Notes

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Linear Algebra classes

Class # 1: August 17, 2017

Discussed syllabus

Definition of a field: Chapter 1, page 5. Examples:
real numbers $\mathbb{R}$; also complex numbers $\mathbb{C}$ and rational numbers $\mathbb{Q}$.
Read the rest of Chapter 1.

Important points on Linear Algebra:
Matrices: definition and the following operations
transpose, scalar multiplication, addition and multiplication of matrices.
Typical exercise for this material:

Exercise 1 *For* $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 11 & -1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0 & 8 \\ 4 & -2 \\ 5 & 1 \end{bmatrix}$ *find* A^T, $2A - 3B$, $A^T B$, *and* BA^T.

Example:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \text{ but } \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

Class # 2

Last class we defined: matrices and the following operations
transpose, scalar multiplication, addition and multiplication of matrices;

zero matrix θ ($A + \theta = \theta + A = A$);

Define the identity matrix I ($AI = A$ and $IB = B$);

Properties (multiplication, addition properties latter):
$A(BC) = (AB)C$; $AB = I$ implies $BA = I$; however, AB need not be equal BA:
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \text{ but } \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

Chapter 3, Vector spaces: State Definition 1, Chapter 3 page 2.
Examples:

- $K^{n\times m}$ – the family of all $n \times m$-matrices over the field K; e.g. $\mathbb{R}^{n\times m}$
- $\mathbb{R}^{n\times 1}$ – the family of all n-dimensional (column) matrices $[x_1 \cdots x_n]^T$; often denotes as $\mathbb{R}^n$;
- $\mathbb{R}^2$, the classical plane vectors $[x\ y]^T$ often identified with $[x\ y]$ and written as $\langle x, y\rangle$; similarly $3D$-vectors $\mathbb{R}^3$;
- The family $\mathcal{F}(D, \mathbb{R})$ of all functions from a set $D \subset \mathbb{R}$ into $\mathbb{R}$; also the classes of: all polynomials; of all differentiable functions; of solutions of some differential equations; etc;

Subspaces: Definition 1, Chapter 3 page 10.

Theorem 1 *If V is a vector space (over the field K) and W is non-empty subset of V, then W is a subspace if, and only if, $v + w$ and cv are in W for every c from K and $v, w \in W$.*

Examples:

- $W = \{\langle x, 3x\rangle : x \in \mathbb{R}\}$, a line in the plane $\mathbb{R}^2$ is a vector subspace of $\mathbb{R}^2$;
- polynomials forms a vector subspace of $\mathcal{F}(D, \mathbb{R})$; so are differentiable functions;

Chapter 4, System of linear equations $A\mathbf{x} = \mathbf{b}$:
For a system $A\mathbf{x} = \mathbf{b}$ of m equations with n unknowns $x_1, \ldots, x_n$, A is $m \times n$ coefficient matrix, $\mathbf{x} = [x_1, \ldots, x_n]^T$, and $\mathbf{b} = [b_1, \ldots, b_n]^T$.

Given one simple example of solving $A\mathbf{x} = \mathbf{b}$ via Gauss elimination.

Class # 3: August 24, 2017

Solutions of $A\mathbf{x} = \mathbf{b}$ via Gauss elimination:

Use of Gauss elimination, that is, using augmented matrix approach. If the system is consistent (i.e., has at least one solution), the solution must be expressed in the vertical vector form:

$$\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \text{ or } \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ 5 \\ 11 \end{pmatrix} \text{ or } \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ 5 \\ 11 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}.$$

From the text: Example # 1, Ch. 4, Pg. 8 (also Pg. 19)
From the text: Example # 2, Ch. 4, Pg. 19
Solve exercise 2 from the Sample Test # 1, via Gauss elimination.
Solve exercise 1 from the Sample Test # 1, via Gauss elimination.

