

**Topology 1, Math 581, Fall 2016: Notes and homework**

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**Class of August 18:**

Course and syllabus overview.

Topology is an abstract geometry, sometimes referred to as *Rubber Sheet Geometry*. Material, in this course, will be presented “from abstract definitions and results to specific examples.”

Notation:

- Do not confuse  $A \in \mathcal{A}$  (which reads “ $A$  is an element of  $\mathcal{A}$ ”) with  $A \subset \mathcal{A}$  (which reads “ $A$  is a subset of  $\mathcal{A}$ ” and means “every element of  $A$  is also an element of  $\mathcal{A}$ ”).

Notice that  $A \subset B \subset C$  implies  $A \subset C$ , but  $A \in B \in C$  does not imply  $A \in C$ . You will never see in this course a pair  $A$  and  $B$ , for which we will have simultaneously  $A \in B$  and  $A \subset B$ .

- Notation  $f: X \rightarrow Y$  means that  $f$  is a function from a set  $X$ , domain of the function, into the set  $Y$ . For any set  $C$  (usually,  $C \subset Y$ ), the preimage  $f^{-1}(C)$  (of  $C$  under  $f$ ) is defined as

$$f^{-1}(C) = \{x \in X: f(x) \in C\}.$$

**Example 1**  $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$  for every  $A, B$ , and function  $f$ .

PROOF.  $x \in f^{-1}(A \cap B) \Leftrightarrow f(x) \in A \cap B \Leftrightarrow f(x) \in A \ \& \ f(x) \in B$   
 $\Leftrightarrow x \in f^{-1}(A) \ \& \ x \in f^{-1}(B) \Leftrightarrow x \in f^{-1}(A) \cap f^{-1}(B).$  ■

**Motivation:**

Let  $\mathbb{R}$  be the set of real numbers and for  $x \in \mathbb{R}$  and  $\varepsilon > 0$  let

$$B(x, \varepsilon) = \{r \in \mathbb{R}: |x - r| < \varepsilon\}.$$

We will refer to  $B(x, \varepsilon)$  as an *open ball*, although for this case it is just an open interval  $(x - \varepsilon, x + \varepsilon)$ . Let  $\mathcal{T}$  be the family of all subsets  $U$  of  $\mathbb{R}$  such that for every  $x \in U$  there is an  $\varepsilon > 0$  such that  $x \in B(x, \varepsilon) \subset U$ :

$$\mathcal{T} = \{U \subset \mathbb{R}: \forall x \in U \exists \varepsilon > 0 (B(x, \varepsilon) \subset U)\}.$$

Latter, we will refer to  $\mathcal{T}$  as the *standard topology* on  $\mathbb{R}$  and its elements  $U \in \mathcal{T}$  will be called *open sets*.

**Theorem 2** (Motivational) Let  $f: \mathbb{R} \rightarrow \mathbb{R}$ . The following two definitions of continuity of  $f$  are equivalent:

- (a) (Topological definition)  $f^{-1}(U) \in \mathcal{T}$  for every  $U \in \mathcal{T}$ .
- (b) ( $\varepsilon$ - $\delta$  definition) For every  $x \in \mathbb{R}$  and every  $\varepsilon > 0$  there is a  $\delta > 0$  such that for every  $r \in \mathbb{R}$ , if  $|x - r| < \delta$ , then  $|f(x) - f(r)| < \varepsilon$ .

PROOF. Latter today.

For functions  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$  their composition  $g \circ f: X \rightarrow Z$  is defined via formula:  $(g \circ f)(x) = g(f(x))$  for every  $x \in X$ . Also, if  $A \subset X$ , then the image  $f[A]$  of  $A$  under  $f$  is defined as  $\{f(a): a \in A\}$ .

**Theorem 3** We have the following properties:

- (a)  $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$
- (b)  $(g \circ f)[A] = g[f[A]]$

PROOF. (a)  $x \in (g \circ f)^{-1}(C) \Leftrightarrow (g \circ f)(x) \in C \Leftrightarrow g(f(x)) \in C \Leftrightarrow f(x) \in g^{-1}(C) \Leftrightarrow x \in f^{-1}(g^{-1}(C))$ .

Proof of (b) is left as an exercise. (Not homework assignment.) ■

The next theorem gives a motivation of defining continuity of a functions via property (a) of Theorem 2. Note, that the proof is considerably easier than a standard  $\varepsilon$ - $\delta$  proof.

**Theorem 4** If functions  $f, g: \mathbb{R} \rightarrow \mathbb{R}$  are continuous, then so is their composition  $g \circ f: \mathbb{R} \rightarrow \mathbb{R}$ .

PROOF. Let  $U \in \mathcal{T}$ . By Theorem 2 it is enough to prove that  $(g \circ f)^{-1}(U) \in \mathcal{T}$ . By Theorem 3(a),  $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ . Now,  $W = g^{-1}(U) \in \mathcal{T}$  by the continuity of  $g$  and Theorem 2. Therefore, by the continuity of  $f$  (and Theorem 2 used once again),  $(g \circ f)^{-1}(U) = f^{-1}(W) \in \mathcal{T}$ , as required. ■

The same proof will work for arbitrary continuous functions defined via a general notion of defined below. (See section 12 in the text.)

PROOF OF THEOREM 2. (a) $\implies$ (b): Fix an  $x \in \mathbb{R}$  and an  $\varepsilon > 0$ . Using (a), we need to find a  $\delta$  satisfying (b).

Let  $U = B(f(x), \varepsilon) = (f(x) - \varepsilon, f(x) + \varepsilon)$ . Notice that  $U \in \mathcal{T}$ . (This requires checking, that  $U$  satisfies the definition of sets in  $\mathcal{T}$ .) So, by (a),

$f^{-1}(U) \in \mathcal{T}$ . Note also, that  $x \in f^{-1}(U)$ , as  $f(x) \in (f(x) - \varepsilon, f(x) + \varepsilon) = U$ . Therefore, we have  $x \in f^{-1}(U) \in \mathcal{T}$  and, by the definition of  $\mathcal{T}$ , there is a  $\delta > 0$  such that  $B(x, \delta) \subset f^{-1}(U)$ . We show, that this  $\delta$  satisfies (b).

Indeed, let  $r \in \mathbb{R}$  be such that  $|x - r| < \delta$ . Then,  $r \in (x - \delta, x + \delta) = B(x, \delta) \subset f^{-1}(U)$ . Therefore,  $f(r) \in U = (f(x) - \varepsilon, f(x) + \varepsilon)$  and so,  $|f(x) - f(r)| < \varepsilon$ , as required.

(b) $\implies$ (a): Fix a  $U \in \mathcal{T}$ . We need to show that  $f^{-1}(U)$  is in  $\mathcal{T}$ . For this, take an  $x \in f^{-1}(U)$ . We need to find a  $\delta > 0$  for which  $B(x, \delta) \subset f^{-1}(U)$ .

We have  $f(x) \in U$ , as  $x \in f^{-1}(U)$ . Since  $U \in \mathcal{T}$ , there exists an  $\varepsilon > 0$  for which  $B(f(x), \varepsilon) \subset U$ . Using (b) for this  $x$  and  $\varepsilon$ , we can find a  $\delta > 0$  such that  $|f(x) - f(r)| < \varepsilon$  provided  $|x - r| < \delta$ . We will show that for this choice of  $\delta$  we indeed have  $B(x, \delta) \subset f^{-1}(U)$ .

To see this, take an  $r \in B(x, \delta) = (x - \delta, x + \delta)$ . We need to show that  $r \in f^{-1}(U)$ . Since  $r \in (x - \delta, x + \delta)$ , we have  $|x - r| < \delta$ . So, by the choice of  $\delta$ ,  $|f(x) - f(r)| < \varepsilon$ . In particular,  $f(r) \in (f(x) - \varepsilon, f(x) + \varepsilon) = B(f(x), \varepsilon) \subset U$ . Thus,  $r \in f^{-1}(U)$ , as required. ■

**Reading assignment:** Read Sections 1-7.

It is assumed that you are familiar with the material presented there. Therefore, we will not cover this material in class. (If necessary, we will be reviewing these notion on “as needed” basis.)

**Written assignment:** Write for the next class:

**Exercise 1** Prove that  $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$  for every sets  $A$  and  $B$ , and a function  $f: X \rightarrow Y$ .

Prove, for Tuesday, September 6, the following version of Theorem 2. Provide direct proof, that is, without using condition (b) of Theorem 2.

**Note:** Exercise 2 will be treated as **bonus exercise**. (This is, really, a real analysis problem. So, do not feel bad, if you cannot solve this.)

**Exercise 2** (Motivational Theorem Part 2) Let  $f: \mathbb{R} \rightarrow \mathbb{R}$ . The following two definitions of continuity of  $f$  are equivalent:

(a) (Topological definition)  $f^{-1}(U) \in \mathcal{T}$  for every  $U \in \mathcal{T}$ .

(c) (Sequential definition)  $\lim_{n \rightarrow \infty} f(x_n) = f(x)$  for every sequence  $\langle x_n \rangle_{n=1}^{\infty}$  in  $\mathbb{R}$  converging to  $x \in \mathbb{R}$ .

**Class of August 23:**

What we covered last class: For  $x \in \mathbb{R}$  and  $\varepsilon > 0$  we define an open ball

$$B(x, \varepsilon) = \{r \in \mathbb{R}: |x - r| < \varepsilon\} = (x - \varepsilon, x + \varepsilon).$$

Let  $\mathcal{T}$  be the family of all subsets  $U$  of  $\mathbb{R}$  such that for every  $x \in U$  there is an  $\varepsilon > 0$  such that  $x \in B(x, \varepsilon) \subset U$ :

$$\mathcal{T} = \{U \subset \mathbb{R}: \forall x \in U \exists \varepsilon > 0 (B(x, \varepsilon) \subset U)\}.$$

We will refer to  $\mathcal{T}$  as the *standard topology* on  $\mathbb{R}$  and its elements  $U \in \mathcal{T}$  will be called *open sets*.

