

Problem Set 1.1, page 8

- 1 The combinations give (a) a line in \mathbf{R}^3 (b) a plane in \mathbf{R}^3 (c) all of \mathbf{R}^3 .
- 2 $\mathbf{v} + \mathbf{w} = (2, 3)$ and $\mathbf{v} - \mathbf{w} = (6, -1)$ will be the diagonals of the parallelogram with \mathbf{v} and \mathbf{w} as two sides going out from $(0, 0)$.
- 3 This problem gives the diagonals $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$ of the parallelogram and asks for the sides: The opposite of Problem 2. In this example $\mathbf{v} = (3, 3)$ and $\mathbf{w} = (2, -2)$.
- 4 $3\mathbf{v} + \mathbf{w} = (7, 5)$ and $c\mathbf{v} + d\mathbf{w} = (2c + d, c + 2d)$.
- 5 $\mathbf{u} + \mathbf{v} = (-2, 3, 1)$ and $\mathbf{u} + \mathbf{v} + \mathbf{w} = (0, 0, 0)$ and $2\mathbf{u} + 2\mathbf{v} + \mathbf{w} = (-2, 3, 1)$. The vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are in the same plane because a combination gives $(0, 0, 0)$. Stated another way: $\mathbf{u} = -\mathbf{v} - \mathbf{w}$ is in the plane of \mathbf{v} and \mathbf{w} .
- 6 The components of every $c\mathbf{v} + d\mathbf{w}$ add to zero. $c = 3$ and $d = 9$ give $(3, 3, -6)$.
- 7 The nine combinations $c(2, 1) + d(0, 1)$ with $c = 0, 1, 2$ and $d = (0, 1, 2)$ will lie on a lattice. If we took all whole numbers c and d , the lattice would lie over the whole plane.
- 8 The other diagonal is $\mathbf{v} - \mathbf{w}$ (or else $\mathbf{w} - \mathbf{v}$). Adding diagonals gives $2\mathbf{v}$ (or $2\mathbf{w}$).
- 9 The fourth corner can be $(4, 4)$ or $(4, 0)$ or $(-2, 2)$. Three possible parallelograms!
- 10 $\mathbf{i} - \mathbf{j} = (1, 1, 0)$ is in the base (x - y plane). $\mathbf{i} + \mathbf{j} + \mathbf{k} = (1, 1, 1)$ is the opposite corner from $(0, 0, 0)$. Points in the cube have $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$.
- 11 Four more corners $(1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)$. The center point is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Centers of faces are $(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2}, 1)$ and $(0, \frac{1}{2}, \frac{1}{2}), (1, \frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, 0, \frac{1}{2}), (\frac{1}{2}, 1, \frac{1}{2})$.
- 12 A four-dimensional cube has $2^4 = 16$ corners and $2 \cdot 4 = 8$ three-dimensional faces and 24 two-dimensional faces and 32 edges in Worked Example 2.4 A.
- 13 Sum = zero vector. Sum = $-2:00$ vector = $8:00$ vector. $2:00$ is 30° from horizontal = $(\cos \frac{\pi}{6}, \sin \frac{\pi}{6}) = (\sqrt{3}/2, 1/2)$.
- 14 Moving the origin to $6:00$ adds $\mathbf{j} = (0, 1)$ to every vector. So the sum of twelve vectors changes from $\mathbf{0}$ to $12\mathbf{j} = (0, 12)$.
- 15 The point $\frac{3}{4}\mathbf{v} + \frac{1}{4}\mathbf{w}$ is three-fourths of the way to \mathbf{v} starting from \mathbf{w} . The vector $\frac{1}{4}\mathbf{v} + \frac{1}{4}\mathbf{w}$ is halfway to $\mathbf{u} = \frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{w}$. The vector $\mathbf{v} + \mathbf{w}$ is $2\mathbf{u}$ (the far corner of the parallelogram).
- 16 All combinations with $c + d = 1$ are on the line that passes through \mathbf{v} and \mathbf{w} . The point $\mathbf{V} = -\mathbf{v} + 2\mathbf{w}$ is on that line but it is beyond \mathbf{w} .
- 17 All vectors $c\mathbf{v} + d\mathbf{w}$ are on the line passing through $(0, 0)$ and $\mathbf{u} = \frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{w}$. That line continues out beyond $\mathbf{v} + \mathbf{w}$ and back beyond $(0, 0)$. With $c \geq 0$, half of this line is removed, leaving a ray that starts at $(0, 0)$.
- 18 The combinations $c\mathbf{v} + d\mathbf{w}$ with $0 \leq c \leq 1$ and $0 \leq d \leq 1$ fill the parallelogram with sides \mathbf{v} and \mathbf{w} . For example, if $\mathbf{v} = (1, 0)$ and $\mathbf{w} = (0, 1)$ then $c\mathbf{v} + d\mathbf{w}$ fills the unit square.
- 19 With $c \geq 0$ and $d \geq 0$ we get the infinite "cone" or "wedge" between \mathbf{v} and \mathbf{w} . For example, if $\mathbf{v} = (1, 0)$ and $\mathbf{w} = (0, 1)$, then the cone is the whole quadrant $x \geq 0, y \geq 0$. Question: What if $\mathbf{w} = -\mathbf{v}$? The cone opens to a half-space.

