MATH 441.001 NAME (print): ____________________
Instr. K. Ciesielski
Spring 2011

FINAL TEST Review

Final Test will start with: "Solve the following exercises. **Show your work.** (No credit will be given for an answer with no supporting work shown.)"

Remember, That Final Test is comprehensive!

This review is based on a combination of the actual tests I have given you, as well as in the in-class reviews. Some problems that I consider especially important to study for the final are marked by read rectangle.

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TEST # 1

Solve the following exercises. **Show your work.** (No credit will be given for an answer with no supporting work shown.)

Ex. 1. Evaluate

(a) $\begin{bmatrix} 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} =$

(9 pts) *Solution:*

$= [1 - 8 - 3] = [-10]$

(b) $\begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \end{bmatrix} =$

(9 pts) *Solution:*

$$= \begin{bmatrix} 1 & -2 & 3 \\ 5 & -10 & 15 \\ -1 & 2 & -3 \end{bmatrix}$$

(c) $(AB)^{-1}$, if $A^{-1} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 6 & 3 \\ 0 & 1 & 2 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Do not calculate A or B.

(13 pts) *Solution:*

$$(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ -1 & 6 & 3 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 9 & 6 \\ -1 & 7 & 5 \\ 0 & 1 & 2 \end{bmatrix}$$

Ex. 2. Find the matrix A for which $A\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} x+y \\ y+z \\ z-t \\ x-t \end{bmatrix}$

(10 pts) *Solution:*

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

Ex. 3. Using block multiplication, evaluate

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 4 & 3 & 4 \\ 1 & 2 & 1 & 2 \\ 3 & 4 & 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 0 & 5 & 0 \\ 1 & 2 & 1 & 2 \\ 5 & 0 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

(13 pts) *Solution:*

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 4 & 3 & 4 \\ 1 & 2 & 1 & 2 \\ 3 & 4 & 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 0 & 5 & 0 \\ 1 & 2 & 1 & 2 \\ 5 & 0 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} A & A \\ A & A \end{bmatrix}\begin{bmatrix} B & B \\ B & B \end{bmatrix} = \begin{bmatrix} AB+AB & AB+AB \\ AB+AB & AB+AB \end{bmatrix},$$

where $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 0 \\ 1 & 2 \end{bmatrix}$. Since $AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 19 & 8 \end{bmatrix}$ and $AB + AB = \begin{bmatrix} 7 & 4 \\ 19 & 8 \end{bmatrix} + \begin{bmatrix} 7 & 4 \\ 19 & 8 \end{bmatrix} = \begin{bmatrix} 14 & 8 \\ 38 & 16 \end{bmatrix}$, the final answer is

$$\begin{bmatrix} 14 & 8 & 14 & 8 \\ 38 & 16 & 38 & 16 \\ 14 & 8 & 14 & 8 \\ 38 & 16 & 38 & 16 \end{bmatrix}$$

Ex. 4. Using Gauss-Jordan elimination, find the inverse of the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 2 & 1 & 0 \end{bmatrix}$

(13 pts) *Solution:*

$$[A : I] = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -2 & 1 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 0 & -1 & -2 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 & -1 & -1 \end{bmatrix}, \text{ so } A^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 0 & -1 \\ 2 & -1 & -1 \end{bmatrix}$$

Ex. 5.

(a) Use Guassian elimination to represent the linear system $A\vec{x} = \vec{b}$ in an upper triangular form $U\vec{x} = \vec{c}$, where $A = \begin{bmatrix} 1 & -1 & -1 \\ 2 & 2 & -2 \\ 3 & 3 & 3 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix}$.

(12 pts) *Solution:* $[A : \vec{b}] =$

$$\begin{bmatrix} 1 & -1 & -1 & 2 \\ 2 & 2 & -2 & 0 \\ 3 & 3 & 3 & 6 \end{bmatrix} \begin{matrix} \\ -2\vec{r}_1 \\ \\ \end{matrix} \rightarrow \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 4 & 0 & -4 \\ 3 & 3 & 3 & 6 \end{bmatrix} \begin{matrix} \\ \\ -3\vec{r}_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 4 & 0 & -4 \\ 0 & 6 & 6 & 0 \end{bmatrix} \begin{matrix} \\ \\ -\frac{3}{2}\vec{r}_2 \end{matrix} \rightarrow \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 4 & 0 & -4 \\ 0 & 0 & 6 & 6 \end{bmatrix}$$

So, in $U\vec{x} = \vec{c}$ we have $U = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$ and $\vec{c} = \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$.