We proved

Let  $f: \mathbb{R} \rightarrow \mathbb{R}$ . The following two definitions of continuity of  $f$  are equivalent:

- (a) (Topological definition)  $f^{-1}(U) \in \mathcal{T}$  for every  $U \in \mathcal{T}$ .
- (b) ( $\varepsilon$ - $\delta$  definition) For every  $x \in \mathbb{R}$  and every  $\varepsilon > 0$  there is a  $\delta > 0$  such that for every  $r \in \mathbb{R}$ , if  $|x - r| < \delta$ , then  $|f(x) - f(r)| < \varepsilon$ .

*New material:*

**Definition 1** Let  $X$  be an arbitrary set having at least two elements. A *topology* on  $X$  is any family  $\mathcal{T}$  of subsets of  $X$  having the following properties:

- (1)  $\emptyset, X \in \mathcal{T}$ .
- (2) The union of any subfamily of  $\mathcal{T}$  is in  $\mathcal{T}$ , that is,  $\bigcup \mathcal{U} \in \mathcal{T}$  for every  $\mathcal{U} \subset \mathcal{T}$ .
- (3) The intersection of any *finite* subfamily of  $\mathcal{T}$  is in  $\mathcal{T}$ , that is,  $\bigcap \mathcal{U} \in \mathcal{T}$  for every finite  $\mathcal{U} \subset \mathcal{T}$ .

The pair  $\langle X, \mathcal{T} \rangle$  is called a *topological space*. For a fixed topological space  $\langle X, \mathcal{T} \rangle$ , the sets belonging to the family  $\mathcal{T}$  will be referred to as the *open sets* (with respect to this topology).

In the above definition, we used the following notation:

- Arbitrary unions and intersections of sets: Let  $\mathcal{A}$  be a family of sets, say  $\mathcal{A} = \{A_t: t \in T\}$ . Then  $\bigcup \mathcal{A} = \bigcup_{t \in T} A_t$  denotes the same set:  $\{x: \exists A \in \mathcal{A} (x \in A)\}$ , that is,  $\{x: \exists t \in T (x \in A_t)\}$ .

- Similarly,  $\bigcap \mathcal{A} = \bigcap_{t \in T} A_t$  denotes the same set:  $\{x: \forall A \in \mathcal{A}(x \in A)\}$ , that is,  $\{x: \forall t \in T(x \in A_t)\}$ .

**Remark 5** In the definition, condition (3) can be replaced with

- (3') The intersection of any two sets in  $\mathcal{T}$  is in  $\mathcal{T}$ , that is, if  $U, V \in \mathcal{T}$ , the also  $U \cap V \in \mathcal{T}$ .

PROOF. Easy induction. ■

**Example 6** Here are some examples of topological spaces  $\langle X, \mathcal{T} \rangle$ , where  $X$  is an arbitrary set.

- $\mathcal{T} = \mathcal{P}(X)$ , where  $\mathcal{P}(X)$  is the **power set of  $X$** , that is, the family of all subsets of  $X$ . This topology is called the **discrete topology**.
- $\mathcal{T} = \{\emptyset, X\}$ . This topology is called **trivial or indiscrete topology**.
- **The standard topology  $\mathcal{T}$  on  $\mathbb{R}$** , defined for Theorem 2.

More examples:

**Example 7** Examples of topologies on a set  $X$ :

- For a three elements set  $X = \{a, b, c\}$ , there are many different possible topologies. (Nine are indicated in Example 1, page 76). E.g.  $\{\emptyset, \{a\}, \{a, b\}, X\}$ . Other examples from the text, section 12.
- **Finite complement topology  $\mathcal{T}_f = \{\emptyset\} \cup \{X \setminus F: F \text{ is finite}\}$** . Notice that  $\langle X, \mathcal{T}_f \rangle$  is discrete, for finite  $X$ .
- **Countable complement topology  $\mathcal{T}_C = \{\emptyset\} \cup \{X \setminus F: F \text{ is countable}\}$** . Notice that  $\langle X, \mathcal{T}_C \rangle$  is discrete, for countable  $X$ .

Definition of *finer* and *coarser* topologies.

Proof that  $f^{-1}(\bigcup_{t \in T} A_t) = \bigcup_{t \in T} f^{-1}(A_t)$ :

$$\begin{aligned}
 x \in f^{-1}\left(\bigcup_{t \in T} A_t\right) &\Leftrightarrow f(x) \in \bigcup_{t \in T} A_t \quad (\text{by the definition of preimage}) \\
 &\Leftrightarrow \exists t \in T \ f(x) \in A_t \quad (\text{by the definition of union}) \\
 &\Leftrightarrow \exists t \in T \ x \in f^{-1}(x)A_t \quad (\text{by the definition of preimage}) \\
 &\Leftrightarrow x \in \bigcup_{t \in T} f^{-1}(A_t) \quad (\text{by the definition of union}).
 \end{aligned}$$

**Class of August 25:**

Recall that a *topology* on  $X$  is a family  $\mathcal{T}$  of subsets of  $X$  such that

- (1)  $\emptyset, X \in \mathcal{T}$ ;
- (2)  $\bigcup \mathcal{U} \in \mathcal{T}$  for every  $\mathcal{U} \subset \mathcal{T}$ ;
- (3)  $\bigcap \mathcal{U} \in \mathcal{T}$  for every finite  $\mathcal{U} \subset \mathcal{T}$ .

Examples of topological spaces  $\langle X, \mathcal{T} \rangle$ :

- **Discrete topology**  $\mathcal{T} = \mathcal{P}(X)$ , where  $\mathcal{P}(X)$  is the **power set** of  $X$ .
- **Trivial or indiscrete topology**  $\mathcal{T} = \{\emptyset, X\}$ .
- **The standard topology**  $\mathcal{T}$  on  $\mathbb{R}$ , defined for Theorem 2.
- **Finite complement topology**  $\mathcal{T}_f = \{\emptyset\} \cup \{X \setminus F : F \text{ is finite}\}$ . Notice that  $\langle X, \mathcal{T}_f \rangle$  is discrete, for finite  $X$ .
- **Countable complement topology**  $\mathcal{T}_C = \{\emptyset\} \cup \{X \setminus F : F \text{ is countable}\}$ . Notice that  $\langle X, \mathcal{T}_C \rangle$  is discrete, for countable  $X$ .

*New material:*

**Section 13: Basis for a Topology**

**Definition 2** *Basis* — *Two related definitions*

FROM A BASIS TO TOPOLOGY — **Basis for a topology:** A collection  $\mathcal{B}$  of a subsets of a set  $X$  such that

- (B1) For every  $x \in X$  there is a  $B \in \mathcal{B}$  with  $x \in B$  (i.e.,  $\bigcup \mathcal{B} = X$ ).
- (B2) For every  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \cap B_2$  there is a  $B \in \mathcal{B}$  with  $x \in B \subset B_1 \cap B_2$ .

[FROM A TOPOLOGY TO ITS BASIS — **Basis for a given topology**  $\mathcal{T}$ :

Let  $\langle X, \mathcal{T} \rangle$  be a fixed topological space. A basis for  $\mathcal{T}$  is any collection  $\mathcal{B} \subset \mathcal{T}$  such that for every  $U \in \mathcal{T}$  and every  $x \in U$  there exists a  $B \in \mathcal{B}$  with  $x \in B \subset U$ .

The first of these notion is used to create new topologies. The second is used to easier deal with a given, fixed topology  $\mathcal{T}$ . This second notion is used considerably more often than the first one.

**Fact 1** *If  $\mathcal{B}$  satisfies (B1) and (B2), then the family*

$$\mathcal{T}(\mathcal{B}) = \{U \subset X: \forall x \in U \exists B \in \mathcal{B}(x \in B \subset U)\} = \left\{ \bigcup \mathcal{U}: \mathcal{U} \subset \mathcal{B} \right\}$$

*is a topology on  $X$ . The family  $\mathcal{B}$  is a basis to the topology  $\mathcal{T}(\mathcal{B})$ .*

**Fact 2** (Lemma 13.2) *If  $\mathcal{B}$  is a basis for a topology  $\mathcal{T}$ , then  $\mathcal{T} = \mathcal{T}(\mathcal{B})$ .*

Discuss examples 1–3.

Go over Lemma 13.3.

There may be more than one basis for a given topology: Example 4 (from Examples 1 and 2).

**Example 8** *Two examples of topologies on  $\mathbb{R}$ :*

- **Standard topology**, generated by basis  $\mathcal{B}_{st} = \{(a, b): a, b \in \mathbb{R}, a < b\}$ , that is, the topology  $\mathcal{T}_{st} = \mathcal{T}(\mathcal{B}_{st})$ . We usually write just  $\mathbb{R}$  for  $\langle \mathbb{R}, \mathcal{T}_{st} \rangle$ .  
Notice, that this is the same topology that was used in Theorem 2.
- **Lower limit (or Sorgenfrey) topology**  $\mathcal{T}_\ell$  is generated by basis  $\mathcal{B}_\ell = \{[a, b): a, b \in \mathbb{R}, a < b\}$ , that is,  $\mathcal{T}_\ell = \mathcal{T}(\mathcal{B}_\ell)$ . We usually write  $\mathbb{R}_\ell$  for  $\langle \mathbb{R}, \mathcal{T}_\ell \rangle$ .

**Written assignment for Tuesday, August 30:** Exercise 8, page 83. (In part (b), do not forget to prove, that  $\mathcal{T}(\mathcal{C})$  is indeed a topology. Do you need to prove, in part (a), that  $\mathcal{T}(\mathcal{B})$  is a topology?)

Bonus question to part (b): What if we replace family  $\mathcal{C}$  with the family  $\mathcal{C}^* = \{[a, b): a < b, \text{ and } a \text{ and } b \text{ are irrational}\}$ ? How  $\mathcal{T}(\mathcal{C})$  and  $\mathcal{T}(\mathcal{C}^*)$  compare to each other?

