MATH 441.001 NAME (print): ______________________________
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Spring 2011

Review for TEST # 3

Solve the following exercises. **Show your work.** (No credit will be given for an answer with no supporting work shown.)

Ex. 1.

(a) Find the orthogonal completion $V^\perp$ of the vector space V spanned by the following three vectors. The description of $V^\perp$ is understood as: (i) giving (explicitly) a basis for $V^\perp$ and (ii) stating the dimension of $V^\perp$.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

(b) Give a basis for V and state the dimension of V. Justify, why the vectors in the basis provided as your answer indeed form a basis of V.

(c) Find the matrices P and $P^\perp$ whose application result in the orthogonal projections on V and on $V^\perp$, respectively.

(d) For what value of a parameter p, p being a real number, the projection of a vector $\vec{w} = \begin{bmatrix} 3p^2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, onto space V is the shortest? What is the length of such shortest vector?

Solution: (a) $V^\perp$ is equal to the left null space of $A = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$, that is, all vectors $\vec{y}$ satisfying $A^T\vec{y} = 0$. To solve it, note that reduction of A^T is as follows

$$A^T = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}\begin{matrix} \\ -\vec{r}_1 \rightarrow \\ -\vec{r}_1 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 \end{bmatrix}\begin{matrix} \\ \\ -\vec{r}_2 \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}\begin{matrix} \\ +\vec{r}_3 \\ \times(-1) \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Therefore, system $A^T\vec{y} = 0$ is equivalent to $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = 0$, so y_4 is a free variable (non-pivot column) and $y_1 = -y_4$, $y_2 = y_4$, $y_3 = 0$. This gives the solution of $A^T\vec{y} = 0$ as $\vec{y} = y_4\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

Answer: $V^{\perp}$ is the line spanned by a vector $\vec{a} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$. A basis $\mathcal{B}$ for $V^{\perp}$ is a vector $\vec{a}$, that is, $\mathcal{B} = \{\vec{a}\}$. The dimension of $V^{\perp}$ is 1.

(b) The dimension of V is "the dimension of the large space, $\mathbb{R}^4$, minus the dimension of its perpendicular complement $V^{\perp}$." Therefore, the dimension of V is $= 4 - 1 = 3$.

A basis $\mathcal{B}$ for V can be formed either by columns of A, or by rows of any reduced form of A^T. Thus, it can be given, as $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

or as $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Both versions of $\mathcal{B}$ span space V (the first by definition, second by the property of matrix reduction). They must be independent, since the dimension of V is 3. (This can be deduced either as above, or by counting number of pivots in the reduced version of the matrix A^T.)

(c) The formula for the matrix representing the orthogonal projection onto the column space of a matrix A is $A(A^TA)^{-1}A^T$. We will use it first to find $P^{\perp}$, for which the matrix A is given by a single column $\vec{a}$. So, $A^TA = (-1, 1, 0, 1) \cdot (-1, 1, 0, 1) = 1 + 1 + 0 + 1 = 3$ and

$$A(A^TA)^{-1}A^T = \tfrac{1}{3}AA^T = \tfrac{1}{3} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 1 \end{bmatrix} = \tfrac{1}{3} \begin{bmatrix} 1 & -1 & 0 & -1 \\ -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

Answer: $P^{\perp} = \begin{bmatrix} 1/3 & -1/3 & 0 & -1/3 \\ -1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 0 \\ -1/3 & 1/3 & 0 & 1/3 \end{bmatrix}$ and $P = I - P^{\perp} = \begin{bmatrix} 2/3 & 1/3 & 0 & 1/3 \\ 1/3 & 2/3 & 0 & -1/3 \\ 0 & 0 & 1 & 0 \\ 1/3 & -1/3 & 0 & 2/3 \end{bmatrix}$.