(b) Use part (a) and back substitution to solve the above linear system $A\vec{x} = \vec{b}$.

(10 pts) *Solution:* $U\vec{x} = \vec{c}$ gives us the equations:

$6z = 6$, so $z = 1$;

$4y = -4$, so $y = -1$;

$x - y - z = 2$, so $x = 2 + y + z = 2 - 1 + 1 = 2$.

Answer: $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

(c) Use part (a) to find a lower triangular matrix L for which $A = LU$.

(11 pts) *Solution:* The elimination matrices were, consecutively,

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \text{ and } E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{3}{2} & 1 \end{bmatrix}.$$

Their respective inverses are

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}, \text{ and } E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \end{bmatrix}.$$

So, $L = (E_{32}E_{31}E_{21})^{-1} = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & \frac{3}{2} & 1 \end{bmatrix}$.

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TEST # 2 Review

To be solved in class.

Test will start with: "Solve the following exercises. **Show your work.** (No credit will be given for an answer with no supporting work shown.)"

Ex. 1. Find a basis for a vector space V generated by the following vectors:
$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 5 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 4 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 0 \\ 4 \\ 2 \\ -2 \end{bmatrix}$.

(**pts)** *Solution:* The space V is equal to the column space $C(A)$ of $A = [\vec{v}_1\vec{v}_2\vec{v}_2]$. Since A reduces as follows:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 1 & 2 & 2 \\ 5 & 4 & -2 \end{bmatrix} \begin{matrix} \\ \times 1/2 \\ -\vec{r}_1 \\ -5\vec{r}_1 \end{matrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix} \begin{matrix} -\vec{r}_2 \\ \\ -\vec{r}_2 \\ +\vec{r}_2 \end{matrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = R$$

the pivot columns of A are the same as of R: columns #1 and #2. Therefore these columns of **matrix A** form the basis of $V = C(A)$.

Answer: $\{\vec{v}_1, \vec{v}_2\}$ is a basis of V.

Ex. 2. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 1 & 2 & 2 \\ 5 & 4 & -2 \end{bmatrix}$. (a) For what value of number c vector $\vec{w} = \begin{bmatrix} 2 \\ -4 \\ 0 \\ c \end{bmatrix}$ belongs to the column space of A? (b) Find the most general solution of $A\vec{x} = \vec{w}$ for this value of c. (c) Without further calculation, identify a basis for a row space of A (i.e., for $C(A^T)$).

(**pts)** *Solution:* (a) Vector $\vec{w}$ belongs to $C(A)$ precisely when system $A\vec{x} = \vec{w}$ has a solution. Thus, we need to solve it. We will do it by reducing augmented matrix of this system:

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 2 & 4 & -4 \\ 1 & 2 & 2 & 0 \\ 5 & 4 & -2 & c \end{bmatrix} \begin{matrix} \\ \times 1/2 \\ -\vec{r}_1 \\ -5\vec{r}_1 \end{matrix} \to \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & -2 \\ 0 & 1 & 2 & -2 \\ 0 & -1 & -2 & c-10 \end{bmatrix} \begin{matrix} -\vec{r}_2 \\ \\ -\vec{r}_2 \\ +\vec{r}_2 \end{matrix} \to \begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c-12 \end{bmatrix}$$

For this system to have the solution we need to have $c - 12 = 0$, that is, $\mathbf{c = 12}$.

(b) We already have done all reduction work for this problem in part (a). The free variable is x_3, since the third column is the only non-pivot column. From the first reduced equation $x_1 - 2x_3 = 4$, we get $x_1 = 4 + 2x_3$. The second equation $x_2 + 2x_3 = -2$ gives $x_2 = -2 - 2x_3$.

Answer: The most general solution of $A\vec{x} = \vec{w}$ with $c = 12$ is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$.

(c) The basis is formed by pivot rows of either A or its reduced form. (But not augmented form, that is, we need to ignore the last column in our calculation.) Thus, the basis can be given either as $\{(1, 1, 0), (0, 2, 4)\}$ or as $\{(1, 0, -2), (0, 1, 2)\}$.

Ex. 3. Let A be an 7×11 matrix with the rank $r = 4$. What is the dimension of the following four spaces. (a) Column space $C(A)$ of A. (b) Row space $C(A^T)$ of A. (c) Null space $N(A)$ of A. (d) Left-null space $N(A^T)$ of A.