*Be ready for a quiz next class time!*

**Class of August 30:**

Recall that (rephrasing):

**Basis for a given topology  $\mathcal{T}$ :** Let  $\langle X, \mathcal{T} \rangle$  be a fixed topological space. A basis for  $\mathcal{T}$  is any collection  $\mathcal{B} \subset \mathcal{T}$  such that for every  $U \in \mathcal{T}$  and every  $x \in U$  there exists a  $B \in \mathcal{B}$  with  $x \in B \subset U$ .

**Fact 3** For a collection  $\mathcal{B}$  of subsets of  $X$ , let

$$\mathcal{T}(\mathcal{B}) = \{U \subset X : \forall x \in U \exists B \in \mathcal{B} (x \in B \subset U)\}.$$

If  $\mathcal{B}$  satisfies the following two conditions:

(B1) For every  $x \in X$  there is a  $B \in \mathcal{B}$  with  $x \in B$  (i.e.,  $\bigcup \mathcal{B} = X$ ).

(B2) For every  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \cap B_2$  there is a  $B \in \mathcal{B}$  with  $x \in B \subset B_1 \cap B_2$ .

then  $\mathcal{T}(\mathcal{B})$  is a topology on  $X$  and  $\mathcal{B}$  is a basis for  $\mathcal{T}(\mathcal{B})$ .

Restate Lemma 13.3.

**(Partially) new material:**

**Example 9** Three examples of topologies on  $\mathbb{R}$ , defined via bases:

- **Standard topology**, generated by basis  $\mathcal{B}_{st} = \{(a, b) : a, b \in \mathbb{R}, a < b\}$ , that is, the topology  $\mathcal{T}_{st} = \mathcal{T}(\mathcal{B}_{st})$ . We usually write just  $\mathbb{R}$  for  $\langle \mathbb{R}, \mathcal{T}_{st} \rangle$ .  
Notice, that this is the same topology that was used in Theorem 2.
- **Lower limit (or Sorgenfrey) topology  $\mathcal{T}_\ell$**  is generated by basis  $\mathcal{B}_\ell = \{[a, b) : a, b \in \mathbb{R}, a < b\}$ , that is,  $\mathcal{T}_\ell = \mathcal{T}(\mathcal{B}_\ell)$ . We usually write  $\mathbb{R}_\ell$  for  $\langle \mathbb{R}, \mathcal{T}_\ell \rangle$ .
- **K-topology  $\mathcal{T}_K$ :** Let  $K = \{1/n : n = 1, 2, 3, \dots\}$ . Then  $\mathcal{T}_K$  is generated by basis  $\mathcal{B}_K = \mathcal{B}_{st} \cup \{(a, b) \setminus K : a, b \in \mathbb{R}, a < b\}$ , that is,  $\mathcal{T}_K = \mathcal{T}(\mathcal{B}_K)$ . We usually write  $\mathbb{R}_K$  for  $\langle \mathbb{R}, \mathcal{T}_K \rangle$ .

**Fact 4** (Lemma 13.4)  $\mathcal{T}_\ell$  and  $\mathcal{T}_K$  are strictly finer than  $\mathcal{T}_{st}$ .

Definition of *subbasis* for a topology.

Note that  $\mathcal{S} = \{(a, \infty) : a \in \mathbb{R}\} \cup \{(-\infty, b) : b \in \mathbb{R}\}$  is a subbasis for  $\mathbb{R}$  (with the standard topology).

Go over exercises 1, 3, 6. Possibly, also exercises 4 and/or 5.

**Class of September 1:**

Go briefly over:

**Section 14: Order Topology:** For linearly ordered set  $\langle X, \leq \rangle$ , order topology is generated by subbasis  $\mathcal{S} = \{(a, \infty): a \in X\} \cup \{(-\infty, b): b \in X\}$ .

Describe basis for  $X$ . (Definition, page 84.)

Go over examples 1-4.

**Section 15: Product Topology on  $X \times Y$** 

**Definition 3** For topological spaces  $\langle X, \mathcal{T}_1 \rangle$  and  $\langle Y, \mathcal{T}_2 \rangle$  let  $\mathcal{B}_{\mathcal{T}_1, \mathcal{T}_2}$  be the family of all open rectangles, that is,

$$\mathcal{B}_{\mathcal{T}_1, \mathcal{T}_2} = \{U \times V: U \in \mathcal{T}_1 \text{ \& } V \in \mathcal{T}_2\}.$$

Note that  $\mathcal{B}_{\mathcal{T}_1, \mathcal{T}_2}$  satisfies conditions (B1) and (B2) for a topology on  $X \times Y$ . So, the family  $\mathcal{T}(\mathcal{B}_{\mathcal{T}_1, \mathcal{T}_2})$  is a topology on  $X \times Y$ .

The topology  $\mathcal{T}(\mathcal{B}_{\mathcal{T}_1, \mathcal{T}_2})$  is called the *product topology* on  $X \times Y$ .

Note that, in general,

$$\mathcal{T}(\mathcal{B}_{\mathcal{T}_1, \mathcal{T}_2}) \neq \mathcal{B}_{\mathcal{T}_1, \mathcal{T}_2},$$

since,  $\mathcal{B}_{\mathcal{T}_1, \mathcal{T}_2}$  is not closed under unions, as, usually,  $(U_1 \times V_1) \cup (U_2 \times V_2)$  is not a rectangle (so, it does not belong to  $\mathcal{B}_{\mathcal{T}_1, \mathcal{T}_2}$ ).

**Theorem 10** If  $\mathcal{B}_1$  is a basis for  $\langle X, \mathcal{T}_1 \rangle$  and  $\mathcal{B}_2$  is a basis for  $\langle Y, \mathcal{T}_2 \rangle$ , then the family

$$\mathcal{B}_{\mathcal{B}_1, \mathcal{B}_2} = \{U \times V: U \in \mathcal{B}_1 \text{ \& } V \in \mathcal{B}_2\}$$

is a basis for the product topology on  $X \times Y$ .

**Corollary 11** (Example 1) The family  $\mathcal{B} = \{(a, b) \times (c, d): a, b, c, d \in \mathbb{R}\}$  is a basis for the product topology on  $\mathbb{R} \times \mathbb{R}$ , where  $\mathbb{R}$  is considered with the standard topology. Thus, the product topology on  $\mathbb{R} \times \mathbb{R}$  coincides with the standard topology  $\mathcal{T}(\mathcal{B})$  on  $\mathbb{R} \times \mathbb{R}$ .

**Definition 4** For the Cartesian product  $X_1 \times X_2$  define the *projection function*  $\pi_1: X_1 \times X_2 \rightarrow X_1$  onto the first coordinate as  $\pi_1(x_1, x_2) = x_1$ . Similarly, the projection onto the second coordinate is the function  $\pi_2: X_1 \times X_2 \rightarrow X_2$  defined as  $\pi_2(x_1, x_2) = x_2$ .

Notice that for  $U \subset X_1$  and  $V \subset X_2$  we have

$$\pi_1^{-1}(U) = U \times X_2 \quad \text{and} \quad \pi_2^{-1}(V) = X_1 \times V.$$

In particular, for topological spaces  $\langle X_1, \mathcal{T}_1 \rangle$  and  $\langle X_2, \mathcal{T}_2 \rangle$ , the family

$$\mathcal{S} = \{\pi_i^{-1}(W) : i \in \{1, 2\} \ \& \ W \in \mathcal{T}_i\}$$

forms a subbasis for the product topology on  $X_1 \times X_2$ , since we have the identity  $\pi_1^{-1}(U) \cap \pi_2^{-1}(V) = U \times V$ .

**Class of September 6:**

Recall that:

- For the topological spaces  $\langle X, \mathcal{T}_1 \rangle$  and  $\langle Y, \mathcal{T}_2 \rangle$ , the *product topology* on  $X \times Y$  is generated by a basis:  $\mathcal{B}_{\mathcal{T}_1, \mathcal{T}_2} = \{U \times V : U \in \mathcal{T}_1 \text{ \& } V \in \mathcal{T}_2\}$ .
- If  $\mathcal{B}_1$  is a basis for  $\langle X, \mathcal{T}_1 \rangle$  and  $\mathcal{B}_2$  is a basis for  $\langle Y, \mathcal{T}_2 \rangle$ , then the family  $\mathcal{B}_{\mathcal{B}_1, \mathcal{B}_2} = \{U \times V : U \in \mathcal{B}_1 \text{ \& } V \in \mathcal{B}_2\}$  is a basis for the product topology on  $X \times Y$ .

**Section 16: Subspace Topology**

**Definition 5** Let  $\langle X, \mathcal{T} \rangle$  be a topological space and  $Y$  be any subset of  $X$  (containing at least two points). Then the family

$$\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\}$$

forms a topology on  $Y$  called the *subspace topology*.

**Lemma 12** If  $\mathcal{B}$  is a basis for a topological space  $\langle X, \mathcal{T} \rangle$  and  $Y \subset X$ , then the family

$$\mathcal{B}_Y = \{Y \cap B : B \in \mathcal{B}\}$$

is a basis for  $\langle Y, \mathcal{T}_Y \rangle$ .

Go over Lemma 16.2 and Example 1.

Discuss briefly Theorem 16.4.

**Theorem 13** (Theorem 16.3) Let  $\langle A, \mathcal{T}_A \rangle$  be a subspace of  $\langle X, \mathcal{T}_1 \rangle$  and  $\langle B, \mathcal{T}_B \rangle$  be a subspace of  $\langle Y, \mathcal{T}_2 \rangle$ . Then the following two topologies on  $A \times B$  coincide:

- $\mathcal{T}_{A \times B}$ , the subspace topology of the product topology on  $X \times Y$ ;
- $\mathcal{T}(\mathcal{B}_{\mathcal{T}_A, \mathcal{T}_B})$ , the product topology for the spaces  $\langle A, \mathcal{T}_A \rangle$  and  $\langle B, \mathcal{T}_B \rangle$ .

Go over Examples 2 & 3 and discuss Theorem 13.

Go over Exercises 1 and 4.

**Written assignment for Tuesday, September 13:** Exercises 8 and 9, page 92.

**Class of September 8:**

Administer Quiz # 2. Hand solution to bonus homework of 8/18.

Next class I will hand you solutions for homework assigned on 8/25.