Next class: Quiz # 1. Material as in Exercises 1 and 2 of the Sample Test # 1, just solved. The Sample Test # 1 is available at

http://www.math.wvu.edu/~kcies/teach/current/CurrentTeaching.html

System of linear equations $A\mathbf{x} = \mathbf{b}$, revisited:

For a system $A\mathbf{x} = \mathbf{b}$ of m equations with n unknowns $x_1, \ldots, x_n$, A is $m \times n$ coefficient matrix, $\mathbf{x} = [x_1, \ldots, x_n]^T$, and $\mathbf{b} = [b_1, \ldots, b_n]^T$.

When $\mathbf{b} = \mathbf{0} = [b_1, \ldots, b_n]^T$, then the system $A\mathbf{x} = \mathbf{b}$ (i.e., $A\mathbf{x} = \mathbf{0}$) is a homogeneous system.

The solutions $\mathbf{x}$ of the homogeneous system $A\mathbf{x} = \mathbf{0}$, that is, $V = \{\mathbf{x}\colon A\mathbf{x} = \mathbf{0}\}$, is a *vector space:*

$\mathbf{0} \in V$ and $\alpha\mathbf{x} + \beta\mathbf{y} \in V$ for every $\mathbf{x}, \mathbf{y} \in V$.

In other words, V is a *null space* of the operator A: $\mathbf{x} \mapsto A\mathbf{x}$.

[A function T from a vector space into another is a linear operator when

$$T(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha T(\mathbf{x}) + \beta T(\mathbf{y}).$$

Its *null space* is the set of all vectors $\mathbf{x}$ for which $T(\mathbf{x}) = \mathbf{0}$. Null space of any linear operator is also a vector space.]

In particular $A\mathbf{x} = \mathbf{0}$ has either one, or infinitely many solutions.

If $\mathbf{x}_p$ is a solution for $A\mathbf{x} = \mathbf{b}$, then

$\mathbf{x}$ solution for $A\mathbf{x} = \mathbf{b}$ if, and only if, it is of the form $\mathbf{x}_p + \mathbf{x}_h$, where $\mathbf{x}_h$ is a solution for $A\mathbf{x} = \mathbf{0}$.

Class # 4: August 29, 2017

Linear independence of vectors and basis

Inverse of a square, $n \times n$, matrix A If there exists a matrix B such that $BA = I$, then also $AB = I$ and B is unique. It is denoted as A^{-1} and referred to as the inverse of A. Example: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad - bc \neq 0$, then the inverse of A exists and $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Note, $A^{-1} \neq \frac{1}{A}$. In fact, the quotient $\frac{1}{A}$ has no sense at all!
A is *singular* if A^{-1} does not exist; otherwise, it is *non-singular*.

Q. What A^{-1} is useful for?

A. Many uses. E.g.: $A\mathbf{x} = \mathbf{b}$ if, and only if, $\mathbf{x} = A^{-1}\mathbf{b}$.

Also, in determining when vectors $\mathbf{b}_1, \ldots, \mathbf{b}_n \in \mathbb{R}^n$ are *linearly independent* (form a basis) — notions to be discussed.

Q. When does A^{-1} exist (i.e., when A is non-singular)?

A. E.g.: when the *determinant* of A, denoted $|A|$ or $\det A$, is $\neq 0$. Calculation of the determinants to be discussed, chapter 7.

Q. When A is non-singular, how to find A^{-1}?

A. Gaussian elimination (again), to be explained.

Finding A^{-1} via Gaussian elimination: Chapter 9. To find A^{-1}: (1) write augmented matrix $[A; I]$; (2) Gaussian elimination to transform it to a matrix $[I; B]$; (3) declare that A^{-1} equals B.

Go over Exercises 4, 5 from the sample test and Example 1, Ch. 9, Pg 5.

Class # 5: August 30, 2017

Calculation of the determinant: Via arbitrary row (or column) expansion (known as Laplace Expansion Method), definition (not in the "textbook"), Example on page Ch. 7, Pg 4. Take a look at Theorem Ch. 7, Pg 2, the properties of the determinant – leads to Gaussian elimination. Solve (the same problem) using Gaussian elimination, see Ch. 7, Pg 6.

Solving $A\mathrm{x} = \mathbf{b}$ via *Cramer Rule*: application of determinants. Just state (Ch. 6, Pg 7), no exercises.