and indeed  $\bigcup \mathcal{H}$  is a proper filter.

Now, since  $\mathcal{G}$  satisfies the assumption of Hausdorff Maximal Principle, there exists a maximal  $\mathcal{F} \in \mathcal{G}$ . Clearly,  $\mathcal{F}$  extends  $\mathcal{F}_0$ . So, to finish the proof, it is enough to show that  $\mathcal{F}$  is prime. To see this, choose an arbitrary  $A \subset X$ . If  $X \setminus A$  is in  $\mathcal{F}$ , then we are done. So, assume that  $X \setminus A \notin \mathcal{F}$ . We need to show, that  $A \in \mathcal{F}$ .

To see this, notice that  $\mathcal{F}_1 = \{D \in \mathcal{P}(X): D \supset F \cap A \text{ for some } F \in \mathcal{F}\}$  belongs to  $\mathcal{G}$ . Indeed, it satisfies (1) since for any  $B', C' \in \mathcal{F}_1$  there are  $B, C \in \mathcal{F}$  with  $B \cap A \subset B'$  and  $C \cap A \subset C'$ ; therefore we have  $B \cap C \in \mathcal{F}$ ,  $B' \cap C' \supset (B \cap C) \cap A$ , and  $B \cap C \in \mathcal{F}_1$ , as desired. The property (2) for  $\mathcal{F}_1$  is clearly satisfied. Finally,  $\mathcal{F}_1$  is proper, since  $\emptyset \notin \mathcal{F}_1$  — otherwise we would have  $\emptyset = F \cap A$  for some  $F \in \mathcal{F}$ , implying that  $F \subset X \setminus A$ , that is,  $X \setminus A \in \mathcal{F}$ , contradicting our assumption.

To finish the proof, it is enough to notice that maximality of  $\mathcal{F}$  and  $\mathcal{F} \subset \mathcal{F}_1 \in \mathcal{G}$  imply that  $\mathcal{F}_1 = \mathcal{F}$ , so  $A \in \mathcal{F}$ , as needed.

*Exercise 4 page 73: If  $\kappa$  is an infinite cardinal and  $|X_\alpha| \leq \kappa$  for all  $\alpha < \kappa$ , then  $|\bigcup_{\alpha < \kappa} X_\alpha| \leq \kappa$ .*

SOLUTION, HANDED OCTOBER 15. If  $\bigcup_{\alpha < \kappa} X_\alpha = \emptyset$ , then clearly the inequality holds. So, assume that  $\bigcup_{\alpha < \kappa} X_\alpha \neq \emptyset$  and choose  $x \in \bigcup_{\alpha < \kappa} X_\alpha$ . For every  $\alpha < \kappa$  let  $X_\alpha^* = X_\alpha$  if  $X_\alpha \neq \emptyset$  and  $X_\alpha = \{x\}$ , otherwise. Then  $\bigcup_{\alpha < \kappa} X_\alpha = \bigcup_{\alpha < \kappa} X_\alpha^*$  and  $|X_\alpha^*| \leq \kappa$  for all  $\alpha < \kappa$ .

For every  $\alpha < \kappa$  let  $f_\alpha$  be a surjection from  $\kappa$  onto  $X_\alpha^*$ . (We needed  $X_\alpha^* \neq \emptyset$  to ensure existence of  $f_\alpha$ .) Then function  $F: \kappa \times \kappa \rightarrow \bigcup_{\alpha < \kappa} X_\alpha^*$ ,  $F(\alpha, \beta) = f_\alpha(\beta)$  is a surjection. Therefore  $|\bigcup_{\alpha < \kappa} X_\alpha| = |\bigcup_{\alpha < \kappa} X_\alpha^*| \leq |\kappa \times \kappa| = \kappa$ , as required.

**Class of October 15:**

Finish section 5.2, starting with the proof of Theorem 5.2.4.

Hand additional exercises for the Mid Term Test.