(d) The projection is equal to

$$P\vec{b} = \begin{bmatrix} 2/3 & 1/3 & 0 & 1/3 \\ 1/3 & 2/3 & 0 & -1/3 \\ 0 & 0 & 1 & 0 \\ 1/3 & -1/3 & 0 & 2/3 \end{bmatrix} \begin{bmatrix} 3p^2 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2p^2 \\ p^2 \\ 1 \\ p^2 \end{bmatrix}.$$

Its lenght is $||P\vec{b}|| = \sqrt{(2p^2)^2 + (p^2)^2 + 1 + (p^2)^2} = \sqrt{6(p^2)^2 + 1}$. It is the smallest, when $6p^4+1$ is the smallest, that is, when $p = 0$. In this case the vector has length $\sqrt{6(0)^2 + 1} = 1$.

2. (8 pts) Find an orthonormal basis for the subspace V spanned by

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

This is just standard Gram-Schmidt.

$$\bar{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \bar{u}_2 = \bar{v}_2 - \frac{\bar{u}_1 \bullet \bar{v}_2}{\bar{u}_1 \bullet \bar{u}_1}\bar{u}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix} - \frac{2}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\bar{u}_3 = \bar{v}_3 - \frac{\bar{u}_1 \bullet \bar{v}_3}{\bar{u}_1 \bullet \bar{u}_1}\bar{u}_1 - \frac{\bar{u}_2 \bullet \bar{v}_3}{\bar{u}_2 \bullet \bar{u}_2}\bar{u}_2 = \begin{bmatrix} 3 \\ 0 \\ 0 \\ -1 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} - \frac{3}{3}\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \\ -1 \end{bmatrix}$$

(check your calculations by confirming orthogonality - it works)
Finally, normalize each vector:

$$\bar{q}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \bar{q}_2 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \bar{q}_3 = \frac{1}{\sqrt{5/2}}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \\ -1 \end{bmatrix}$$

3. (33 points) This problem is about the n by n matrix A_n that has zeros on its main diagonal and all other entries equal to -1. In MATLAB $A_n = \text{eye}\,(n) - \text{ones}\,(n)$.

(a) Find the determinant of A_n. Here is a suggested approach:

Start by adding all rows (except the last) to the last row, and then factoring out a constant. (You could check $n = 3$ to have a start on part b.)

$\boxed{\text{Solution}}$ Following the hint, add all of the rows to the last row (which does not change the determinant). Thus the matrix becomes

$$\begin{bmatrix} 0 & -1 & -1 & \cdots & -1 \\ -1 & 0 & -1 & \cdots & -1 \\ -1 & -1 & 0 & \cdots & -1 \\ \vdots & \vdots & \vdots & & \vdots \\ -(n-1) & -(n-1) & -(n-1) & \cdots & -(n-1) \end{bmatrix}.$$

Next, pull out the factor of $-(n-1)$ from the last row. As the determinant is linear in each row separately, we get

$$\begin{vmatrix} 0 & -1 & -1 & \cdots & -1 \\ -1 & 0 & -1 & \cdots & -1 \\ -1 & -1 & 0 & \cdots & -1 \\ \vdots & \vdots & \vdots & & \vdots \\ -(n-1) & -(n-1) & -(n-1) & \cdots & -(n-1) \end{vmatrix} = (1-n) \begin{vmatrix} 0 & -1 & -1 & \cdots & -1 \\ -1 & 0 & -1 & \cdots & -1 \\ -1 & -1 & 0 & \cdots & -1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{vmatrix}.$$

Next, add the last row back to each of the other rows (which again keeps the determinant the same). So now we want to find

$$(1-n) \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{vmatrix}.$$

This matrix is lower triangular. So its determinant is the product of the entries on its diagonal. Thus the above quantity is $(1-n)$.

Least squares:

1) Given the (x,y) data

x	-1	2	3	5	6
y	3	1	-2	0	4

Find the least squares quadratic approximation $y = a + bx + cx^2$ as follows:

a) Write down the system of linear equations $y_i = a + bx_i + cx_i^2$, $i = 1,2,..,5$ you would like a,b,c to exactly satisfy.

In vector form, the equations to be satisfied are

$$\begin{bmatrix} 3 \\ 1 \\ -2 \\ 0 \\ 4 \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} -1 \\ 2 \\ 3 \\ 5 \\ 6 \end{bmatrix} + c\begin{bmatrix} (-1)^2 \\ (2)^2 \\ (3)^2 \\ (5)^2 \\ (6)^2 \end{bmatrix}$$

b) Write the normal equations giving the $a,b,c,$ that come closest to solving the system in the least squares sense.