Recall that:

- If  $\langle X, \mathcal{T} \rangle$  is a topological space and  $Y \subset X$ , then  $\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\}$  is the *subspace topology* on  $Y$ .
- If  $\mathcal{B}$  is a basis for a topological space  $\langle X, \mathcal{T} \rangle$  and  $Y \subset X$ , then the family  $\mathcal{B}_Y = \{Y \cap B : B \in \mathcal{B}\}$  is a basis for  $\langle Y, \mathcal{T}_Y \rangle$ .

New material:

Ex. 10. p. 92: Let  $I = [0, 1]$ . Compare the following topologies on  $I^2$ : the standard product topology  $\tau_{st}$ , the dictionary order topology  $\tau_{\leq}$ , and the subspace topology  $\tau_{\leq}^*$  of  $\mathcal{T}_{\leq}$ .

PROOF. In the proof, we will use the following two facts, mentioned many times in class. (For notation, see lecture for Section 13.)

- If  $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{P}(X)$ , then  $\mathcal{T}(\mathcal{B}_0) \subset \mathcal{T}(\mathcal{B}_1)$ .
- If  $\mathcal{T}_0$  is a topology (on  $X$ ), then  $\mathcal{T}(\mathcal{T}_0) = \mathcal{T}_0$ .

Property (i) holds, as  $\mathcal{T}(\mathcal{B}_0) = \{\bigcup \mathcal{B} : \mathcal{B} \subset \mathcal{B}_0\} \subset \{\bigcup \mathcal{B} : \mathcal{B} \subset \mathcal{B}_1\} = \mathcal{T}(\mathcal{B}_1)$ . Property (ii) holds, since the family  $\mathcal{B}_0 = \mathcal{T}_0$  is a basis for  $\mathcal{T}_0$ , and so  $\mathcal{T}(\mathcal{T}_0) = \mathcal{T}(\mathcal{B}_0) = \mathcal{T}_0$ .

We will prove, that the only inclusions between the topologies are  $\tau_{st} \subset \tau_{\leq}^*$  and  $\tau_{\leq} \subset \tau_{\leq}^*$ .

$\tau_{st} \subset \tau_{\leq}^*$ : By Theorem 15.1 the family  $\mathcal{B}_{st} = \{(a, b) \times (c, d) : a, b, c, d \in \mathbb{R}\}$  is a basis for  $\mathbb{R}^2$  with the standard topology  $\mathcal{T}_{st}$ . Hence, by Lemma 16.1, the family  $\mathcal{D}_{st} = \{B \cap I^2 : B \in \mathcal{B}_{st}\}$  forms a basis for  $\tau_{st}$ .

Next, notice that  $\mathcal{D}_{st} \subset \tau_{\leq}^*$ . Indeed, if  $[(a, b) \times (c, d)] \cap I^2 \in \mathcal{D}_{st}$  and  $x \in (a, b)$ , then  $\{x\} \times (c, d) = (\langle x, c \rangle, \langle x, d \rangle)$  is a basic open set for the dictionary order topology on  $\mathbb{R}^2$  so  $[(a, b) \times (c, d)] \cap I^2 = \bigcup_{x \in (a, b)} [\{x\} \times (c, d)] \cap I^2 \in \tau_{\leq}^*$ .

Hence, by (i) and (ii),  $\tau_{st} = \mathcal{T}(\mathcal{D}_{st}) \subset \mathcal{T}(\tau_{\leq}^*) = \tau_{\leq}^*$ .

$\tau_{\leq} \subset \tau_{\leq}^*$ : Notice that  $\mathcal{D}_{\leq} = \{(\langle a, b \rangle, \langle c, d \rangle) \cap I^2 : a, b, c, d \in I\}$  is a basis for  $\tau_{\leq}$  (straight from the definition of order topology) while, by Lemma 16.1,  $\mathcal{D}_{\leq}^* = \{(\langle a, b \rangle, \langle c, d \rangle) \cap I^2 : a, b, c, d \in \mathbb{R}\}$  is a basis for  $\tau_{\leq}^*$ . Clearly,  $\mathcal{D}_{\leq} \subset \mathcal{D}_{\leq}^*$ . Therefore, by (i),  $\tau_{\leq} = \mathcal{T}(\mathcal{D}_{\leq}) \subset \mathcal{T}(\mathcal{D}_{\leq}^*) = \tau_{\leq}^*$ , as desired.

To finish the argument, we need to show that the topologies  $\tau_{st}$  and  $\tau_{\leq}$  and not comparable. Indeed,  $\tau_{st} \not\subset \tau_{\leq}$  since a set  $[0, 1]^2 = (-1, 1)^2 \cap I^2 \in \tau_{st}$  but it does not belong to  $\tau_{\leq}$  since there is no  $J \in \mathcal{D}_{\leq}$  with  $\langle .5, 0 \rangle \in J \subset [0, 1]^2$  (as any  $J \in \mathcal{D}_{\leq}$  containing  $\langle .5, 0 \rangle$  must contain also  $\langle x, 1 \rangle$  for some  $x \in (0, .5)$ ).

Similarly,  $\tau_{\leq} \not\subset \tau_{st}$ , as  $\{0\} \times (0, 1) = (\langle 0, 0 \rangle, \langle 0, 1 \rangle) \in \tau_{\leq}$  does not belong to  $\tau_{st}$ . ■

## Section 17: Closed sets; Closure and Interior of a Set

**Definition 6** A set  $A \subset X$  is *closed* in the topological space  $\langle X, \mathcal{T} \rangle$  if its complement  $X \setminus A$  is open.

Go over Examples 1-5.

Go over Theorem 17.1.

Go over Exercise 1.

**Theorem 14** (Theorem 17.2) Let  $Y$  be a subspace of  $X$ . Then,  $A \subset Y$  is closed in  $Y$  iff  $A = Y \cap F$  for some closed subset  $F$  of  $X$ .

Go over Theorem 17.3.

Go over Exercises 2, 3, and 4.

**Class of September 13:**

Recall, from the last class:

- A set  $A \subset X$  is *closed* in the topological space  $\langle X, \mathcal{T} \rangle$  if its complement  $X \setminus A$  is open.
- (Theorem 17.2) Let  $Y$  be a subspace of  $X$ . Then,  $A \subset Y$  is closed in  $Y$  iff  $A = Y \cap F$  for some closed subset  $F$  of  $X$ .

New material:

**Definition 7** Let  $A \subset X$  be a subset of a topological space  $\langle X, \mathcal{T} \rangle$ .

- The *interior* of  $A$ , denoted as  $\text{int}(A)$ , is defined as a union of all open subsets contained in  $A$ , that is,  $\text{int}(A) = \bigcup \{U \in \mathcal{T} : U \subset A\}$ .

Notice that  $\text{int}(A)$  is open and that it is the largest open subset of  $A$ .

- The *closure* of  $A$ , denoted either as  $\text{cl}(A)$  or as  $\bar{A}$ , is defined as an intersection of all closed subsets containing in  $A$ , that is,  $\text{cl}(A) = \bigcap \{F \supset A : F \text{ is closed in } X\}$ .

Notice that  $\text{cl}(A)$  is closed and that it is the smallest closed set containing  $A$ .

We will sometimes use symbols  $\text{int}_X(A)$  and  $\text{cl}_X(A)$  in place of  $\text{int}(A)$  and  $\text{cl}(A)$  to stress that the operation is with respect to the given topology on  $X$ .

Go over Exercise 6(a) and (b).

**Theorem 15** (Theorem 17.4) Let  $Y$  be a subspace of  $X$  and  $A \subset Y$ . Then  $\text{cl}_Y(A) = Y \cap \text{cl}_X(A)$ .

**Theorem 16** (Theorem 17.5) Let  $A \subset X$  be a subset of a topological space  $\langle X, \mathcal{T} \rangle$  and  $\mathcal{B}$  be a basis for  $X$ . Then

$x \in \text{cl}(A)$  if, and only if,  $A \cap B \neq \emptyset$  for every  $B \in \mathcal{B}$  with  $x \in B$ .

In particular, the result is true with  $\mathcal{B} = \mathcal{T}$ .

Go over Exercises 8 and 9; also Examples 6 and 7.

Let  $A = K \cup (2, 3)$ , where  $K = \{1/n : n \in \{1, 2, 3, \dots\}\}$ . Find the closures of  $A$  in:  $\mathbb{R}$  (i.e.,  $\mathbb{R}$  with the standard topology),  $\mathbb{R}_\ell$ ,  $\mathbb{R}_d$  (i.e.,  $\mathbb{R}$  with the discrete topology), and  $\mathbb{R}_K$ .

Answer:  $\text{cl}_{\mathbb{R}}(A) = \{0\} \cup K \cup [2, 3]$ ;  $\text{cl}_{\mathbb{R}_\ell}(A) = \{0\} \cup K \cup [2, 3]$ ;  $\text{cl}_{\mathbb{R}_d}(A) = A$ ;  $\text{cl}_{\mathbb{R}_K}(A) = K \cup [2, 3]$ ;

**Written assignment due Thursday, Sept. 15:** Exercise 6(c), p. 101.

**Class of September 15:**

Recall, from the last class:

- The *interior* of  $A$  is  $\text{int}(A) = \bigcup\{U \in \mathcal{T}: U \subset A\}$ .
- The *closure* of  $A$  is  $\text{cl}(A) = \bigcap\{F \supset A: F \text{ is closed in } X\}$ .
- If  $Y$  be is subspace of  $X$  and  $A \subset Y$ , then  $\text{cl}_Y(A) = Y \cap \text{cl}_X(A)$ .
- If  $A \subset X$  and  $\mathcal{B}$  is a basis for  $X$ , then

$x \in \text{cl}(A)$  if, and only if,  $A \cap B \neq \emptyset$  for every  $B \in \mathcal{B}$  with  $x \in B$ .

**New material: Section 17, Limit Points**

**Definition 8** Let  $A \subset X$  be a subset of a topological space  $\langle X, \mathcal{T} \rangle$ . A point  $x \in X$  is a *limit point* (or *accumulation point*) of  $A$  provided  $x \in \text{cl}(A \setminus \{x\})$ . The set of all limit points of  $A$  is denoted as  $A'$ .