- 20 (a) $\frac{1}{3}\mathbf{u} + \frac{1}{3}\mathbf{v} + \frac{1}{3}\mathbf{w}$ is the center of the triangle between \mathbf{u} , \mathbf{v} and \mathbf{w} ; $\frac{1}{2}\mathbf{u} + \frac{1}{2}\mathbf{w}$ lies between \mathbf{u} and \mathbf{w} (b) To fill the triangle keep $c \geq 0, d \geq 0, e \geq 0$, and $c + d + e = 1$.
- 21 The sum is $(\mathbf{v} - \mathbf{u}) + (\mathbf{w} - \mathbf{v}) + (\mathbf{u} - \mathbf{w}) = \mathbf{zero\ vector}$. Those three sides of a triangle are in the same plane!
- 22 The vector $\frac{1}{2}(\mathbf{u} + \mathbf{v} + \mathbf{w})$ is *outside* the pyramid because $c + d + e = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} > 1$.
- 23 All vectors are combinations of \mathbf{u} , \mathbf{v} , \mathbf{w} as drawn (not in the same plane). Start by seeing that $c\mathbf{u} + d\mathbf{v}$ fills a plane, then adding $e\mathbf{w}$ fills all of \mathbf{R}^3 .
- 24 The combinations of \mathbf{u} and \mathbf{v} fill one plane. The combinations of \mathbf{v} and \mathbf{w} fill another plane. Those planes meet in a *line*: *only the vectors* $c\mathbf{v}$ are in both planes.
- 25 (a) For a line, choose $\mathbf{u} = \mathbf{v} = \mathbf{w} =$ any nonzero vector (b) For a plane, choose \mathbf{u} and \mathbf{v} in different directions. A combination like $\mathbf{w} = \mathbf{u} + \mathbf{v}$ is in the same plane.
- 26 Two equations come from the two components: $c + 3d = 14$ and $2c + d = 8$. *The solution is* $c = 2$ and $d = 4$. Then $2(1, 2) + 4(3, 1) = (14, 8)$.
- 27 The combinations of $\mathbf{i} = (1, 0, 0)$ and $\mathbf{i} + \mathbf{j} = (1, 1, 0)$ fill the xy plane in xyz space.
- 28 There are **6** unknown numbers $v_1, v_2, v_3, w_1, w_2, w_3$. The six equations come from the components of $\mathbf{v} + \mathbf{w} = (4, 5, 6)$ and $\mathbf{v} - \mathbf{w} = (2, 5, 8)$. Add to find $2\mathbf{v} = (6, 10, 14)$ so $\mathbf{v} = (3, 5, 7)$ and $\mathbf{w} = (1, 0, -1)$.
- 29 Two combinations out of infinitely many that produce $\mathbf{b} = (0, 1)$ are $-2\mathbf{u} + \mathbf{v}$ and $\frac{1}{2}\mathbf{w} - \frac{1}{2}\mathbf{v}$. **No**, three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in the x - y plane could fail to produce \mathbf{b} if all three lie on a line that does not contain \mathbf{b} . *Yes*, if one combination produces \mathbf{b} then two (and infinitely many) combinations will produce \mathbf{b} . This is true even if $\mathbf{u} = \mathbf{0}$; the combinations can have different $c\mathbf{u}$.
- 30 The combinations of \mathbf{v} and \mathbf{w} fill the plane *unless* \mathbf{v} and \mathbf{w} lie on the same line through $(0, 0)$. Four vectors whose combinations fill 4-dimensional space: one example is the "standard basis" $(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0),$ and $(0, 0, 0, 1)$.
- 31 The equations $c\mathbf{u} + d\mathbf{v} + e\mathbf{w} = \mathbf{b}$ are