(**pts)** *Solution:* Answers: (a) $= r = 4$. (b) $= r = 4$. (c) $= n - r = 11 - 4 = 7$. (d) $= m - r = 7 - 4 = 3$. Name "left-null" comes from the fact that y is in $N(A^T)$ when it is a solution of $A^T\vec{y} = 0$, which is equivalent to $(A^T\vec{y})^T = 0^T$, that is, to $\vec{y}^T A = 0$.

Ex. 4. Let A be an $m \times n$ matrix of rank r. Describe precisely the possible number of solutions of a system $A\vec{x} = \vec{b}$ under the following assumptions:

(a) $r < m$ and $r < n$

(b) $r = m$ and $r < n$

(c) $r < m$ and $r = n$

(d) $r = m = n$

(**pts)** *Solution:* Answers: (a) none or infinitely many; (b) one or infinitely many; (c) none or one; (d) one.

Ex. 5. Finish the following sentence. Give a short explanation for your answer.

An $n \times n$ matrix A is invertible if, and only if, the dimension of its column space $C(A)$ is . . .

(**pts)** *Solution:* Answer: "the dimension of $C(A)$ is equal n."

This is the case, since A is invertible when every column of A is its pivot column, that is, when rank r of A is equal n. But we know, that the dimension of $C(A)$ is equal to r.

In the test, there will be also a bonus exercise.

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Review for TEST # 3

Solve the following exercises. **Show your work.** (No credit will be given for an answer with no supporting work shown.)

Ex. 1.

(a) Find the orthogonal completion $V^\perp$ of the vector space V spanned by the following three vectors. The description of $V^\perp$ is understood as: (i) giving (explicitly) a basis for $V^\perp$ and (ii) stating the dimension of $V^\perp$.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

(b) Give a basis for V and state the dimension of V. Justify, why the vectors in the basis provided as your answer indeed form a basis of V.

(c) Find the matrices P and $P^\perp$ whose application result in the orthogonal projections on V and on $V^\perp$, respectively.

(d) For what value of a parameter p, p being a real number, the projection of a vector $\vec{w} = \begin{bmatrix} 3p^2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, onto space V is the shortest? What is the length of such shortest vector?

Solution: (a) $V^\perp$ is equal to the left null space of $A = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$, that is, all vectors $\vec{y}$ satisfying $A^T\vec{y} = 0$. To solve it, note that reduction of A^T is as follows

$$A^T = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{matrix} \\ -\vec{r}_1 \\ -\vec{r}_1 \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{matrix} \\ \\ -\vec{r}_2 \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{matrix} \\ +\vec{r}_3 \\ \times(-1) \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Therefore, system $A^T\vec{y} = 0$ is equivalent to $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = 0$, so y_4 is a free variable (non-pivot column) and $y_1 = -y_4$, $y_2 = y_4$, $y_3 = 0$. This gives the solution of $A^T\vec{y} = 0$ as $\vec{y} = y_4 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

Answer: $V^{\perp}$ is the line spanned by a vector $\vec{a} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$. A basis $\mathcal{B}$ for $V^{\perp}$ is a vector $\vec{a}$, that is, $\mathcal{B} = \{\vec{a}\}$. The dimension of $V^{\perp}$ is 1.

(b) The dimension of V is "the dimension of the large space, $\mathbb{R}^4$, minus the dimension of its perpendicular complement $V^{\perp}$." Therefore, the dimension of V is $= 4 - 1 = 3$.

A basis $\mathcal{B}$ for V can be formed either by columns of A, or by rows of any reduced form of A^T. Thus, it can be given, as $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

or as $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Both versions of $\mathcal{B}$ span space V (the first by definition, second by the property of matrix reduction). They must be independent, since the dimension of V is 3. (This can be deduced either as above, or by counting number of pivots in the reduced version of the matrix A^T.)