Go over Example 8.

**Theorem 17** (Theorem 17.6) Let  $A$  be a subset of a topological space  $\langle X, \mathcal{T} \rangle$ . Then  $\text{cl}(A) = A \cup A'$ .

**Theorem 18** (Theorem 17.7) Let  $A$  be a subset of a topological space  $\langle X, \mathcal{T} \rangle$ . Then  $A$  is closed in  $X$  if, and only if,  $A' \subset A$ .

**Section 17: Hausdorff spaces**

**Definition 9** Let  $\langle X, \mathcal{T} \rangle$  be a topological space. We say that:

- $X$  is *Hausdorff* (or a  $T_2$  space) provided for every distinct  $x, y \in X$  there exists disjoint open sets  $U, V \subset X$  such that  $x \in U$  and  $y \in V$ .
- $X$  is a  $T_1$  space provided for every distinct  $x, y \in X$  there exists an open set  $U \subset X$  such that  $x \in U$  and  $y \notin U$ .
- $X$  is a  $T_0$  space provided for every distinct  $x, y \in X$  there exists an open set  $U \subset X$  such that either  $x \in U$  and  $y \notin U$  or  $y \in U$  and  $x \notin U$  (i.e., such that  $U$  contains precisely one of the points  $x$  and  $y$ ).

Notice that if  $X$  is  $T_2$  then it is also  $T_1$ , and if  $X$  is  $T_1$  then it is also  $T_0$ .

Examples:

- A space  $X$  with a trivial topology  $\mathcal{T} = \{\emptyset, X\}$  is not  $T_0$ .
- $X = \{0, 1\}$  with a topology  $\mathcal{T} = \{\emptyset, \{0\}, X\}$  is  $T_0$  but not  $T_1$ .
- $X = \mathbb{R}$  with a cofinite topology  $\mathcal{T} = \{\emptyset\} \cup \{X \setminus F : F \text{ is finite}\}$  is  $T_1$  but not  $T_2$ .
- The following spaces are  $T_2$ : any space with the discrete topology,  $\mathbb{R}$  with the standard topology,  $\mathbb{R}_\ell$ ,  $\mathbb{R}_K$ .

**Theorem 19** (Exercise 15) *A space  $X$  is  $T_1$  if, and only if, every finite subset of  $X$  is closed.*

**Corollary 20** (Theorem 17.8) *Every finite subset in a Hausdorff space is closed.*

**Theorem 21** (Theorem 17.9) *Let  $X$  be a  $T_1$  topological space and  $A \subset X$ . Then  $x \in A'$  if, and only if,  $U \cap A$  is infinite for every open  $U$  containing  $x$ .*

**Definition 10** Let  $X$  be a topological. We say that  $x \in X$  is an *isolated point* provided  $\{x\}$  is open in  $X$ .

**Remark 22** If  $X$  is  $T_1$  and an open set  $U$  is finite, then every  $x \in U$  is isolated.

**Definition 11** Let  $X$  be a topological. We say that a sequence  $\langle x_n \rangle_{n=1}^\infty$  of points of  $X$  *converges* to an  $x \in X$  provided for every open set  $U$  containing  $x$  there exists an  $N$  such that  $x_n \in U$  for every  $n \geq N$ .

If this is the case, we say also, that  $x$  is a *limit* of a sequence  $\langle x_n \rangle_{n=1}^\infty$ .

**Theorem 23** (Theorem 17.10) *If  $X$  is a Hausdorff topological space, then any sequence  $\langle x_n \rangle_{n=1}^\infty$  of points of  $X$  converges to at most one point in  $X$ .*

**Class of September 20:**

Recall that for a topological space  $\langle X, \mathcal{T} \rangle$  and  $A \subset X$ :

- $A' = \{x \in X : x \in \text{cl}(A \setminus \{x\})\}$ .
- $X$  is a  $T_1$  space provided for every distinct  $x, y \in X$  there exists an open set  $U \subset X$  such that  $x \in U$  and  $y \notin U$ . Equivalently,  $X$  is  $T_1$  if, and only if, every singleton is closed in  $X$ .
- If  $X$  is a  $T_1$  space and  $A \subset X$ , then  
 $x \in A'$  if, and only if,  $U \cap A$  is infinite for every open  $U$  containing  $x$ .
- $X$  is Hausdorff (or a  $T_2$  space) provided for every distinct  $x, y \in X$  there exists disjoint open sets  $U, V \subset X$  such that  $x \in U$  and  $y \in V$ .
- A sequence  $\langle x_n \rangle_{n=1}^{\infty}$  of points of  $X$  converges to an  $x \in X$  provided for every open  $U \ni x$  there exists an  $N$  such that  $x_n \in U$  for every  $n \geq N$ .
- (Theorem 17.10) If  $X$  is a Hausdorff topological space, then any sequence  $\langle x_n \rangle_{n=1}^{\infty}$  of points of  $X$  converges to at most one point in  $X$ .

**New material**

Review the proof of Theorem 21 (17.10).

Example: (Exercise 14) Theorem 21 (17.10) is false for  $T_1$  spaces. For example, if  $X = \mathbb{R}$  is considered with the cofinite topology (which is  $T_1$ ) and  $x_n = 1/n$  for every  $n$ , then every real number is a limit of  $\langle x_n \rangle_{n=1}^{\infty}$ .

**Theorem 24** (Theorem 17.11) *The product of two Hausdorff topological spaces is a Hausdorff space. A subspace of a Hausdorff topological space is a Hausdorff space.*

Go over Exercise 10.

**Written assignment due Thursday, Sept. 22:**    Exercise 11, page 101.

**Written assignment due Tuesday, Sept. 27:**    Exercise 13, page 101.

## Section 18: Continuous functions

**Definition 12** Let  $X$  and  $Y$  be the topological spaces. A function  $f: X \rightarrow Y$  is *continuous* provided  $f^{-1}(V)$  is open in  $X$  for every open subset  $V$  of  $Y$ .

Notice, that the definition agrees with (a) from Theorem 2.

**Theorem 25** Let  $X$  and  $Y$  be the topological spaces and  $\mathcal{B}$  a basis for  $Y$ . Then  $f: X \rightarrow Y$  is continuous if, and only if,  $f^{-1}(B)$  is open in  $X$  for every  $B \in \mathcal{B}$ .

Similarly, if  $\mathcal{S}$  is a subbasis for  $Y$ , then  $f: X \rightarrow Y$  is continuous if, and only if,  $f^{-1}(S)$  is open in  $X$  for every  $S \in \mathcal{S}$ .

Example 3:

- $f: \mathbb{R} \rightarrow \mathbb{R}_\ell$ ,  $f(x) = x$ , is discontinuous, as  $f^{-1}([0, 1)) = [0, 1)$  is not open in  $\mathbb{R}$ .
- $f: \mathbb{R}_\ell \rightarrow \mathbb{R}$  is continuous, as  $f^{-1}(U) = U \in \mathcal{T}_{st} \subset \mathcal{T}_\ell$  for every  $U \in \mathcal{T}_{st}$ .

Go over Exercise 3(a).

**Class of September 22:**

Recall that:

- The product of two Hausdorff topological spaces is a Hausdorff space. A subspace of a Hausdorff topological space is a Hausdorff space.
- A function  $f: X \rightarrow Y$  is *continuous* provided  $f^{-1}(V)$  is open in  $X$  for every open subset  $V$  of  $Y$ .
- If  $\mathcal{B}$  a basis for  $Y$ , then  $f: X \rightarrow Y$  is continuous if, and only if,  $f^{-1}(B)$  is open in  $X$  for every  $B \in \mathcal{B}$ .

**New material**

Go over Theorem 18.1. (Very important!)

Stress continuity at a point.

Go over Exercise 2.

**Section 18: Homeomorphisms**

**Definition 13** Let  $X$  and  $Y$  be the topological spaces and let  $f: X \rightarrow Y$  be a bijection (i.e., one-to-one and onto). Then  $f$  is a *homeomorphism* (from  $X$  onto  $Y$ ) provided both  $f$  and  $f^{-1}: Y \rightarrow X$  are continuous.

Topological spaces  $X$  and  $Y$  are *homeomorphic* provided there is a homeomorphism from  $X$  onto  $Y$ .

**Fact.** If  $f: X \rightarrow Y$  is homeomorphism, then  $U \subset X$  is open in  $X$ , if, and only if,  $f[U]$  is open in  $Y$ . In particular, if  $\tau$  is a topology on  $X$  and  $\mathcal{T}$  is a topology on  $Y$ , then  $\mathcal{T} = \{f[U]: U \in \tau\}$  and  $\tau = \{f^{-1}[V]: V \in \mathcal{T}\}$ .

PROOF. Notice that  $(f^{-1})^{-1} = f$ .

If  $U \in \tau$ , then, since  $f^{-1}: Y \rightarrow X$  is continuous,  $f[U] = (f^{-1})^{-1}(U) \in \mathcal{T}$ .

If  $f[U] \in \mathcal{T}$ , then, since  $f: X \rightarrow Y$  is continuous,  $U = f^{-1}(f[U]) \in \tau$ . ■

Go over Examples 4-6.

**Class of September 27:****In class mid term test will be on Tuesday, October 4**

We will start test 15-30 minutes earlier, that is, between 6:00 and 6:15pm.  
You will be able to take up to 2 hours to complete it.

**Solutions for the remaining homework will be given next class**

Recall that:

- Spaces  $X$  and  $Y$  are *homeomorphic* provided there exists a *homeomorphism*  $f: X \rightarrow Y$ , that is, a bijection such that both  $f$  and  $f^{-1}: Y \rightarrow X$  are continuous.
- **Fact.** If  $f: \langle X, \tau \rangle \rightarrow \langle Y, \mathcal{T} \rangle$  is a homeomorphism, then  $U \in \tau$ , if, and only if,  $f[U] \in \mathcal{T}$ . In particular,  $\mathcal{T} = \{f[U]: U \in \tau\}$  and  $\tau = \{f^{-1}[V]: V \in \mathcal{T}\}$ .