**Class of October 20:**

Test review.

**Class of October 22:**

In class Mid Term Test.

**Class of October 27:**

Return of the Mid Term Test.

Went over section 5.3 up to, including, Proposition 5.3.6.

**Class of October 29:**

Quickly review and finish section 5.3.

Solved the first exercise from additional exercises for Mid Term Test.

Started section 6.1: went only over Theorem 6.1.1.

**Class of November 3:**

Proved Theorems 6.1.2-4; sketched the proof of Theorem 6.1.5.

**Written assignment for November 5:** Exercise 4 page 88.

**Written assignment for November 10:** Exercises 2 and 3 page 88.

**Solution of the assignment of November 3, for November 5:**

*Exercise 4 p. 88: Modify Mazurkiewicz's theorem by proving that there exists a subset  $A$  of the plane  $\mathbb{R}^2$  that intersects every circle in exactly three points.*

SOLUTION. Let  $\{C_\xi: \xi < \mathfrak{c}\}$  be an enumeration of all (non-trivial) circles in the plane  $\mathbb{R}^2$ . By transfinite induction on  $\xi < \mathfrak{c}$  we will construct a sequence  $\{A_\xi: \xi < \mathfrak{c}\}$  of subsets of  $\mathbb{R}^2$  such that for every  $\xi < \mathfrak{c}$

- (I)  $A_\xi$  has at most three points;
- (P)  $\bigcup_{\zeta \leq \xi} A_\zeta$  does not have four points on the same circle;
- (D)  $\bigcup_{\zeta \leq \xi} A_\zeta$  contains precisely three points of  $C_\xi$ .

Then the set  $A = \bigcup_{\xi < \mathfrak{c}} A_\xi$  will have the desired property. This is the case since the *preservation* condition (P) implies that every circle contains at most three points of  $A$ , while the *diagonal* condition (D) makes sure that every circle  $C$  contains at least three points from  $A$ . Thus, it is enough to show that we can choose a set  $A_\xi$  satisfying (I), (P), and (D) for every  $\xi < \mathfrak{c}$ . This will be proved by induction on  $\xi < \mathfrak{c}$ .

So assume that for some  $\xi < \mathfrak{c}$  the sequence  $\{A_\zeta: \zeta < \xi\}$  is already constructed. By condition (I) the set  $B = \bigcup_{\zeta < \xi} A_\zeta$  has cardinality  $< \mathfrak{c}$ , being a union of  $|\xi| < \mathfrak{c}$  many finite sets. Similarly, the family  $\mathcal{G}$  of all circles containing three points from  $B$  has cardinality  $\leq |B^3| < \mathfrak{c}$ . Notice that by the inductive assumption (P) the set  $B \cap C_\xi$  has at most three points.

**If  $|B \cap C_\xi| = 3$**  put  $A_\xi = \emptyset$ . Then, (P) and (D) are satisfied.

Otherwise, choose  $x \in C_\xi \setminus \bigcup \mathcal{G}$  and notice that  $B_1 = B \cup \{x\}$  does not have four points on the same circle. The family  $\mathcal{G}_1$  of all circles containing three points from  $B_1$  has cardinality  $\leq |B_1^3| < \mathfrak{c}$ .

**If  $|B \cap C_\xi| = 2$**  put  $A_\xi = \{x\}$ . Then, (P) and (D) are satisfied.

Otherwise, choose  $y \in C_\xi \setminus \bigcup \mathcal{G}_1$ ,  $y \neq x$ , and notice that  $B_2 = B \cup \{x, y\}$  does not have four points on the same circle. The family  $\mathcal{G}_2$  of all circles containing three points from  $B_2$  has cardinality  $\leq |B_2^3| < \mathfrak{c}$ .

