Topology, Math 681, Spring 2010

Krzysztof Chris Ciesielski

Solutions for Homework

Ex. 10. p. 186: Show that is X is a Hausdorff space that is locally compact at the point x, then for every neighborhood U of x, there is a neighborhood V of x such that $\mathrm{cl}(V)$ is compact and $\mathrm{cl}(V) \subset U$.

PROOF. Note, that we cannot use one point compactification of X, since we do not know if X is locally compact at all points.

By definition, there exist compact set C and open set W such that $x \in W \subset C$. Let $F = X \setminus (W \cap U)$ and $Y = C \cap F$. Both sets are closed in X. Also, Y is compact, as a closed subset of a compact set C. Note also that $x \notin Y$.

Now, we work in the compact space C. Using Lemma 26.4, there are open disjoint sets V and V_1 in C with $x \in V$ and $Y \subset V_1$. (This is regularity of the space C.) We will show that V is as desired.

First, note that V is open in X, since its complement $X \setminus V$ is equal $(C \setminus V) \cup (X \setminus C) = (C \setminus V) \cup F$ and both of these last sets are closed in X. ($C \setminus V$ is closed, since it is closed in C, so compact in C, so compact in X, so closed in X, as X is Hausdorff.)

Second, note that $\mathrm{cl}_X(V) = \mathrm{cl}_C(V)$, as C is closed in X, and clearly $\mathrm{cl}_C(V)$ is compact, as a closed subset of a compact space C. Finally, $\mathrm{cl}(V) \subset U$, since it is disjoint with $V_1 \cup (X \setminus C) \supset (F \cap C) \cup (X \setminus C) = F \supset X \setminus U$. ∎

Exercise 4, page 213 (one direction): If X is normal, then

(a) for every closed G_δ set $A \subset X$ there is continuous $f\colon X \to [0,1]$ such that $f^{-1}(0) = A$.

PROOF. Let $\{U_n\colon n \in \mathbb{N}\}$ be a family of open subsets of X such that $A = \bigcap_{n\in\mathbb{N}} U_n$. By Urysohn Lemma, for every $n \in \mathbb{N}$ there is a continuous function $f_n\colon X \to [0,1]$ such that $A \subset f_n^{-1}(0)$ and $X \setminus U_n \subset f_n^{-1}(1)$. Let $f = \sum_{n\in\mathbb{N}} 2^{-n} f_n$. Then f is continuous, as a uniform limit of continuous functions, $f(a) = 0$ for every $a \in A$, and for every $x \notin A$ there is an $n \in \mathbb{N}$ with $x \in X \setminus U_n$, implying that $f(x) \geq 2^{-n} > 0$. So, $f^{-1}(0) = A$. ∎

Exercise 5, page 213 (one direction): (This was not a homework.) If X is normal, then

(d) for every closed disjoin G_δ subsets A and B of X there is continuous $f\colon X \to [0,1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$.

Proof. First note that, by (a),

(b) there is continuous $g\colon X \to [0,1]$ such that $g^{-1}(0) = A \cup B$

Indeed, if f_A and f_B are the functions as in (a) for A and B, respectively, then $g = f_A \cdot f_B$ is as desired.

Next note that

(c) there is continuous function $h_{AB}\colon X \to [0,1]$ such that $h_{AB}^{-1}(0) = A$ and $h_{AB}^{-1}(1) \supset B$.

Indeed, if $f_0\colon X \to [0,1]$ is a function from Urysohn Lemma (i.e., such that $f_0^{-1}(0) \supset A$ and $f_0^{-1}(1) \supset B$), g is as in (b), and we put $h = f_0 + g$, then $h^{-1}(0) = A$ and $h^{-1}(1) \supset B$. Thus, function $h_{AB} = \min\{1, h\}$ is as desired.

Finally, put $f = \frac{1}{2}[h_{AB} + (1 - h_{BA})]$. Then function f satisfies (d), as $(1 - h_{AB})\colon X \to [0,1]$, $(1 - h_{AB})^{-1}(0) \supset A$, and $(1 - h_{AB})^{-1}(1) = B$. ∎

Exercise 8, page 213: Let X be completely regular; let A and B be disjoint closed subsets of X. Show that if A is compact, there is a continuous function $f\colon X \to [0,1]$ such that $f[A] \subset \{0\}$ and $f[B] \subset \{1\}$.

Proof. By the definition of complete regularity, for every $a \in A$ there is a continuous function $f_a\colon X \to [0,1]$ with $f_a(a) = 1$ and $f_a[B] \subset \{0\}$.

Let $g_a(x) = \max\{0, 1 - 2f(x)\}$. Then function $g_a\colon X \to [0,1]$ is continuous, $g_a[B] \subset \{1\}$, and $g_a[U_a] \subset \{0\}$, where $U_a = f_a^{-1}([0,1/2))$ is an open set containing a.

The family $\{U_a\colon a \in A\}$ is an open cover of A, so we can find a finite subcover $\{U_a\colon a \in A_0\}$. Let $f\colon X \to [0,1]$ be defined by $f(x) = \prod_{a\in A_0} g_a(x)$. Then f is continuous (as a finite product of continuous functions), $f(b) = 1$ for every $b \in B$ (since $g_a(b) = 1$ for every $a \in A_0$), and $f(x) = 1$ for every $x \in A$ (since there is an $a \in A_0$ with $x \in U_a$ and so $g_a(x) = 0$). So, f is as desired. ∎

Exercise. A space Y has the *universal extension property* if for every normal space X, a closed subset A of X, and continuous function $f\colon A \to Y$, there is a continuous extension $F\colon X \to Y$ of f.

Show that if Y has the universal extension property and Z is homeomorphic to Y, then Y also has the universal extension property.

PROOF. Assume that Y has the universal extension property and let $h\colon Y \to Z$ be a homeomorphism. Take normal space X, a closed subset A of X, and continuous function $f\colon A \to Z$. We need to find a continuous extension $F\colon X \to Y$ of f.

Function $g = h^{-1} \circ f\colon A \to Y$ is continuous. Since Y has the universal extension property, there is a continuous function extension $G\colon X \to Y$ of g. Let $F = h \circ G\colon X \to Z$. Then F is continuous, and it extends f, since for every $x \in A$: $F(x) = h(G(x)) = h(g(x)) = h(h^{-1} \circ f(x)) = f(x)$. ∎