Math 441 Review topics for first exam

Vectors, linear combinations of vectors
$A\bar{x}$ as a linear combination of the columns of $A,$ as the dot product of $\bar{x}$ with the rows of $A$
Solve $A\bar{x} = \bar{b}$ $\leftrightarrow$ Express $\bar{b}$ as a linear combination of vectors (the columns of $A$ )

Matrix multiplication:
$AB$
You can build up $AB$ column by column:
its $j^{th}$ column is $A * \left(\text{col}\, j \text{ of } B\right),$ in turn, a linear combination of the columns of $A$
You can build up $AB$ row by row:
its $i^{th}$ row is $\left(\text{row } i \text{ of } A\right) * B,$ in turn, a linear combination of the rows of $B$
You can compute individual elements of $AB$ as required:
its $(i,j)$ element is $\left(\text{row } i \text{ of } A\right) * \left(\text{col } j \text{ of } B\right) = \left(\text{row } i \text{ of } A\right) \cdot \left(\text{col } j \text{ of } B\right)$

$\bar{y}^T\bar{x}$ is 1x1, the dot product of the vectors $\bar{y}$ and $\bar{x}$
$\bar{y}\bar{x}^T$ is $n \times n,$ the rows are multiples of $\bar{x},$ the columns are multiples of $\bar{y}$
$A^T$ the transpose, $(A^T)_{ij} = (A)_{ji}$
$(AB)^T = B^T A^T$

Block multiplication of matrices: In $AB$ you can partition the rows of $A$ and the columns of $B$ any way you want, a partition of the columns of $A$ must match the partition of the rows of $B$

Extreme case: Partition $A$ into columns and $B$ into rows

Matrices to perform elementary operations:

Permutation matrix: A matrix in which the rows or columns of the identity matrix have been permuted (order changed) - an easy description is that there should be exactly one ’1’ in each row and column, everything else zero. $PA$ permutes the rows of $A,$ $AP$ permutes the columns of $A$

Subtract a multiple of a row from another row: $E_{ij}$ subtracts a multiple $\ell_{ij}$ of row $j$ from row $i$ , it is the identity matrix together with $-\ell_{ij}$ in the $(i,j)$ position. Then $E_{ij}A$ performs the desired operation on $A$.

You should be able to "see" how an elementary matrix does its job. However, note that if you want an elementary matrix that performs a certain operation, just apply that operation on $I$ and whatever you get is $E.$ Indeed, if it is a row operation, you know that $EA$ will perform that operation on any $A$ of compatible size so in particular $EI = E$ will be the result of performing that operation on $I.$

Solving square linear systems -
Gaussian elimination with back substitution

$$\left[A \,\vdots\, \bar{b}\right] \rightarrow \left[U \,\vdots\, \bar{b}'\right]$$

Eliminating all the way: Gauss-Jordan elimination

$\left[A \vdots \bar{b}\right] \to \left[U \vdots \bar{b}'\right] \to \left[I \vdots \bar{x}\right]$ when $A$ has been turned into $I$, the solution $\bar{x}$ is on the right

Nonsingular matrix $A$:

has a full set of pivots
is invertible
$A\bar{x} = \bar{b}$ has a solution for each $\bar{b}$
$A\bar{x} = \bar{b}$ can have only one possible solution
$A\bar{x} = \bar{0}$ has only $\bar{x} = \bar{0}$ as a solution

Elementary matrix interpretation of elimination:

$E_k \cdot \cdot E_1 A = U$ or $E_k \cdot \cdot E_1 A = I$

when we do elimination, we are multiplying on the left by a sequence of elementary matrices.

Inverse matrices (only for square matrices):

$AA^{-1} = A^{-1}A = I$

if either way works, so does the other.

$(AB)^{-1} = B^{-1}A^{-1}$

If $AB$ is invertible ($A, B$ both square) then both $A$ and $B$ are invertible.

(One argument: Since $AB$ is invertible, $(AB)\bar{x} = \bar{b}$ has a solution $\bar{x}$ for each $\bar{b}$, so that $A(B\bar{x}) = \bar{b}$. This means that $A\bar{x} = \bar{b}$ has a solution for each $\bar{b}$ and so $A$ is invertible. Then $A^{-1}(AB) = B$ shows that $B$ is the product of two invertible matrices (namely $A$ and $AB$ ) so that $B$ is invertible as well.)