**If  $|B \cap C_\xi| = 1$**  put  $A_\xi = \{x, y\}$ . Then, (P) and (D) are satisfied.

Otherwise, choose  $z \in C_\xi \setminus \bigcup \mathcal{G}_2$ ,  $z \notin \{x, y\}$ , and put  $A_\xi = \{x, y, z\}$ . Then, once again, (P) and (D) are satisfied, completing the proof. ■

**Class of November 5:**

Proved Lemma 6.1.6 and Theorem 6.1.7.

**Class of November 10:**

**Written assignment for November 17:** Exercise 6 page 89. (Hint: Use the proof method that was used in the proof of Theorem 6.1.7, utilizing Lemma 6.1.6.) **Cancelled as a homework** on November 12.

**Written assignment for November 17:** Exercise 2 page 103.

- Collect homework, exercises 2 and 3 p. 88. (The solutions for this will be given next class.)  
Discuss briefly the solutions of graded homework, exercise 4 p. 88.
- Prove Theorem 6.1.8. Discuss its role in analysis: it is not measurable in Lebesgue sense and its characteristic functions show the importance of assumptions in Fubini theorem.
- Define perfect sets in  $\mathbb{R}^n$  and Theorem 6.2.3 on their cardinalities. Notice that there is only continuum many such sets.
- Construct *Bernstein set*, Theorem 6.3.7.
- List basic properties of Lebesgue measure on  $\mathbb{R}^n$ .
- State Theorem 6.3.6 and use it to prove (Corollary 6.3.8) that Bernstein set is neither measurable nor does it have Baire property.

In the remainder of the semester we will finish sections 6.2 and 6.3.

**Solutions of the assignment of November 3, for November 10:**

*Exercise 2 page 88: Prove that  $\mathbb{R}^3 \setminus \mathbb{Q}^3$  is a union of disjoint lines.*

SOLUTION. Choose an enumeration  $\{p_\xi: \xi < \mathfrak{c}\}$ , without repetitions, of the set  $\mathbb{R}^3 \setminus \mathbb{Q}^3$ . By induction on  $\xi < \mathfrak{c}$  we will construct a sequence  $\langle L_\xi: \xi < \mathfrak{c} \rangle$  of pairwise disjoint lines in  $\mathbb{R}^3 \setminus \mathbb{Q}^3$  such that for every  $\xi < \mathfrak{c}$  we have

$$(D) \quad p_\xi \in \bigcup_{\zeta \leq \xi} L_\zeta.$$

Then,  $\mathbb{R}^3 \setminus \mathbb{Q}^3$  will be a union of disjoint lines  $\{L_\xi: \xi < \mathfrak{c}\}$ .

To finish the argument, we need to construct a sequence  $\langle L_\xi: \xi < \mathfrak{c} \rangle$  of pairwise disjoint lines in  $\mathbb{R}^3 \setminus \mathbb{Q}^3$  satisfying (D) for every  $\xi < \mathfrak{c}$ . So assume that for some  $\xi < \mathfrak{c}$  we already have a sequence  $\langle L_\zeta: \zeta < \xi \rangle$  of lines. By the recursion theorem it is enough to find a line  $L_\xi$  in  $\mathbb{R}^3 \setminus \mathbb{Q}^3$  disjoint with  $\bigcup_{\zeta < \xi} L_\zeta$  such that (D) is satisfied.