(c) The formula for the matrix representing the orthogonal projection onto the column space of a matrix A is $A(A^TA)^{-1}A^T$. We will use it first to find $P^{\perp}$, for which the matrix A is given by a single column $\vec{a}$. So, $A^TA = (-1, 1, 0, 1) \cdot (-1, 1, 0, 1) = 1 + 1 + 0 + 1 = 3$ and

$$A(A^TA)^{-1}A^T = \tfrac{1}{3}AA^T = \tfrac{1}{3}\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 & 1 \end{bmatrix} = \tfrac{1}{3}\begin{bmatrix} 1 & -1 & 0 & -1 \\ -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

Answer: $P^{\perp} = \begin{bmatrix} 1/3 & -1/3 & 0 & -1/3 \\ -1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 0 \\ -1/3 & 1/3 & 0 & 1/3 \end{bmatrix}$ and $P = I - P^{\perp} = \begin{bmatrix} 2/3 & 1/3 & 0 & 1/3 \\ 1/3 & 2/3 & 0 & -1/3 \\ 0 & 0 & 1 & 0 \\ 1/3 & -1/3 & 0 & 2/3 \end{bmatrix}$.

(d) The projection is equal to

$$P\vec{b} = \begin{bmatrix} 2/3 & 1/3 & 0 & 1/3 \\ 1/3 & 2/3 & 0 & -1/3 \\ 0 & 0 & 1 & 0 \\ 1/3 & -1/3 & 0 & 2/3 \end{bmatrix}\begin{bmatrix} 3p^2 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2p^2 \\ p^2 \\ 1 \\ p^2 \end{bmatrix}.$$

Its lenght is $||P\vec{b}|| = \sqrt{(2p^2)^2 + (p^2)^2 + 1 + (p^2)^2} = \sqrt{6(p^2)^2 + 1}$. It is the smallest, when $6p^4+1$ is the smallest, that is, when $p = 0$. In this case the vector has length $\sqrt{6(0)^2 + 1} = 1$.

Ex. 1. Find the equation of the line in the plane that fits the best (in the least square approximation sense) the data points $(-3,-2)$, $(0,0)$, $(1,0)$, $(2,2)$.

(12 pts) *Solution:* The solution is of the form $y = C + Dt$, where $\hat{x} = \begin{bmatrix} C \\ D \end{bmatrix}$ constitute the solution of the system $A^T A\hat{x} = A^T\vec{b}$, where $A = \begin{bmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 2 \end{bmatrix}$. Then, $A^T A = \begin{bmatrix} m & \sum_i t_i \\ \sum_i t_i & \sum_i t_i^2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 14 \end{bmatrix}$ and $A^T\vec{b} = \begin{bmatrix} \sum_i b_i \\ \sum_i t_i b_i \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$, so we get system $\begin{bmatrix} 4 & 0 \\ 0 & 14 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$. Hence, $4C = 0$, that is, $C = 0$, and $14D = 10$, that is, $D = 5/7$.

Answer: The equation of the line is $y = 0 + 5/7t$.

Ex. 2. Evaluate the determinant of the matrix $A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$. Show your work!

Hint. In the reduction first step add rows 1-4 to row # 5. In the second step, transform matrix to lower triangular. Then calculate determinant.

(12 pts) *Solution:*

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} \begin{matrix} \\ \\ \\ \\ +\sum_{i=1}^4 \vec{r}_i \end{matrix} \rightarrow \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 4 & 4 & 4 & 4 & 4 \end{vmatrix} \begin{matrix} -.25\vec{r}_5 \\ -.25\vec{r}_5 \\ -.25\vec{r}_5 \\ -.25\vec{r}_5 \\ \\ \end{matrix} \rightarrow \begin{vmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 4 & 4 & 4 & 4 & 4 \end{vmatrix}$$

Answer: $|A| = (-1)(-1)(-1)(-1)4 = 4$

Ex. 3. Use Gram-Schmidt process to find an orthonormal basis for the subspace V of $\mathbb{R}^4$ spanned by the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \text{ and } \vec{v}_3 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}.$$

(12 pts) *Solution:* First, we find orthogonal vectors $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$, and, at the end of the process, normalize each of them to $\vec{q}_1$, $\vec{q}_2$, and $\vec{q}_3$, respectively.