**PROOF OF Fact.** Notice that  $(f^{-1})^{-1} = f$ .

If  $U \in \tau$ , then, since  $f^{-1}: Y \rightarrow X$  is continuous,  $f[U] = (f^{-1})^{-1}(U) \in \mathcal{T}$ .

If  $f[U] \in \tau$ , then, since  $f: X \rightarrow Y$  is continuous,  $U = f^{-1}(f[U]) \in \tau$ . ■

A mapping  $f: X \rightarrow Y$  is an *imbedding* provided  $f$  is injective (i.e., one-to-one), continuous, and  $f^{-1}: f[X] \rightarrow X$  is also continuous. In such a case a mapping  $f': X \rightarrow f[X]$ ,  $f'(x) = f(x)$ , is a homeomorphism (from  $X$  onto  $f[X]$ ).

Go over Exercises 4, 5, and 6.

**Section 18: Constructing Continuous Functions**

Go over Theorem 18.2.

Go over Theorem 18.3 (The pasting Lemma).

**Class of September 29:**

Recall that the next class will be the mid term test, starting 15-30 minutes earlier. Recall also:

- (i) Composition of two continuous functions is continuous.
- (ii) If  $f: X \rightarrow Y$  is continuous and  $A \subset X$ , then  $f \upharpoonright A: A \rightarrow Y$ , the restriction of  $f$  to  $A$ , is also continuous.
- (iii) Let  $\{U_j: j \in J\}$  be a family of open sets such that  $X = \bigcup_{j \in J} U_j$ . Then, the map  $f: X \rightarrow Y$  is continuous if, and only if,  $f \upharpoonright U_i$  is continuous for every  $j \in J$ .
- (iv) **The pasting lemma.** Let  $X = A \cup B$ , where  $A$  and  $B$  are closed in  $X$ . Let  $f: A \rightarrow Y$  and  $g: B \rightarrow Y$  be continuous. If  $f(x) = g(x)$  for every  $x \in A \cap B$ , then the function  $h: X \rightarrow Y$ , defined as  $h(x) = f(x)$  for  $x \in A$  and  $h(x) = g(x)$  for  $x \in B$ , is continuous.

PROOF OF (iii): Implication  $\implies$  is (ii). To show the other implication, let  $V$  be an arbitrary open subset of  $Y$ . It is enough to show, that  $f^{-1}(V)$  is open in  $X$ . Indeed, this is the case, since

$$f^{-1}(V) = f^{-1}(V) \cap \bigcup_{j \in J} U_j = \bigcup_{j \in J} (U_j \cap f^{-1}(V)) = \bigcup_{j \in J} (U_j \cap (f \upharpoonright U_j)^{-1}(V))$$

is open in  $X$ , since each set  $U_j \cap (f \upharpoonright U_j)^{-1}(V)$  is open in  $U_j$  (as  $f \upharpoonright U_i$  is continuous) and so, it is also open in  $X$ , as  $U_j$  is open in  $X$ .

PROOF OF (iv) is similar.

**New material**

Go over Example 8; Theorem 18.4, and Exercise 11.

Variant of Exercise 12, with  $f(x, y) = \frac{xy^2}{x^2+y^4}$  for  $\langle x, y \rangle \neq \langle 0, 0 \rangle$  and  $f(0, 0) = 0$ . Show that it is discontinuous (on curve  $y^2 = x$ ), but  $f \upharpoonright L$  is continuous for every straight line  $L$ .

**Key elements to review for the test:** (1) Any homework assignment. (They will not be a part of the test. However, some problems will be similar to homework.) (2) All definitions. (3) All theorems, with special emphasis on: continuous functions, closure and interior operations, Hausdorff and  $T_1$  spaces, as well as subspaces and product spaces.

**Class of October 4:** In class Mid Term Test.

**Class of October 6:**

Hand the results of the test and the solutions for its exercises.

Discuss test results and, in general, the course standings.

New materisl:

### Section 19: The product topology

**Definition 14** For sets  $J$  and  $X$  let  $X^J$  denotes the family of all functions  $f: J \rightarrow X$ .

Let  $\{A_\alpha\}_{\alpha \in J}$  be an arbitrary indexed family of sets and let  $X = \bigcup_{\alpha \in J} A_\alpha$ . (Notice that the index set  $J$  may be uncountable!) The *Cartesian product* of the family  $\{A_\alpha\}_{\alpha \in J}$ , denoted by  $\prod_{\alpha \in J} A_\alpha$ , is defined as

$$\prod_{\alpha \in J} A_\alpha = \{f \in X^J : f(j) \in A_j \text{ for all } j \in J\}.$$

Elements of  $\{A_\alpha\}_{\alpha \in J}$  will be also sometimes denotes as  $\langle a_\alpha \rangle_{\alpha \in J}$  and referred to as *J-tuples*.

Notice that  $X^J = \prod_{\alpha \in J} A_\alpha$ , where  $A_\alpha = X$  for every  $\alpha \in J$ .

Notice, that this definition agrees the definition of the finite cartesian product (over the set  $J = \{1, \dots, n\}$ )  $\prod_{i=1}^n A_i = A_1 \times \dots \times A_n$  as the set of all sequences  $\langle a(1), \dots, a(n) \rangle$  with  $a(i) \in A_i$ , since any such sequence can be considered as a function  $a: \{1, \dots, n\} \rightarrow X$ . Similar agreement is also for  $J = \{1, 2, 3, \dots\}$ .

**Definition 15** Let  $\{X_\alpha\}_{\alpha \in J}$  be an indexed family of topological spaces. Then, on the product space  $X = \prod_{\alpha \in J} X_\alpha$ , we define the following two kinds of topologies.

**box topology**  $\mathcal{T}_{box}$ : Generated by a basis  $\mathcal{B}_{box}$  formed by all sets of the form

$$\prod_{\alpha \in J} U_\alpha \quad \text{where each } U_\alpha \text{ is open in } X_\alpha.$$

**product topology**  $\mathcal{T}_{prod}$ : Generated by a subbasis  $\mathcal{S}$  formed by all sets of the form

$$\pi_\beta^{-1}(U_\beta) \quad \text{for all } \beta \in J \text{ and open subsets } U_\beta \text{ of } X_\beta,$$

where  $\pi_\beta: X \rightarrow X_\beta$  is the *projection* onto  $\beta$ th coordinate, that is, defined as  $\pi_\beta(x) = x(\beta)$ .

Notice that  $\pi_\beta^{-1}(U_\beta) = \prod_{\alpha \in J} U_\alpha$ , where  $U_\alpha = X_\alpha$  for all  $\alpha \neq \beta$ .

A natural basis,  $\mathcal{B}_{prod}$  associated with  $\mathcal{S}$  is formed by finite intersections of sets from  $\mathcal{S}$ , that is, all sets of the form  $\prod_{\alpha \in J} U_\alpha$  where each  $U_\alpha$  is open in  $X_\alpha$  and *the set*  $\{\alpha \in J: U_\alpha \neq X_\alpha\}$  *is finite*.

Go over Theorem 19.6 and Example 2.

**Class of October 11:**

Recall that for  $X = \prod_{\alpha \in J} X_\alpha$ , each  $X_\alpha$  being a topological space,

- *Box topology*  $\mathcal{T}_{box}$  on  $X$  is generated by basis  
 $\mathcal{B}_{box} = \{\prod_{\alpha \in J} U_\alpha : \text{each } U_\alpha \text{ is open in } X_\alpha\}$ .
- *Product topology*  $\mathcal{T}_{prod}$  on  $X$  is generated by subsbasis  
 $\mathcal{S}_{prod} = \{\pi_\beta^{-1}(U_\beta) \text{ for all } \beta \in J \text{ and open subsets } U_\beta \text{ of } X_\beta\}$   
 or, equivalently, by a basis  
 $\mathcal{B}_{prod} = \{\prod_{\alpha \in J} U_\alpha \in \mathcal{B}_{box} : U_\alpha = X_\alpha \text{ for all but finitely many } \alpha\}$ .
- $\mathcal{T}_{prod} \subset \mathcal{T}_{box}$ ; equation holds when  $J$  is finite (or all but finitely many spaces  $X_\alpha$  have trivial topology  $\{\emptyset, X_\alpha\}$ );
- If  $f_\alpha: A \rightarrow X_\alpha$  and  $f: A \rightarrow X$  is given by  $f(a)(\alpha) = f_\alpha(a)$ , then  
*continuity of  $f$  implies the continuity of each  $f_\alpha$ ;*  
*continuity of all  $f_\alpha$ 's implies the continuity of  $f: A \rightarrow \langle X, \mathcal{T}_{prod} \rangle$ ;*  
*continuity of all  $f_\alpha$ 's does not imply continuity of  $f: A \rightarrow \langle X, \mathcal{T}_{box} \rangle$*   
*(as  $f: \mathbb{R} \rightarrow \langle \mathbb{R}^\omega, \mathcal{T}_{box} \rangle$ ,  $f(x) = \langle x, x, x, \dots \rangle$  is discontinuous).*

**New material**

Go over Theorem 19.2: bases for  $\mathcal{T}_{box}$  and  $\mathcal{T}_{prod}$  in term of basis for  $X_\alpha$ 's.

Go over Theorem 19.3: subspace topology on  $A = \prod_{\alpha \in J} A_\alpha \subset X$ .

Theorem 19.4: product of Hausdorff spaces is Hausdorff ( $\mathcal{T}_{box}$  and  $\mathcal{T}_{prod}$ ).

State Theorem 19.5:  $\prod_{\alpha \in J} \text{cl}(A_\alpha) = \text{cl}(\prod_{\alpha \in J} A_\alpha)$  (in  $\mathcal{T}_{box}$  and  $\mathcal{T}_{prod}$ ).

Solve Exercise 7.

**Class of October 13:**

Recall that for  $X = \prod_{\alpha \in J} X_\alpha$ , each  $X_\alpha$  being a topological space,

- Product of Hausdorff spaces is Hausdorff ( $\mathcal{T}_{box}$  and  $\mathcal{T}_{prod}$ ).