$$\begin{array}{rcl} 2c - d & = & 1 \\ -c + 2d - e & = & 0 \\ -d + 2e & = & 0 \end{array} \quad \begin{array}{l} \text{So } d = 2e \\ \text{then } c = 3e \\ \text{then } 4e = 1 \end{array} \quad \begin{array}{l} c = 3/4 \\ d = 2/4 \\ e = 1/4 \end{array}$$

Problem Set 1.2, page 19

- 1 $\mathbf{u} \cdot \mathbf{v} = -1.8 + 3.2 = 1.4, \mathbf{u} \cdot \mathbf{w} = -4.8 + 4.8 = 0, \mathbf{v} \cdot \mathbf{w} = 24 + 24 = 48 = \mathbf{w} \cdot \mathbf{v}$.
- 2 $\|\mathbf{u}\| = 1$ and $\|\mathbf{v}\| = 5$ and $\|\mathbf{w}\| = 10$. Then $1.4 < (1)(5)$ and $48 < (5)(10)$, confirming the Schwarz inequality.
- 3 Unit vectors $\mathbf{v}/\|\mathbf{v}\| = (\frac{3}{5}, \frac{4}{5}) = (.6, .8)$ and $\mathbf{w}/\|\mathbf{w}\| = (\frac{4}{5}, \frac{3}{5}) = (.8, .6)$. The cosine of θ is $\frac{\mathbf{v}}{\|\mathbf{v}\|} \cdot \frac{\mathbf{w}}{\|\mathbf{w}\|} = \frac{24}{25}$. The vectors $\mathbf{w}, \mathbf{u}, -\mathbf{w}$ make $0^\circ, 90^\circ, 180^\circ$ angles with \mathbf{w} .
- 4 (a) $\mathbf{v} \cdot (-\mathbf{v}) = -1$ (b) $(\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{v} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{w} - \mathbf{w} \cdot \mathbf{w} = 1 + () - () - 1 = 0$ so $\theta = 90^\circ$ (notice $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$) (c) $(\mathbf{v} - 2\mathbf{w}) \cdot (\mathbf{v} + 2\mathbf{w}) = \mathbf{v} \cdot \mathbf{v} - 4\mathbf{w} \cdot \mathbf{w} = 1 - 4 = -3$.