The system above asks that $\begin{bmatrix} 3 \\ 1 \\ -2 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 5 & 25 \\ 1 & 6 & 36 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

This is overdetermined, so we provide values (a,b,c) that minimize the sum of the squares of the errors. The equation has the form $\bar{b} = A\bar{x}$ so the least squares solution is obtained from the normal equations $A^T\bar{b} = (A^TA)\bar{x}$ which, numerically, becomes

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 5 & 25 \\ 1 & 6 & 36 \end{bmatrix}^T\begin{bmatrix} 3 \\ 1 \\ -2 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 17 \\ 133 \end{bmatrix} = \begin{bmatrix} 5 & 15 & 75 \\ 15 & 75 & 375 \\ 75 & 375 & 2019 \end{bmatrix}\bar{x}$$

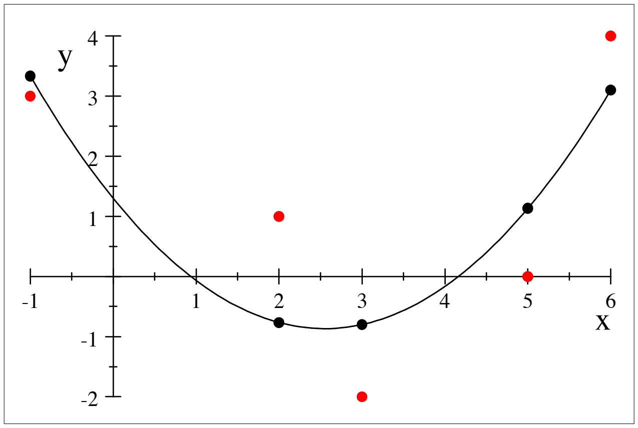
We solve for $\bar{x} = (A^TA)^{-1}A^T\bar{b} = \begin{bmatrix} 5 & 15 & 75 \\ 15 & 75 & 375 \\ 75 & 375 & 2019 \end{bmatrix}^{-1}\begin{bmatrix} 6 \\ 17 \\ 133 \end{bmatrix} = \begin{bmatrix} \frac{13}{10} \\ -\frac{17}{10} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

c) Find the least squares quadratic approximation to the data and plot it and the data (use whatever software you want for this part)

The quadratic approximation to the data is

$$\begin{bmatrix}3\\1\\-2\\0\\4\end{bmatrix} = \bar{y} \cong \begin{bmatrix}1&-1&1\\1&2&4\\1&3&9\\1&5&25\\1&6&36\end{bmatrix}\begin{bmatrix}\frac{13}{10}\\-\frac{17}{10}\\\frac{1}{3}\end{bmatrix} = \begin{bmatrix}\frac{10}{3}\\-\frac{23}{30}\\-\frac{4}{5}\\\frac{17}{15}\\\frac{31}{10}\end{bmatrix}$$

The quadratic itself is, of course, $\frac{13}{10} + \left(-\frac{17}{10}\right)x + \frac{1}{3}x^2$. Below, the original data is in red, the quadratic data is black, as is the graph of the quadratic.



d) From a linear algebra view point, what vector $\bar{b}$ is being projected onto what subspace V? How would you describe the projection $\bar{p}$ as it relates to the data of the problem?

The vector $\bar{b}$ is the data $\bar{y}$, being projected onto the subspace spanned by the powers x^0, x^1, x^2 evaluated at the points in x. The projection $\bar{p}$ is the quadratic data that comes closest to the y data at the points in x.

2) The least-squares approximation of data (x_i, y_i) with a constant function $f(x) = a$ is obtained when we set $a = y_{av}$, the average value of the y_i. Prove it. (What are the normal equations for a ?)

The equation to be "solved" is $\begin{bmatrix}y_1\\y_2\\\vdots\\y_n\end{bmatrix} = \begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}a$

The normal equations are $\begin{bmatrix}1 & 1 & \vdots & 1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\\vdots\\y_n\end{bmatrix} = \begin{bmatrix}1 & 1 & \vdots & 1\end{bmatrix}\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}a$