Elimination algorithm for finding $A^{-1}$ :

$\left[A \vdots I\right] \to \left[I \vdots B\right]$ and $B = A^{-1}$

Dual interpretations:

1) Each column on the right, say column $j$ of $B$, must be the solution of $A\bar{x} = \bar{e}_j = (\text{col } j \text{ } of\, I)$ so multiplying out $AB$ column by column must give $AB = I$ and so $B = A^{-1}$.

2) We got from $A$ to $I$ by multiplying by a sequence of elementary matrices: $E_k \cdot \cdot E_1\left[A \vdots I\right] = \left[E_k \cdot \cdot E_1 A \vdots E_k \cdot \cdot E_1 I\right] = \left[I \vdots B\right]$ so on the left side we have $E_k \cdot \cdot E_1 A = I$ which means $E_k \cdot \cdot E_1 = A^{-1}$ and on the right we have $E_k \cdot \cdot E_1 I = E_k \cdot \cdot E_1 = A^{-1}$ so sitting on the right is our inverse.

Note that these dual interpretations show immediately that $\left[A \vdots I\right] \to \left[I \vdots B\right]$ implies that $AB = I$ and $BA = I$.

$LU$ decomposition:

$A = LU$

Works if no row interchanges take us from $A$ to $U$. Start with $I$ and then just store the multipliers $\ell_{ij}$ in the $(i,j)$ position where the zero is produced during elimination. At the end you'll have $L$.

$LU$ reconstructs $A$ by adding back into each row of $U$ the scalar multiplies of the rows of $U$ that lie above it that were subtracted during elimination.

$L$ can also be obtained by inverting the elementary matrices that take us from $A$ to $U$.

$E_k \cdot \cdot E_1 A = U$ so $A = E_1^{-1} \cdot \cdot E_k^{-1} U$

$A = LDU$ where $L, U$ have 1's down their diagonals, and $D$ is diagonal. This form is unique.

To solve $A\bar{x} = \bar{b}$ if we have $A = LU$ we have $L(U\bar{x}) = \bar{b}$ so first solve $L\bar{c} = \bar{b}$ and then $U\bar{x} = \bar{c}$ to find $\bar{x}$. Since $L, U$ are triangular these can be solved by "front substitution" and "back substitution" respectively, or (equivalently) by elimination.

$\left[L \vdots \bar{b}\right] \to \left[I \vdots \bar{c}\right], \left[U \vdots \bar{c}\right] \to \left[I \vdots \bar{x}\right]$

Note that if we were to start with $A$, then $\left[A \vdots \bar{b}\right] \to \left[U \vdots \bar{c}\right]$ takes us to the same second stage but it is much much easier to do $\left[L \vdots \bar{b}\right] \to \left[I \vdots \bar{c}\right]$ given that we have precomputed $L$ or otherwise have it available. Knowing $L$ tells us exactly what elimination steps to apply to $\bar{b}$ with no computation on the left necessary! The elimination steps that turn $L$ into $I$ turn $A$ into $U$ and $\bar{b}$ into $\bar{c}$. This can be explained in a general way as follows: If $A \to I$ through a sequence of elimination steps, then those same elimination steps applied to the matrix $AB$ produces $AB \to B$. This is because a sequence of elimination steps is equivalent to a left multiplication (in this case, multiplication by $A^{-1}$) so applying these same steps to $AB$ results in $A^{-1}(AB) = B$. So if we apply elimination with $L \to I$ then $A = LU \to U$ using the same elimination steps.

$PA = LU$

This form applies If there are row interchanges along the way from $A \to U$. See the homework solutions for a simple algorithm for carrying this out.

Operation count: Elimination $A \to U$ requires about $\frac{n^3}{3}$ operations (multiplications and subtractions). Taking $\bar{b} \to \bar{c}$ or $\left[L \vdots \bar{b}\right] \to \left[I \vdots \bar{c}\right]$ requires about $\frac{n^2}{2}$ operations (no operations on the left are required!) and then $\left[U \vdots \bar{c}\right] \to \left[I \vdots \bar{x}\right]$ needs about another $\frac{n^2}{2}$ operations. So once we have done the $\frac{n^3}{3}$ operations to do elimination from $A$ to $U$, and obtain $A = LU$ (or $PA = LU$) it takes only $n^2$ operations to process $\bar{b} \to \bar{x}$, the solution of $A\bar{x} = \bar{b}$.