We have $\vec{u}_1 = \vec{v}_1$, $\vec{u}_2 = \vec{v}_2 - \frac{\vec{u}_1 \cdot \vec{v}_2}{\vec{u}_1 \cdot \vec{u}_1}\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} - \frac{1+0+1+0}{1+0+1+0}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, and

$$\vec{u}_3 = \vec{v}_3 - \frac{\vec{u}_1 \cdot \vec{v}_3}{\vec{u}_1 \cdot \vec{u}_1}\vec{u}_1 - \frac{\vec{u}_2 \cdot \vec{v}_3}{\vec{u}_2 \cdot \vec{u}_2}\vec{u}_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} - \frac{2+0+0+0}{1+0+1+0}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} - \frac{0+0+0+0}{0+0+0+1}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$

Since $||\vec{u}_1|| = \sqrt{1+0+1+0} = \sqrt{2}$, $||\vec{u}_2|| = 1$, and $||\vec{u}_3|| = \sqrt{1+1+1+0} = \sqrt{3}$, we have

$$\vec{q}_1 = \frac{\vec{u}_1}{||\vec{u}_1||} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \ \vec{q}_2 = \frac{\vec{u}_2}{||\vec{u}_2||} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \text{ and } \vec{q}_3 = \frac{\vec{u}_3}{||\vec{u}_3||} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$

Answer: An orthonormal basis under question is $\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}$, where vectors $\vec{q}_i$ are as above.

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Review for TEST # 4

Solve the following exercises. **Show your work.** (No credit will be given for an answer with no supporting work shown.)

Ex. 1.

(a) State the Cofactor Formula for finding the inverse $A^{-1} = [x_{ij}]$ of an $n \times n$ matrix $A = [a_{ij}]$.

(?? pts) *Solution:* $x_{ij} = \frac{C_{ji}}{\det(A)}$, where $C_{ji} = (-1)^{j+i}\det(M_{ji})$ and M_{ji} is obtained from A by removing its jth row and ith column.

(b) Use the Cofactor Formula you citated in (a) to find x_{21}, if $A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$.

(?? pts) *Solution:* $\det(A) = \begin{vmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 7 \end{vmatrix} \begin{matrix} \\ -3\vec{r}_1 \\ \\ \\ \end{matrix} = \begin{vmatrix} 1 & 2 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 7 \end{vmatrix} = -2 \cdot 35.$

$C_{12} = (-1)^{1+2} \begin{vmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{vmatrix} = -3 \cdot 35$. So, $x_{21} = \frac{C_{12}}{\det(A)} = \frac{-3 \cdot 35}{-2 \cdot 35} = 1.5$

Ex. 2. Find all eigenvalues and associated eigenvectors of the matrix $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Is matrix A diagonalizable?

(?? pts) *Solution:* $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3 & 4 \\ 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{vmatrix} = (1-\lambda)(-\lambda)(-\lambda) = (1-\lambda)\lambda^2$.

We have two eigenvalues, $\lambda = 1$ and $\lambda = 0$, the second having (algebraic) multiplicity 2.

To find eigenvector associated with $\lambda = 1$ we need to solve $(A - \lambda I)\vec{x} = 0$. As $A - I = \begin{bmatrix} 0 & 3 & 4 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 3 & 4 \end{bmatrix} \to \begin{matrix} \vec{r}_3 \\ \\ +r_1 + 7r_2 \end{matrix} \to \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

indicates one free variable, x_1, and $x_2 = x_3 = 0$, we get a null space $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Thus, eigenvalue $\lambda = 1$ is associated with eigenvector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

To find eigenvector(s) associated with $\lambda = 0$ we need to solve $(A - \lambda I)\vec{x} = 0$. As $A - \lambda I = A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} -4\vec{r}_2 \\ \\ \\ \end{matrix} \to \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ indicates only one free variable, x_2, and $x_1 = -3x_2$, $x_3 = 0$, we get a null space $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$. Thus, eigenvalue $\lambda = 0$ (of algebraic multiplicity 2) is associated with a single eigenvector $\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$.

This means, in particular, that A has only two eigenvectors, so A is not diagonalizable.

Math 441 Exam 4

1. (12 pts) Given $A = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$

a) Find the eigenvalues and eigenvectors of A

$$\det\begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix} = \lambda^2 - \lambda - 2 = 0 \text{ , } \lambda = -1,2$$

$\lambda = -1$:

$$A - \lambda I = \begin{bmatrix} 6 & -6 \\ 3 & -3 \end{bmatrix}, \bar{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda = 2$:

$$A - \lambda I = \begin{bmatrix} 3 & -6 \\ 3 & -6 \end{bmatrix}, \bar{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

b) Write A in diagonalized form, $A = SDS^{-1}$ where D is a diagonal matrix. Then find a formula for A^k. As $k \to \infty$, every column of $\frac{1}{2^k}A^k$ is a multiple of what vector - what is the significance of this vector?

