MATH 441.001 NAME (print): ____________________

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Review for TEST # 4

Solve the following exercises. **Show your work.** (No credit will be given for an answer with no supporting work shown.)

Ex. 1.

(a) State the Cofactor Formula for finding the inverse $A^{-1} = [x_{ij}]$ of an $n \times n$ matrix $A = [a_{ij}]$.

(?? pts) *Solution:* $x_{ij} = \frac{C_{ji}}{\det(A)}$, where $C_{ji} = (-1)^{j+i}\det(M_{ji})$ and M_{ji} is obtained from A by removing its jth row and ith column.

(b) Use the Cofactor Formula you citated in (a) to find x_{21}, if $A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$.

(?? pts) *Solution:* $\det(A) = \begin{vmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 7 \end{vmatrix} \begin{matrix} \\ -3\vec{r}_1 \\ \\ \\ \end{matrix} = \begin{vmatrix} 1 & 2 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 7 \end{vmatrix} = -2 \cdot 35.$

$C_{12} = (-1)^{1+2} \begin{vmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{vmatrix} = -3 \cdot 35.$ So, $x_{21} = \frac{C_{12}}{\det(A)} = \frac{-3 \cdot 35}{-2 \cdot 35} = 1.5$

Ex. 2. Find all eigenvalues and associated eigenvectors of the matrix $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Is matrix A diagonalizable?

(?? pts) *Solution:* $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3 & 4 \\ 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{vmatrix} = (1-\lambda)(-\lambda)(-\lambda) = (1-\lambda)\lambda^2.$
We have two eigenvalues, $\lambda = 1$ and $\lambda = 0$, the second having (algebraic) multiplicity 2.

To find eigenvector associated with $\lambda = 1$ we need to solve $(A - \lambda I)\vec{x} = 0$. As $A - I =$
$\begin{bmatrix} 0 & 3 & 4 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 3 & 4 \end{bmatrix} \to \xrightarrow[+r_1+7r_2]{\vec{r}_3} \to \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$
indicates one free variable, x_1, and $x_2 = x_3 = 0$, we get a null space $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.
Thus, eigenvalue $\lambda = 1$ is associated with eigenvector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

To find eigenvector(s) associated with $\lambda = 0$ we need to solve $(A - \lambda I)\vec{x} = 0$. As
$A - \lambda I = A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-4\vec{r}_2} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ indicates only one free variable, x_2,
and $x_1 = -3x_2$, $x_3 = 0$, we get a null space $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$. Thus, eigenvalue $\lambda = 0$
(of algebraic multiplicity 2) is associated with a single eigenvector $\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$.

This means, in particular, that A has only two eigenvectors, so A is not diagonalizable.

Math 441 Exam 4

1. (12 pts) Given $A = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$

a) Find the eigenvalues and eigenvectors of A

$$\det\begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix} = \lambda^2 - \lambda - 2 = 0\ ,\ \lambda = -1, 2$$

$\lambda = -1$:

$$A - \lambda I = \begin{bmatrix} 6 & -6 \\ 3 & -3 \end{bmatrix}, \bar{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda = 2$:

$$A - \lambda I = \begin{bmatrix} 3 & -6 \\ 3 & -6 \end{bmatrix},\ \bar{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

b) Write A in diagonalized form, $A = SDS^{-1}$ where D is a diagonal matrix. Then find a formula for A^k. As $k \to \infty$, every column of $\frac{1}{2^k}A^k$ is a multiple of what vector - what is the significance of this vector?

$$A = V\Lambda V^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$A^k = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} (-1)^k & 0 \\ 0 & 2^k \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2^{k+1} + (-1)^{k+1} & -2^{k+1} + 2(-1)^k \\ 2^k + 2(-1)^{k+1} & -2^k + 4(-1)^k \end{bmatrix}$$

$\frac{1}{2^k}A^k = \begin{bmatrix} 2 + 2^{-k}(-1)^{k+1} & -2 + 2^{1-k}(-1)^k \\ 1 + 2^{1-k}(-1)^{k+1} & -1 + 2^{2-k}(-1)^k \end{bmatrix} \to \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}$, each column is a multiple of the eigenvector with the largest eigenvalue, $\lambda = 2$.