If  $p_\xi \notin \bigcup_{\zeta < \xi} L_\zeta$  define  $p = p_\xi$ . Otherwise, choose an arbitrary point  $p \in \mathbb{R}^3 \setminus \left( \mathbb{Q}^3 \cup \bigcup_{\zeta < \xi} L_\zeta \right)$ . (This can be done since any circle  $C$  in  $\mathbb{R}^3$  contains  $\mathfrak{c}$ -many points, while the set  $C \cap \left( \mathbb{Q}^3 \cup \bigcup_{\zeta < \xi} L_\zeta \right)$  has cardinality less than  $\mathfrak{c}$ .) Choose a plane  $P$  in  $\mathbb{R}^3$  containing  $p$  but no line  $L_\zeta$  with  $\zeta < \xi$ . (Such a choice can be made, since there are continuum many planes containing  $p$ , while for every  $\zeta < \xi$  there is precisely one line containing both  $p$  and  $L_\zeta$ .) Then, the set  $Z = \mathbb{Q}^3 \cup \left( P \cap \bigcup_{\zeta < \xi} L_\zeta \right)$  has cardinality less than  $\mathfrak{c}$ , as  $P \cap \bigcup_{\zeta < \xi} L_\zeta = \bigcup_{\zeta < \xi} (P \cap L_\zeta)$  is a union of less than  $\mathfrak{c}$ -many finite sets. Let  $\mathcal{L}_0$  be the family of all lines in  $P$  containing  $p$ . Since  $\mathcal{L}_0$  contains  $\mathfrak{c}$ -many lines, any two of which intersect only at  $p$ , one of this lines, call it  $L_\xi$ , must be disjoint with the set  $Z$ . Thus,  $\emptyset = L_\xi \cap Z = L_\xi \cap \left( \mathbb{Q}^3 \cup \bigcup_{\zeta < \xi} L_\zeta \right)$ , so  $L_\xi$  is disjoint with  $\bigcup_{\zeta < \xi} L_\zeta$  and (D) is satisfied. This finishes the inductive construction and the proof. ■

*Exercise 3 page 88: Generalize Theorem 6.6.3 by proving that  $\mathbb{R}^3$  is a union of disjoint circles of radius 1.*

SOLUTION. We construct, by induction on  $\xi < \mathfrak{c}$ , the family  $\mathcal{C} = \{C_\xi: \xi < \mathfrak{c}\}$  of subsets of  $\mathbb{R}^3$  such that (1) each  $C \in \mathcal{C}$  is a circle of radius 1, (2) the family  $\mathcal{C}$  covers  $\mathbb{R}^3$ , and (3) different elements of  $\mathcal{C}$  are disjoint. For this, choose an enumeration  $\{p_\xi: \xi < \mathfrak{c}\}$  of  $\mathbb{R}^3$ , and for each  $\xi < \mathfrak{c}$  choose  $C_\xi$  such that

(D)  $p_\xi \in \bigcup_{\zeta \leq \xi} C_\zeta$ , and

(P)  $C_\xi \cap \bigcup_{\zeta < \xi} C_\zeta = \emptyset$ .

Evidently, if we could construct  $\mathcal{C} = \{C_\xi: \xi < \mathfrak{c}\}$  such that conditions (D) and (P) are satisfied for every  $\xi < \mathfrak{c}$ , then  $\mathcal{C}$  would be as desired.

So assume that for some  $\xi < \mathfrak{c}$  the sequence  $\{C_\zeta: \zeta < \xi\}$  is already constructed. We will prove that there is a circle  $C_\xi$  satisfying (D) and (P).