- 5 $\mathbf{u}_1 = \mathbf{v}/\|\mathbf{v}\| = (3, 1)/\sqrt{10}$ and $\mathbf{u}_2 = \mathbf{w}/\|\mathbf{w}\| = (2, 1, 2)/3$. $\mathbf{U}_1 = (1, -3)/\sqrt{10}$ is perpendicular to \mathbf{u}_1 (and so is $(-1, 3)/\sqrt{10}$). \mathbf{U}_2 could be $(1, -2, 0)/\sqrt{5}$: There is a whole plane of vectors perpendicular to \mathbf{u}_2 , and a whole circle of unit vectors in that plane.
- 6 All vectors $\mathbf{w} = (c, 2c)$ are perpendicular to \mathbf{v} . All vectors (x, y, z) with $x + y + z = 0$ lie on a *plane*. All vectors perpendicular to $(1, 1, 1)$ and $(1, 2, 3)$ lie on a *line*.
- 7 (a) $\cos \theta = \mathbf{v} \cdot \mathbf{w}/\|\mathbf{v}\|\|\mathbf{w}\| = 1/(2)(1)$ so $\theta = 60^\circ$ or $\pi/3$ radians (b) $\cos \theta = 0$ so $\theta = 90^\circ$ or $\pi/2$ radians (c) $\cos \theta = 2/(2)(2) = 1/2$ so $\theta = 60^\circ$ or $\pi/3$ (d) $\cos \theta = -1/\sqrt{2}$ so $\theta = 135^\circ$ or $3\pi/4$.
- 8 (a) False: \mathbf{v} and \mathbf{w} are any vectors in the plane perpendicular to \mathbf{u} (b) True: $\mathbf{u} \cdot (\mathbf{v} + 2\mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + 2\mathbf{u} \cdot \mathbf{w} = 0$ (c) True, $\|\mathbf{u} - \mathbf{v}\|^2 = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})$ splits into $\mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} = 2$ when $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} = 0$.
- 9 If $v_2 w_2 / v_1 w_1 = -1$ then $v_2 w_2 = -v_1 w_1$ or $v_1 w_1 + v_2 w_2 = \mathbf{v} \cdot \mathbf{w} = 0$: perpendicular!
- 10 Slopes $2/1$ and $-1/2$ multiply to give -1 : then $\mathbf{v} \cdot \mathbf{w} = 0$ and the vectors (the directions) are perpendicular.
- 11 $\mathbf{v} \cdot \mathbf{w} < 0$ means angle $> 90^\circ$; these \mathbf{w} 's fill half of 3-dimensional space.
- 12 $(1, 1)$ perpendicular to $(1, 5) - c(1, 1)$ if $6 - 2c = 0$ or $c = 3$; $\mathbf{v} \cdot (\mathbf{w} - c\mathbf{v}) = 0$ if $c = \mathbf{v} \cdot \mathbf{w} / \mathbf{v} \cdot \mathbf{v}$. Subtracting $c\mathbf{v}$ is the key to perpendicular vectors.
- 13 The plane perpendicular to $(1, 0, 1)$ contains all vectors $(c, d, -c)$. In that plane, $\mathbf{v} = (1, 0, -1)$ and $\mathbf{w} = (0, 1, 0)$ are perpendicular.
- 14 One possibility among many: $\mathbf{u} = (1, -1, 0, 0)$, $\mathbf{v} = (0, 0, 1, -1)$, $\mathbf{w} = (1, 1, -1, -1)$ and $(1, 1, 1, 1)$ are perpendicular to each other. "We can rotate those $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in their 3D hyperplane."
- 15 $\frac{1}{2}(x + y) = (2 + 8)/2 = 5$; $\cos \theta = 2\sqrt{16}/\sqrt{10}\sqrt{10} = 8/10$.
- 16 $\|\mathbf{v}\|^2 = 1 + 1 + \dots + 1 = 9$ so $\|\mathbf{v}\| = 3$; $\mathbf{u} = \mathbf{v}/3 = (\frac{1}{3}, \dots, \frac{1}{3})$ is a unit vector in 9D; $\mathbf{w} = (1, -1, 0, \dots, 0)/\sqrt{2}$ is a unit vector in the 8D hyperplane perpendicular to \mathbf{v} .
- 17 $\cos \alpha = 1/\sqrt{2}$, $\cos \beta = 0$, $\cos \gamma = -1/\sqrt{2}$. For any vector \mathbf{v} , $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = (v_1^2 + v_2^2 + v_3^2)/\|\mathbf{v}\|^2 = 1$.
- 18 $\|\mathbf{v}\|^2 = 4^2 + 2^2 = 20$ and $\|\mathbf{w}\|^2 = (-1)^2 + 2^2 = 5$. Pythagoras is $\|(3, 4)\|^2 = 25 = 20 + 5$.
- 19 Start from the rules (1), (2), (3) for $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$ and $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})$ and $(c\mathbf{v}) \cdot \mathbf{w}$. Use rule (2) for $(\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{v} + \mathbf{w}) \cdot \mathbf{v} + (\mathbf{v} + \mathbf{w}) \cdot \mathbf{w}$. By rule (1) this is $\mathbf{v} \cdot (\mathbf{v} + \mathbf{w}) + \mathbf{w} \cdot (\mathbf{v} + \mathbf{w})$. Rule (2) again gives $\mathbf{v} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{v} + 2\mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{w}$. Notice $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$! The main point is to be free to open up parentheses.
- 20 We know that $(\mathbf{v} - \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{v} \cdot \mathbf{v} - 2\mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{w}$. The Law of Cosines writes $\|\mathbf{v}\|\|\mathbf{w}\|\cos \theta$ for $\mathbf{v} \cdot \mathbf{w}$. When $\theta < 90^\circ$ this $\mathbf{v} \cdot \mathbf{w}$ is positive, so in this case $\mathbf{v} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{w}$ is larger than $\|\mathbf{v} - \mathbf{w}\|^2$.
- 21 $2\mathbf{v} \cdot \mathbf{w} \leq 2\|\mathbf{v}\|\|\mathbf{w}\|$ leads to $\|\mathbf{v} + \mathbf{w}\|^2 = \mathbf{v} \cdot \mathbf{v} + 2\mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{w} \leq \|\mathbf{v}\|^2 + 2\|\mathbf{v}\|\|\mathbf{w}\| + \|\mathbf{w}\|^2$. This is $(\|\mathbf{v}\| + \|\mathbf{w}\|)^2$. Taking square roots gives $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$.
- 22 $v_1^2 w_1^2 + 2v_1 w_1 v_2 w_2 + v_2^2 w_2^2 \leq v_1^2 w_1^2 + v_1^2 w_2^2 + v_2^2 w_1^2 + v_2^2 w_2^2$ is true (cancel 4 terms) because the difference is $v_1^2 w_2^2 + v_2^2 w_1^2 - 2v_1 w_1 v_2 w_2$ which is $(v_1 w_2 - v_2 w_1)^2 \geq 0$.