You can also calculate:

$$\frac{1}{2^k}A^k = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} (-1)^k 2^{-k} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}$$

2. (12 pts) If $A = \begin{bmatrix} 3 & 2 & -1 \\ 6 & 4 & -2 \\ -3 & -2 & 1 \end{bmatrix}$

The rank of A is __1___ so one eigenvalue is $\lambda =$_0___ . (the null space of $A - 0I$ is not just $\bar{0}$)

There must be _1 / $\boxed{2}$ / 3_____ eigenvectors corresponding to this value of λ because _$\dim N(A) = 3 - 1 = 2$__________.

Those corresponding eigenvectors are: $\bar{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$ (any two independent vectors in the null space will do)

The characteristic polynomial must have a factor of __λ^2___________ because _$\lambda = 0$ has multiplicity at least two due to two independent eigenvectors for $\lambda = 0$________ .

The characteristic polynomial of A is $p(\lambda) =$ __$\lambda^2(8 - \lambda)$_________________ (calculate below).

$$\det \begin{bmatrix} 3-\lambda & 2 & -1 \\ 6 & 4-\lambda & -2 \\ -3 & -2 & 1-\lambda \end{bmatrix} = 8\lambda^2 - \lambda^3$$

The remaining eigenvalue is _$\lambda = 8$______ and the corresponding eigenvector is __$\bar{v}_3 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$_. (Show your work below)

Add the second row to the first below to aid in calculations:

$$\begin{bmatrix} -5 & 2 & -1 \\ 6 & -4 & -2 \\ -3 & -2 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -3 \\ 6 & -4 & -2 \\ -3 & -2 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -3 \\ 0 & 8 & 16 \\ 0 & -8 & -16 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

3. (6 pts)

If I tell you that the eigenvectors of $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ are $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, how can you easily check? What eigenvalues correspond to these eigenvectors?

Take each proposed eigenvector and multiply by A :

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ so } \lambda_1 = 0$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ so } \lambda_2 = 1$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ so } \lambda_3 = 2$$

What eigenvalues correspond to these eigenvectors? Given that A is symmetric, it should have a complete set of orthonormal eigenvectors? What vectors (explicitly) are those? Write A in the form $A = Q\Lambda Q^T$ where Q is an orthogonal matrix.

Divide each eigenvector by its length and create an orthogonal matrix Q out of them:

$$A = Q\Lambda Q^T = \begin{bmatrix} -\frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \\ 0 & 1 & 0 \\ \frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} -\frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \\ 0 & 1 & 0 \\ \frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \end{bmatrix}^T$$

4. (4 pts) Explain the following: If $A = SDS^{-1}$ where D is diagonal and S is some nonsingular matrix, then the eigenvalues of A are the columns of S and the eigenvalues of A are the diagonal entries of D.

In this problem you don't know anything about $S, D,$ except that $A = SDS^{-1}$. Next $AS = SD$ follows by multiplying both sides on the right by S. If we express S in terms of its colums as $S = \begin{bmatrix} \bar{v}_1 \; \bar{v}_2 \; .. \; \bar{v}_n \end{bmatrix}$ and the diagonal entries of D as d_1 , d_2 ,.., d_n then by matrix multiplication we can write:

$AS = \begin{bmatrix} A\bar{v}_1 \; A\bar{v}_2 \; .. \; A\bar{v}_n \end{bmatrix} = \begin{bmatrix} d_1\bar{v}_1 \; d_2\bar{v}_2 \; .. \; d_n\bar{v}_n \end{bmatrix}$

and matching up the columns, we have $A\bar{v}_1 = d_1\bar{v}_1$,.. , $Av_n = d_n\bar{v}_n$ so that the columns of S contain eigenvectors of A and the diagonal entries of D are the corresponding eigenvalues.