If  $p_\xi \notin \bigcup_{\zeta < \xi} C_\zeta$  define  $p = p_\xi$ . Otherwise, choose an arbitrary point  $p \in \mathbb{R}^3 \setminus \bigcup_{\zeta < \xi} C_\zeta$ . This can be done since for every straight line  $L$  in  $\mathbb{R}^3$  the set  $L \cap \bigcup_{\zeta < \xi} C_\zeta = \bigcup_{\zeta < \xi} (L \cap C_\zeta)$  has cardinality less than continuum, being a union of less than continuum many finite sets  $L \cap C_\zeta$ . We will choose  $C_\xi$  containing  $p$  and satisfying (P). To do this, take a plane  $P$  containing  $p$  that does not contain any of the circles  $C_\zeta$  for  $\zeta < \xi$ . This can be done since there are continuum many planes passing through  $p$  and there are only  $\leq |\xi| < \mathfrak{c}$  many planes containing circles from  $\{C_\zeta: \zeta < \xi\}$ . Now notice that the plane  $P$  intersects each circle  $C_\zeta$  in at most two points. Thus, the set  $T = P \cap \bigcup_{\zeta < \xi} C_\zeta$  has cardinality less than continuum. For every  $\zeta < \xi$  let  $r_\zeta$  be the center of the circle  $C_\zeta$ . Fix a line  $L$  in  $P$  containing  $p$  and disjoint with the set  $W = T \cup \{r_\zeta: \zeta < \xi\}$ —this can be done, since there are  $\mathfrak{c}$ -many such lines and  $|W| < \mathfrak{c}$ . In particular,  $L$  is disjoint with  $\bigcup_{\zeta < \xi} C_\zeta$ , as  $L \cap \bigcup_{\zeta < \xi} C_\zeta = (L \cap P) \cap \bigcup_{\zeta < \xi} C_\zeta = L \cap T = \emptyset$ . Choose a sphere  $S$  of radius 1 containing  $p$  with the center on  $L$  and let  $\mathcal{C}_\xi$  be the family of all circles of radius 1 contained in  $S$  and containing  $p$ . Notice, that the intersection of any two such circles consists of two points on  $L$ , so it is disjoint from  $\bigcup_{\zeta < \xi} C_\zeta$ . Moreover, for every  $\zeta < \xi$  the set  $S \cap C_\zeta$  contains at most two points, as the sphere  $S$  and the circle  $C_\zeta$  have the same radius, equal 1, and their centers are distinct— $S$  having center on  $L$ , while  $C_\zeta$  having center  $r_\zeta$  off  $L$ . Thus,  $E = S \cap \bigcup_{\zeta < \xi} C_\zeta = \bigcup_{\zeta < \xi} (S \cap C_\zeta)$  has cardinality less than  $\mathfrak{c}$ , so there is  $C_\xi \in \mathcal{C}_\xi$  disjoint with  $E$ . Hence,  $C_\xi \cap \bigcup_{\zeta < \xi} C_\zeta = (C_\xi \cap S) \cap \bigcup_{\zeta < \xi} C_\zeta = C_\xi \cap E = \emptyset$  ensuring (D) and (P). ■

**Different exposition of the proof of Theorem 6.1.7**

*Theorem 6.1.7: There is a countable partition  $\{S_i: i < \omega\}$  of  $\mathbb{R}^2$  such that the distance between any two different points of the same set  $S_i$  is irrational.*

SOLUTION. By transfinite induction we are going to prove that the following property hold for every cardinal  $\kappa \leq \mathfrak{c}$ .

$(I_\kappa)$  For every  $X \subset \mathbb{R}^2$  with  $|X| \leq \kappa$  there exists a map  $g: X \rightarrow \omega$  such that for every  $i < \omega$ ,  $g^{-1}(i)$  contains no two points of rational distance.

This will finish the proof, since then, by  $(I_\kappa)$  used with  $X = \mathbb{R}^2$ , there exists a map  $g: X \rightarrow \omega$  such that for every  $i < \omega$  the distance between any two different points of  $g^{-1}(i)$  is irrational. Therefore, the sets  $S_i = g^{-1}(i)$ , with  $i < \omega$ , form the desired decomposition of  $\mathbb{R}^2$ .

Clearly,  $(I_\kappa)$  holds when  $\kappa \leq \omega$ : any injection  $g: X \rightarrow \omega$  is as needed.

So assume  $\kappa > \omega$  and that  $(I_\mu)$  holds for every cardinal  $\mu < \kappa$ . To finish the proof it is enough to show that this implies that  $(I_\kappa)$  holds.