**New material**

Prove Theorem 19.5:  $\prod_{\alpha \in J} \text{cl}(A_\alpha) = \text{cl}(\prod_{\alpha \in J} A_\alpha)$  (in  $\mathcal{T}_{box}$  and  $\mathcal{T}_{prod}$ ).

Go over Exercises 13, page 112.

Go over Exercises 10, page 112.

**Written assignment for Thursday, October 20:** Exercise 8, page 111. Part (a) is treated as a bonus homework problem. (It is easy for  $Y = \mathbb{R}$ , but you are to show the result for any ordered space  $Y$ .) In part (b) you can use the result from the part (a), even if you did not completed this part of the assignment.

**Section 20: The Metric Topology**

Define a *metric* (*distance*) on  $X$  as a function  $d: X \times X \rightarrow [0, \infty)$ .

A *metric space* is a pair  $\langle X, d \rangle$ , where  $d$  is a metric on  $X$ .

In a metric space  $\langle X, d \rangle$ , define an *open ball* (centered at  $x \in X$  with radius  $\varepsilon > 0$ ) as  $B_d(x, \varepsilon) = \{y \in X: d(x, y) < \varepsilon\}$ .

Prove that a family  $\mathcal{B}_d = \{B(x, \varepsilon): x \in X \ \& \ \varepsilon > 0\}$  is a basis for a topology on  $X$ .

Define a metric topology for a metric space  $\langle X, d \rangle$  as  $\mathcal{T}(\mathcal{B}_d)$ , that is, as a topology generated by the family of all open balls in  $\langle X, d \rangle$ .

Go over Example 1 (discrete metric) and 2 (standard metric on  $\mathbb{R}$ ).

**Definition 16** A topological space  $\langle X, \tau \rangle$  is *metrizable* provided there exists a metric  $d$  on  $X$  such that  $\tau = \mathcal{T}(\mathcal{B}_d)$ .

**Class of October 18:**

Recall that:

- A *metric space* is a pair  $\langle X, d \rangle$ , where  $d$  is a metric on  $X$ .
- $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$  is an *open ball* in  $\langle X, d \rangle$ .
- $\mathcal{B}_d = \{B(x, \varepsilon) : x \in X \text{ \& } \varepsilon > 0\}$  is a basis for a topology on  $X$ .
- $\mathcal{T}(\mathcal{B}_d)$  is the metric topology on  $X$  (for metric  $d$ ).
- $\langle X, \tau \rangle$  is *metrizable* provided  $\tau = \mathcal{T}(\mathcal{B}_d)$  for some metric  $d$  on  $X$ .

**New material**

Go over Exercise 3(a):  $d: X \times X \rightarrow \mathbb{R}$  is continuous in  $X^2$ , where  $X$  is considered with the metric topology.

PROOF. Let  $B = (a, b)$  be basic open set in  $\mathbb{R}$ . Need to prove that  $d^{-1}(B)$  is open in  $X^2$ .

Fix  $\langle x, y \rangle \in d^{-1}(B)$ . So,  $d(x, y) \in B$ . We need to find an open set  $U$  in  $X^2$  with  $\langle x, y \rangle \in U \subset d^{-1}(B)$ . Let  $\varepsilon > 0$  be such that  $(d(x, y) - \varepsilon, d(x, y) + \varepsilon) \subset B$ . Define  $U = B(x, \varepsilon/2) \times B(y, \varepsilon/2)$ . It is open in  $X^2$  and contains  $\langle x, y \rangle$ .

So, fix  $\langle z, t \rangle \in U$ . Then  $d(x, z) < \varepsilon/2$  and  $d(y, t) < \varepsilon/2$ . By the triangle inequality we get  $d(z, x) + d(x, y) + d(y, t) \geq d(z, t)$ , so

$$d(z, x) + d(y, t) \geq d(z, t) - d(x, y).$$

Similarly,  $d(x, z) + d(z, t) + d(t, y) \geq d(x, y)$ , so

$$d(x, z) + d(t, y) \geq d(x, y) - d(z, t).$$

Hence,  $|d(z, t) - d(x, y)| \leq d(x, z) + d(t, y) < \varepsilon/2 + \varepsilon/2$  and so we have  $d(z, t) \in (d(x, y) - \varepsilon, d(x, y) + \varepsilon) \subset B$ , as required.

Define: bounded set and its diameter.

Go over Theorem 20.1. (So, boundedness is not a topological property! Recall *topological property*, see page 105.)

Define Euclidean metric and square metric on  $\mathbb{R}^n$ .

Go over Theorem 20.3, using Lemma 20.2.

**Class of October 20:**

Recall

- Product and metric topologies on  $\mathbb{R}^n$  coincide. .

**New material**

Define uniform metric on  $\mathbb{R}^J$ .

State and prove Theorem 20.4 (on relations between box, uniform, and product topologies on  $\mathbb{R}^J$ ).

Go over Exercise 6.

Go over Exercise 5, page 127. Note, that this implies that, on  $\mathbb{R}^\omega$ , box, uniform, and product topologies are distinct.

Suggested to solve at home (no homework): Exercise 4 page 127.

**Class of October 25:**

Recall

- uniform metric on  $\mathbb{R}^J$  is defined as:  $\bar{\rho}(x, y) = \sup\{\bar{d}(x_\alpha, y_\alpha) : \alpha \in J\}$ ,  
where  $\bar{d}(x, y) = \min\{|x - y|, 1\}$
- uniform topology on  $\mathbb{R}^J$ : generated by  $\bar{\rho}$ .

**New material**

Go over Theorem 20.5 (countable product of metric spaces is metrizable).

**Section 21: The Metric Topology continued**

- Subspace of a metric space is metric.
- No relation between ordered topologies and metric topologies.
- Every metrizable space is Hausdorff.
- Finite and countable product of metric spaces is metrizable.

State Theorem 21.1: for metric spaces,  $\varepsilon$ - $\delta$  definition of continuity is equivalent to topological definition of continuity. (This is an obvious generalization of Theorem 2.)

**Definition 17** Let  $\langle X, \tau \rangle$  be a topological space.

- A family  $\mathcal{B}_x \subset \tau$  is a *basis (for  $X$ ) at  $x$*  provided for every open set  $U \ni x$  there is a  $B \in \mathcal{B}_x$  with  $x \in B \subset U$ .
- A topological space  $X$  is *first countable* (or *satisfies the first countability axiom*) provided for every  $x \in X$  there exists a countable basis  $\mathcal{B}_x$  of  $X$  at  $x$ .

**Proposition 26** *Every metrizable space is first countable.*

Note that for first countable spaces, a countable basis  $\{B_n : n = 1, 2, 3, \dots\}$  can be chosen monotone:  $B_1 \supset B_2 \supset B_3 \supset \dots$ .

Go over Lemma 21.2, version for first countable spaces:

**Lemma 27** *Let  $X$  be a first countable topological space and let  $A \subset X$ . Then  $x \in \text{cl}(A)$  if, and only if, there is a sequence of points of  $A$  converging to  $x$ . Moreover, the implication “ $\Leftarrow$ ” does not require the assumption of first countability.*

State and prove Theorem 21.3, version for first countable spaces.

**Class of October 27:**

Hand solution for the last homework. Recall that:

- A topological space  $\langle X, \mathcal{T} \rangle$  is *first countable* (or *satisfies the first countability axiom*) provided for every  $x \in X$  there exists a countable basis  $\mathcal{B}_x$  of  $X$  at  $x$ , that is,  $\mathcal{B}_x \subset \mathcal{T}$  and for every open set  $U \ni x$  there is a  $B \in \mathcal{B}_x$  with  $x \in B \subset U$ .
- **Lemma (21.2)** *Let  $X$  be a first countable topological space and let  $A \subset X$ . Then  $x \in \text{cl}(A)$  if, and only if, there is a sequence of points of  $A$  converging to  $x$ . Moreover, the implication “ $\Leftarrow$ ” does not require the assumption of first countability.*
- **Theorem (21.3)** *Let  $X$  and  $Y$  topological spaces and let  $f: X \rightarrow Y$ . Assume also that  $X$  is first countable. Then  $f$  is continuous if, and only if, for every sequence  $\langle x_n \rangle_n$  in  $X$  converging to an  $x \in X$ ,  $\langle f(x_n) \rangle_n$  converges to  $f(x)$ . Moreover, the implication “ $\Rightarrow$ ” does not require the assumption of first countability.*

**New material**

Go over Lemma 21.4 (no proof) and Theorem 21.5.

**Definition 18** Let  $\langle Y, d \rangle$  be a metric space,  $X$  any set, and  $f_n: X \rightarrow Y$  be a sequence of functions. We say that the sequence  $\langle f_n \rangle_n$  *converges uniformly* to an  $f: X \rightarrow Y$  provided for every  $\varepsilon > 0$  there exists an  $N$  (independent of  $x$ ) such that for every  $x \in X$

$$d(f_n(x), f(x)) < \varepsilon \quad \text{for all } n > N.$$

State Theorem 21.6: uniform limit of continuous functions is continuous.

Go over Exercise 6: *uniform* convergence assumption in Thm 21.6 is essential.

Prove Theorem 21.6.

Discuss Exer 9: the implication in Theorem 21.6 cannot be reversed.

Go over Example 1:  $\mathbb{R}^\omega$  with the box topology is not first countable. In particular, it is not metrizable.

Go over Example 2: uncountable product  $\mathbb{R}^J$ , considered with the product topology, is not first countable. In particular, it is not metrizable.

**Written assignment for Tuesday, November 1:** Exercise 7, p. 134.

Skip the rest of Chapter 2, that is, section 22.

### **Chapter 3: Connectedness and Compactness**

Stress usability of these notions to the proofs of three classical calculus theorems: *Intermediate Value Theorem*, *Maximum Value Theorem*, and *Uniform Continuity Theorem*.

Intermediate Value Theorem is a consequence of *connectedness* property.  
The other two theorems are the consequences of *compactness* property.