$$A = V\Lambda V^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$A^k = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} (-1)^k & 0 \\ 0 & 2^k \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2^{k+1} + (-1)^{k+1} & -2^{k+1} + 2(-1)^k \\ 2^k + 2(-1)^{k+1} & -2^k + 4(-1)^k \end{bmatrix}$$

$$\frac{1}{2^k}A^k = \begin{bmatrix} 2 + 2^{-k}(-1)^{k+1} & -2 + 2^{1-k}(-1)^k \\ 1 + 2^{1-k}(-1)^{k+1} & -1 + 2^{2-k}(-1)^k \end{bmatrix} \to \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}$$

, each column is a multiple of the eigenvector with the largest eigenvalue, $\lambda = 2$.

You can also calculate:

$$\frac{1}{2^k}A^k = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} (-1)^k 2^{-k} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}$$

2. (12 pts) If $A = \begin{bmatrix} 3 & 2 & -1 \\ 6 & 4 & -2 \\ -3 & -2 & 1 \end{bmatrix}$

The rank of A is __1___ so one eigenvalue is $\lambda =$_0___ . (the null space of $A - 0I$ is not just $\bar{0}$)

There must be _1 / [2] / 3_____ eigenvectors corresponding to this value of λ because _$\dim N(A) = 3 - 1 = 2$__________.

Those corresponding eigenvectors are: $\bar{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$ (any two independent vectors in the null space will do)

The characteristic polynomial must have a factor of __λ^2___________ because _$\lambda = 0$ has multiplicity at least two due to two independent eigenvectors for $\lambda = 0$_________ .

The characteristic polynomial of A is $p(\lambda) =$ __$\lambda^2(8 - \lambda)$_________________ (calculate below).

$$\det \begin{bmatrix} 3-\lambda & 2 & -1 \\ 6 & 4-\lambda & -2 \\ -3 & -2 & 1-\lambda \end{bmatrix} = 8\lambda^2 - \lambda^3$$

The remaining eigenvalue is _$\lambda = 8$______ and the corresponding eigenvector is __$\bar{v}_3 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$_. (Show your work below)

Add the second row to the first below to aid in calculations:

$$\begin{bmatrix} -5 & 2 & -1 \\ 6 & -4 & -2 \\ -3 & -2 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -3 \\ 6 & -4 & -2 \\ -3 & -2 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -3 \\ 0 & 8 & 16 \\ 0 & -8 & -16 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

3. (6 pts)

If I tell you that the eigenvectors of $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ are $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, how can you easily check? What eigenvalues correspond to these eigenvectors?

Take each proposed eigenvector and multiply by A :

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ so } \lambda_1 = 0$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ so } \lambda_2 = 1$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ so } \lambda_3 = 2$$

What eigenvalues correspond to these eigenvectors? Given that A is symmetric, it should have a complete set of orthonormal eigenvectors? What vectors (explicitly) are those? Write A in the form $A = Q\Lambda Q^T$ where Q is an orthogonal matrix.

Divide each eigenvector by its length and create an orthogonal matrix Q out of them:

$$A = Q\Lambda Q^T = \begin{bmatrix} -\frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \\ 0 & 1 & 0 \\ \frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} -\frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \\ 0 & 1 & 0 \\ \frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \end{bmatrix}^T$$

4. (4 pts) Explain the following: If $A = SDS^{-1}$ where D is diagonal and S is some nonsingular matrix, then the eigenvalues of A are the columns of S and the eigenvalues of A are the diagonal entries of D.

In this problem you don't know anything about S, D, except that $A = SDS^{-1}$. Next $AS = SD$ follows by multiplying both sides on the right by S. If we express S in terms of its colums as $S = \left[\bar{v}_1\ \bar{v}_2\ ..\ \bar{v}_n\right]$ and the diagonal entries of D as d_1 , d_2 ,.., d_n then by matrix multiplication we can write:

$AS = \left[A\bar{v}_1\ A\bar{v}_2\ ..\ A\bar{v}_n\right] = \left[d_1\bar{v}_1\ d_2\bar{v}_2\ ..\ d_n\bar{v}_n\right]$

and matching up the columns, we have $A\bar{v}_1 = d_1\bar{v}_1$,.. , $Av_n = d_n\bar{v}_n$ so that the columns of S contain eigenvectors of A and the diagonal entries of D are the corresponding eigenvalues.