To see this, for every non-zero  $p, q \in \mathbb{Q}$  define  $F_{pq}: (\mathbb{R}^2)^2 \rightarrow [\mathbb{R}^2]^{<\omega}$  by

$$F_{pq}(x, y) = \{z \in \mathbb{R}^2: |x - z| = p \ \& \ |y - z| = q\}.$$

Notice that indeed  $F_{pq}(x, y) \in [\mathbb{R}^2]^{<\omega}$  since it has at most two points. Let  $\mathcal{F} = \{F_{pq}: p, q \in \mathbb{Q} \setminus \{0\}\}$  and let  $X = \bigcup_{\alpha < \kappa} X_\alpha$ , where  $\langle X_\alpha: \alpha < \kappa \rangle$  is a sequence of subsets closed under the action of  $\mathcal{F}$  as in Lemma 6.1.6(b).

By induction on  $\alpha \leq \kappa$  we will construct a sequence  $\langle g_\alpha \in \omega^{X_\alpha}: \alpha \leq \kappa \rangle$  of functions such that for every  $\beta \leq \alpha \leq \kappa$  and  $i < \omega$ ,

- $g_\alpha$  extends  $g_\beta$  and no two points in  $g_\alpha^{-1}(i)$  are of rational distance.

Then, clearly  $g_\kappa$  satisfies  $(I_\kappa)$ .

To construct the sequence  $\langle g_\alpha \in \omega^{X_\alpha}: \alpha \leq \kappa \rangle$ , assume that for some  $\alpha \leq \kappa$  the desired sequence  $\langle g_\beta \in \omega^{X_\beta}: \beta < \alpha \rangle$  is already constructed. We just need to construct appropriate  $g_\alpha$ .

If  $\alpha$  is a limit ordinal, then  $g_\alpha = \bigcup_{\beta < \alpha} g_\beta$  is as needed. So, assume that  $\alpha$  is a successor ordinal, say  $\alpha = \beta + 1$ . Then  $g_\beta$  is already defined on  $X_\beta$ . We have to extend  $g_\alpha \upharpoonright \beta = g_\beta$  to the set  $Z = X_{\beta+1} \setminus X_\beta$ .

Since  $|Z| \leq |X_{\beta+1}| < \kappa$ , by  $(I_{|X_{\beta+1}|})$ , we can find an appropriate function  $h: Z \rightarrow \omega$ . Since  $X_\alpha$  is closed under the action of  $\mathcal{F}$ , for every  $z \in Z$  there is at most one  $x \in X_\alpha$  such that  $|z - x| \in \mathbb{Q}$ .

For every  $n < \omega$  and  $z \in h^{-1}(n)$  define  $g_\alpha(z) = 2n$  if there is no  $x \in g_\beta^{-1}(2n)$  with  $|z - x| \in \mathbb{Q}$  and let  $g_\alpha(z) = 2n + 1$  otherwise. Then  $g_\alpha \supset g_\beta$  is as needed.

**Class of November 12:**

**Cancel Exercise 6 page 89 as a homework** – it is too difficult.

Give three handouts, including solutions of graded homework and infirmation on the final test.

Return homework and give some comments on it.

Start section 6.2: covered up to, including, Theorem 6.2.3.

**Class of November 17:**

Collect the homework.

Continued covering section 6.2, starting with reviewing Theorem 6.2.3.

Note, the  $\Sigma_\xi^0$  and  $\Pi_\xi^0$  notation was not covered and Theorem 6.2.7 was skipped. Theorem 6.2.8 was proved using the result stated in Exercise 4, which was proved in class.

**Solution of the (remaining) assignment of November 10, handed November 19:**

*Exercise 2 page 103: Show that there exists a Bernstein set  $B$  such that  $B+B$  is also a Bernstein set.*

HOW TO COME UP WITH THE SOLUTION PRESENTED IN THE NEXT PAGE.

As in the construction of the Bernstein set  $B$  in Theorem 6.3.7, we let  $\{P_\xi: \xi < \mathfrak{c}\}$  be an enumeration of the family  $\mathcal{P}$  all perfect subsets of  $\mathbb{R}$  and define  $B$  as  $B = \{b_\xi: \xi < \mathfrak{c}\}$  for some sequence  $\langle b_\xi: \xi < \mathfrak{c} \rangle$  chosen inductively.