**Class of November 1:**

**Section 23: Connected spaces**

**Definition 19** Let  $X$  be a topological space. A *separation* of  $X$  is any pair  $\langle U, V \rangle$  of open, non-empty, disjoint sets for which  $X = U \cup V$ . A topological space  $X$  is *connected* provided it **does not** exist a separation of  $X$ .

Example 1: Any  $X$  with indiscrete topology is connected.

Any  $X$  with discrete topology is *disconnected*, that is, not connected.

Fact: A space is connected, when  $\emptyset$  and  $X$  are its only subsets that are simultaneously closed and open.

**Definition 20** Let  $Y$  be a subspace of  $X$ . A *separation* of  $Y$  is any pair  $A, B \subset Y$  non-empty sets such that  $Y = A \cup B$  and  $\text{cl}(A) \cap B = A \cap \text{cl}(B) = \emptyset$ .

**Lemma 28** A subspace  $Y$  of  $X$  is connected is, and only if, there is no separation of  $Y$ .

Go over Examples 2, 3, 4, and 5.

**Lemma 29** Assume that sets  $C$  and  $D$  forms separation of  $X$ . If a subspace  $Y$  of  $X$  is connected, then either  $Y \subset C$  or  $Y \subset D$ .

**Theorem 30 (Star Lemma)** Let  $\{A_\alpha\}_{\alpha \in J}$  be a family of connected subspaces of  $X$ . If  $\bigcap_{\alpha \in J} A_\alpha \neq \emptyset$ , then  $\bigcup_{\alpha \in J} A_\alpha$  is connected.

**Theorem 31 (Theorem 23.4)** Let  $A$  be a connected subspace of  $X$ . If  $A \subset B \subset \text{cl}(A)$ , then  $B$  is connected.

**Theorem 32 (Theorem 23.5)** Continuous image of connected space is connected.

This, together with the fact that intervals are connected, is the Intermediate Value Theorem.

**Theorem 33 (Thm 23.6)** Finite product of connected spaces is connected.

Actually, arbitrary product of connected spaces, considered with the product topology, is connected. We show this only for  $\mathbb{R}^\omega$ , Example 7. (In general, this is Exercise 10.)

Example 6:  $\mathbb{R}^\omega$  with the box topology is disconnected.

Go over Exercises 2, 7, and 8, page 152.

Suggestion to students: Look over Exercises 3, 4, and 9, page 152.

**Class of November 3:**

Recall

- A topological space  $X$  is *connected* provided it **does not** exist a separation of  $X$ , where a *separation* of  $X$  is any pair  $U, V$  of open, non-empty disjoint sets with  $X = U \cup V$ .
- **(Star Lemma)** Let  $\{A_\alpha\}_{\alpha \in J}$  be a family of connected subspaces of  $X$ . If  $\bigcap_{\alpha \in J} A_\alpha \neq \emptyset$ , then  $\bigcup_{\alpha \in J} A_\alpha$  is connected.
- A closure of a connected space is connected.
- Continuous image of connected space is connected.
- Finite product of connected spaces is connected.
- $\mathbb{R}^\omega$  with the box topology is disconnected, while with the product topology is connected.

**New material****Section 24: Connected spaces of the Real Line**

Recall that  $\mathbb{R}$  has the *least upper bound property* provided every non-empty bounded above subset  $A$  of  $\mathbb{R}$  has an upper bound  $\sup(A) \in \mathbb{R}$ .

**Theorem 34 (Theorem 24.1, for  $\mathbb{R}$  only)** *A subset  $A$  of  $\mathbb{R}$  (considered with the standard topology) is connected if, and only if,  $A$  is an interval (possibly degenerated).*

Go over the Intermediate Value Theorem, Theorem 24.3.

Go over Exercises 2 and 1.

Define *path connectedness*.

Note that every path connected space is connected.

Go over Examples 3, 4, and 5.

Go over Examples 7, *topologists sine curve*: it is connected but not path connected.

**Written assignment for Thursday, November 10:** Exercise 8, p. 158.

Suggestion to students: Look over Exercises 9, 10, and 11 page 158.

**Class of November 10:**

Recall

- Continuous image of connected space is connected.
- $A \subset \mathbb{R}$  is connected if, and only if,  $A$  is convex (an interval).
- Intermediate Value Theorem.

**New material**

Go over Exercise 3 page 158.

Section 25 will be covered next semester.

**Sections 26 and part of 27, with mixed order: compactness**

**Definition 21** Let  $Y$  be a subset of a topological space  $X$ . A family  $\mathcal{U}$  of subsets of  $X$  is a *covering* of  $Y$  provided  $Y \subset \bigcup \mathcal{U}$ . A covering  $\mathcal{U}$  of  $Y$  is an *open covering* of  $Y$  provided every  $U \in \mathcal{U}$  is open in  $X$ .

**Definition 22** A topological space  $X$  is *compact* provided for every open cover  $\mathcal{U}$  of  $X$  there exists a finite subfamily  $\mathcal{U}_0$  of  $\mathcal{U}$  that covers  $X$  (i.e.,  $\mathcal{U}_0 \subset \mathcal{U}$  is finite and  $X = \bigcup \mathcal{U}_0$ ). Such a family  $\mathcal{U}_0$  will be referred to as a (finite) *subcover* of  $\mathcal{U}$ .

**Note:** Although subcover  $\mathcal{U}_0$  of  $\mathcal{U}$  is defined in term of a union  $\bigcup \mathcal{U}_0$ , this union usually does not belong to  $\mathcal{U}$ !

Go over Examples 1 and 4: Neither  $\mathbb{R}$  nor  $(0, 1]$  are compact.Go over Examples 2 and 3: Every finite space  $X$  is compact. So is  $X = \{L\} \cup \{a_n : n = 1, 2, 3, \dots\} \subset \mathbb{R}$ , provided  $\lim_n a_n = L$ .**Lemma 35 (Lemma 26.1)**

**Theorem 36 (Theorem 26.2)** *Closed subspace of compact space is compact.*

**Theorem 37 (Theorem 26.3)** *Every compact subspace of a Hausdorff space is closed.*

Go over Example 6. Proof of the theorem is based on:

**Lemma 38 (Lemma 26.4)** *Let  $X$  be Hausdorff. For every compact subspace  $Y$  of  $X$  and every  $x \in X \setminus Y$  there exists disjoint open sets  $U$  and  $V$  in  $X$  such that  $x \in U$  and  $Y \subset V$ .*

**Theorem 39 (Theorem 27.1)** *Every closed interval  $[a, b]$  in  $\mathbb{R}$  is compact.*

**Corollary 40 (Corollary 27.3 for  $\mathbb{R}$ )** *A subspace  $X$  of  $\mathbb{R}$  is compact if, and only if, it is closed and bounded.*

**Theorem 41 (Thm 26.5)** *Continuous image of a compact space is compact.*

**Class of November 15:**

Recall

- $X$  is *compact* provided for every open cover  $\mathcal{U}$  of  $X$  contains a finite subcover  $\mathcal{U}_0$  of  $\mathcal{U}$  that covers  $X$ .
- Closed subspace of compact space is compact.
- Every compact subspace of a Hausdorff space is closed.
- For every compact subspace  $Y$  of a Hausdorff space  $X$  and every  $x \in X \setminus Y$  there are disjoint open  $U$  and  $V$  such that  $x \in U$  and  $Y \subset V$ .
- Every closed interval  $[a, b]$  in  $\mathbb{R}$  is compact.
- A subspace  $X$  of  $\mathbb{R}$  is compact if, and only if, it is closed and bounded.
- Continuous image of a compact space is compact.

**New material**

**Theorem 42 (Theorem 26.6)** *If  $f: X \rightarrow Y$  is a continuous bijection,  $X$  is compact, and  $Y$  is Hausdorff, then  $f$  is a homeomorphism.*

**Corollary 43 (Thm 27.4: Extreme Value Theorem)** *For every compact space  $X$  and continuous function  $f: X \rightarrow \mathbb{R}$  there exist  $c, d \in X$  such that  $f(c) \leq f(x) \leq f(d)$  for every  $x \in X$ . In particular, this holds for  $X = [a, b]$ .*

**Theorem 44 (Thm 26.7)** *Finite product of compact spaces is compact.*

Remark: Actually, arbitrary product of compact spaces is compact. This is Tychonoff Theorem. But its proof is more difficult.

Proof of Theorem 44 based on **very important**

**Lemma 45 (Lem 26.8: The Tube Lemma)** *Let  $Y$  be compact and  $x \in X$ . If an open set  $W$  of  $X \times Y$  contains  $\{x\} \times Y$ , then there is an open set  $U$  in  $X$  such that  $\{x\} \times Y \subset U \times Y \subset W$ .*

**Corollary 46 (Corollary 27.3 for  $\mathbb{R}^n$ )** *A subspace  $X$  of  $\mathbb{R}^n$  is compact if, and only if, it is closed and bounded.*

**Corollary 47 (Extreme Value Theorem for  $\mathbb{R}^n$ )** *If  $R$  is a closed bounded subset of  $\mathbb{R}^n$ , then for every continuous function  $f: R \rightarrow \mathbb{R}$  there exist  $c, d \in R$  such that  $f(c) \leq f(x) \leq f(d)$  for every  $x \in R$ .*

**Definition 23** A collection  $\mathcal{C}$  of subsets of  $X$  has *finite intersection property*, *fip*, provided  $\bigcap \mathcal{C}_0 \neq \emptyset$  for every finite  $\mathcal{C}_0 \subset \mathcal{C}$ .

**Theorem 48 (Thm 26.9)**  *$X$  is compact if, and only if,  $\bigcap \mathcal{C} \neq \emptyset$  for every family  $\mathcal{C}$  of closed subsets of  $X$  having fip.*

Pages 175-177 from section 27 will be covered next semester.

Go over some of Exercises 1, 2, 3, 4, 5, 6, 7, 8, 9, and 11 section 26. Also, Exercises 2, 3, and 4 from section 27. (Try to solve by your own the exercises not solved in class.)

No class on November 17.

### **Classes of November 29 and December 1:**

Review for Final Test. This will include solving the exercises indicated during the last two classes which were not solved yet.

### **Class of December 6:**

Administration of Final Test.