We need  $B \cap P \neq \emptyset$  for every  $P \in \mathcal{P}$ . For this, it is enough to choose

$$(1) \quad b_\xi \in P_\xi.$$

Notice that this will also ensure that  $B + B = \{b_\zeta + b_\eta: \zeta, \eta < \mathfrak{c}\}$  intersects every  $P \in \mathcal{P}$ , since for every  $P$  there exists a  $\xi < \mathfrak{c}$  with  $P_\xi = -b_0 + P$  (a translation  $-b_0 + P$  of a perfect set  $P$  is perfect) so that  $b_\xi \in P_\xi = -b_0 + P$  and  $b_0 + b_\xi \in P \cap (B + B)$ .

We need also to ensure that both  $\mathbb{R} \setminus B$  and  $\mathbb{R} \setminus (B + B)$  intersect every  $P \in \mathcal{P}$ . This will be done by choosing, for every  $\xi < \mathfrak{c}$

$$(2) \quad a_\xi \in P_\xi \text{ and making sure that}$$

$$(\bullet) \quad a_\xi \notin B \cup (B + B).$$

This will finish the proof. However, ensuring  $(\bullet)$  requires some additional delicacy in the choice of  $a_\xi$  and  $b_\xi$ . Basically, we need to make sure that  $a_\xi \neq b_\zeta$  and  $a_\xi \neq b_\zeta + b_\eta$  for every  $\xi, \zeta, \eta < \mathfrak{c}$ . Since we actually construct the sequence  $\langle \langle a_\xi, b_\xi \rangle: \xi < \mathfrak{c} \rangle$ , the choice of  $b_\xi$  will depend on the sets  $\{a_\zeta: \zeta < \xi\}$  and  $\{b_\zeta: \zeta < \xi\}$ , while the choice of  $a_\xi$  will depend on the sets  $\{a_\zeta: \zeta < \xi\}$  and  $\{b_\zeta: \zeta \leq \xi\}$ .

Thus, we choose  $b_\xi$  such that  $b_\xi \neq a_\zeta$ ,  $b_\xi + b_\xi \neq a_\zeta$ , and  $b_\xi + b_\eta \neq a_\zeta$  for any  $\zeta, \eta < \xi$ , leading to requirement:

$$(3) \quad b_\xi \notin \{a_\zeta: \zeta < \xi\} \cup \{a_\zeta/2: \zeta < \xi\} \cup (\{a_\zeta: \zeta < \xi\} + \{-b_\eta: \eta < \xi\}).$$

Similarly, we choose  $a_\xi$  such that  $a_\xi \neq b_\zeta$  and  $a_\xi \neq b_\zeta + b_\eta$  for any  $\zeta, \eta \leq \xi$ , leading to requirement:

$$(4) \quad a_\xi \notin \{b_\zeta: \zeta \leq \xi\} \cup (\{b_\zeta: \zeta \leq \xi\} + \{b_\eta: \eta \leq \xi\}).$$

Combining requirements (1)-(4), which is all that is needed, leads to the following proof.

THE ACTUAL SOLUTION OF EXERCISE 2 PAGE 103. Let  $\{P_\xi: \xi < \mathfrak{c}\}$  be an enumeration of all perfect subsets of  $\mathbb{R}$ . Aiming for  $B = \{b_\xi: \xi < \mathfrak{c}\}$ , by transfinite induction on  $\xi < \mathfrak{c}$  we define the sequences  $\langle a_\xi: \xi < \mathfrak{c} \rangle$  and  $\langle b_\xi: \xi < \mathfrak{c} \rangle$  by choosing in step  $\xi < \mathfrak{c}$ :

( $I_\xi$ )  $b_\xi \in P_\xi \setminus D_\xi$ , where

$$D_\xi = \{a_\zeta: \zeta < \xi\} \cup \{a_\zeta/2: \zeta < \xi\} \cup (\{a_\zeta: \zeta < \xi\} + \{-b_\eta: \eta < \xi\});$$

( $J_\xi$ )  $a_\xi \in P_\xi \setminus E_\xi$ , where  $E_\xi = \{b_\zeta: \zeta \leq \xi\} \cup (\{b_\zeta: \zeta \leq \xi\} + \{b_\eta: \eta \leq \xi\})$ .

This can be done since  $|P_\xi| = \mathfrak{c}$ , while  $|D_\xi| \leq |\xi| \oplus |\xi| \oplus (2 \otimes |\xi|) < \mathfrak{c}$  and  $|E_\xi| \leq |\xi| \oplus (2 \otimes |\xi|) < \mathfrak{c}$ . This finishes the inductive construction of the sequences.

The purpose of sets  $D_\xi$  and  $E_\xi$  is to ensure that the set  $A = \{a_\xi: \xi < \mathfrak{c}\}$  is disjoint with  $B \cup (B + B)$ . Indeed, this is the case, since for any  $\xi, \zeta, \eta < \mathfrak{c}$  we have  $a_\xi \neq b_\zeta$  and  $a_\xi \neq b_\zeta + b_\eta$ :

- $a_\xi \neq b_\zeta$  is guaranteed by ( $J_\xi$ ) when  $\xi \geq \zeta$  and by ( $I_\zeta$ ) when  $\zeta > \xi$ ;
- $a_\xi \neq b_\zeta + b_\eta$  is ensured by ( $J_\xi$ ) when  $\xi \geq \max\{\zeta, \eta\}$  and by ( $I_\alpha$ ) when  $\alpha = \max\{\zeta, \eta\} > \xi$  (the case  $\alpha = \zeta = \eta$  handled by the set  $\{a_\xi/2: \xi < \alpha\}$  and case  $\alpha = \zeta > \eta$  by  $\{a_\xi: \xi < \alpha\} + \{-b_\eta: \eta < \alpha\}$ ).

Now,  $B$  is a Bernstein set since for every perfect set  $P$ :

- $P \not\subset B$ , as  $a_\xi \in P \cap A \subset P \cap (\mathbb{R} \setminus B)$ , where  $\xi < \mathfrak{c}$  is such that  $P = P_\xi$ ;
- $P \not\subset \mathbb{R} \setminus B$ , as  $b_\xi \in P \cap B$ , where  $\xi < \mathfrak{c}$  is such that  $P = P_\xi$ .

Similarly,  $B + B$  is a Bernstein set since for every perfect set  $P$ :

- $P \not\subset B + B$ , as  $a_\xi \in P \cap A \subset P \cap (\mathbb{R} \setminus (B + B))$ , where  $\xi < \mathfrak{c}$  is such that  $P = P_\xi$ ;
- $P \not\subset \mathbb{R} \setminus (B + B)$ , as  $b_0 + b_\xi \in P \cap (B + B)$ , where  $\xi < \mathfrak{c}$  is such that  $P_\xi = -b_0 + P$ . ■

**Class of November 19:**

Started going over section 6.3; covered up to, including, Proposition 6.3.1.  
(Theorems 6.3.7 and 6.3.8 covered earlier.)

**Class of December 1:**

Hand over the first exercise preparing for the Final Test.  
Continue going over section 6.3 — cover pages 100 and 101.

**Class of December 3:**

Hand over the remaining exercises preparing for the Final Test.  
State and prove Theorem 6.3.6. (If necessary, skip the proof of the class of Baire class.)  
Review for the Final Test.

**Class of December 8:**

Administration of the Final Test, 7pm-9pm.

**December 9:**

Administration of an optional “Rescue” Mid Term Test, 7pm-9pm.