- 23** $\cos \beta = w_1/\|\mathbf{w}\|$ and $\sin \beta = w_2/\|\mathbf{w}\|$. Then $\cos(\beta - \alpha) = \cos \beta \cos \alpha + \sin \beta \sin \alpha = v_1 w_1/\|\mathbf{v}\|\|\mathbf{w}\| + v_2 w_2/\|\mathbf{v}\|\|\mathbf{w}\| = \mathbf{v} \cdot \mathbf{w}/\|\mathbf{v}\|\|\mathbf{w}\|$. This is $\cos \theta$ because $\beta - \alpha = \theta$.
- 24** Example 6 gives $|u_1|U_1 \leq \frac{1}{2}(u_1^2 + U_1^2)$ and $|u_2|U_2 \leq \frac{1}{2}(u_2^2 + U_2^2)$. The whole line becomes $.96 \leq (.6)(.8) + (.8)(.6) \leq \frac{1}{2}(.6^2 + .8^2) + \frac{1}{2}(.8^2 + .6^2) = 1$. True: $.96 < 1$.
- 25** The cosine of θ is $x/\sqrt{x^2 + y^2}$, near side over hypotenuse. Then $|\cos \theta|^2$ is not greater than 1: $x^2/(x^2 + y^2) \leq 1$.
- 26** The vectors $\mathbf{w} = (x, y)$ with $(1, 2) \cdot \mathbf{w} = x + 2y = 5$ lie on a line in the xy plane. The shortest \mathbf{w} on that line is $(1, 2)$. (The Schwarz inequality $\|\mathbf{w}\| \geq \mathbf{v} \cdot \mathbf{w}/\|\mathbf{v}\| = \sqrt{5}$ is an equality when $\cos \theta = 0$ and $\mathbf{w} = (1, 2)$ and $\|\mathbf{w}\| = \sqrt{5}$.)
- 27** The length $\|\mathbf{v} - \mathbf{w}\|$ is between 2 and 8 (triangle inequality when $\|\mathbf{v}\| = 5$ and $\|\mathbf{w}\| = 3$). The dot product $\mathbf{v} \cdot \mathbf{w}$ is between -15 and 15 by the Schwarz inequality.
- 28** Three vectors in the plane could make angles greater than 90° with each other: for example $(1, 0)$, $(-1, 4)$, $(-1, -4)$. Four vectors could *not* do this (360° total angle). How many can do this in \mathbf{R}^3 or \mathbf{R}^n ? Ben Harris and Greg Marks showed me that the answer is $n + 1$. The vectors from the center of a regular simplex in \mathbf{R}^n to its $n + 1$ vertices all have negative dot products. If $n + 2$ vectors in \mathbf{R}^n had negative dot products, project them onto the plane orthogonal to the last one. Now you have $n + 1$ vectors in \mathbf{R}^{n-1} with negative dot products. Keep going to 4 vectors in \mathbf{R}^2 : no way!
- 29** For a specific example, pick $\mathbf{v} = (1, 2, -3)$ and then $\mathbf{w} = (-3, 1, 2)$. In this example $\cos \theta = \mathbf{v} \cdot \mathbf{w}/\|\mathbf{v}\|\|\mathbf{w}\| = -7/\sqrt{14}\sqrt{14} = -1/2$ and $\theta = 120^\circ$. This always happens when $x + y + z = 0$:

$$\mathbf{v} \cdot \mathbf{w} = xz + xy + yz = \frac{1}{2}(x + y + z)^2 - \frac{1}{2}(x^2 + y^2 + z^2)$$

$$\text{This is the same as } \mathbf{v} \cdot \mathbf{w} = 0 - \frac{1}{2}\|\mathbf{v}\|\|\mathbf{w}\|. \text{ Then } \cos \theta = \frac{1}{2}.$$

- 30** Wikipedia gives this proof of geometric mean $G = \sqrt[3]{xyz} \leq$ arithmetic mean $A = (x + y + z)/3$. First there is equality in case $x = y = z$. Otherwise A is somewhere between the three positive numbers, say for example $z < A < y$. Use the known inequality $g \leq a$ for the *two* positive numbers x and $y + z - A$. Their mean $a = \frac{1}{2}(x + y + z - A)$ is $\frac{1}{2}(3A - A) =$ same as A ! So $a \geq g$ says that $A^3 \geq g^2 A = x(y + z - A)A$. But $(y + z - A)A = (y - A)(A - z) + yz > yz$. Substitute to find $A^3 > xyz = G^3$ as we wanted to prove. Not easy!
- There are many proofs of $G = (x_1 x_2 \cdots x_n)^{1/n} \leq A = (x_1 + x_2 + \cdots + x_n)/n$. In calculus you are maximizing G on the plane $x_1 + x_2 + \cdots + x_n = n$. The maximum occurs when all x 's are equal.
- 31** The columns of the 4 by 4 "Hadamard matrix" (times $\frac{1}{2}$) are perpendicular unit vectors:

$$\frac{1}{2}H = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

- 32** The commands $V = \mathbf{randn}(3, 30)$; $D = \mathbf{sqrt}(\mathbf{diag}(V' * V))$; $U = V \setminus D$; will give 30 random unit vectors in the columns of U . Then $u' * U$ is a row matrix of 30 dot products whose average absolute value may be close to $2/\pi$.

Problem Set 1.3, page 29

1 $2s_1 + 3s_2 + 4s_3 = (2, 5, 9)$. The same vector \mathbf{b} comes from S times $\mathbf{x} = (2, 3, 4)$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} (\text{row 1}) \cdot \mathbf{x} \\ (\text{row 2}) \cdot \mathbf{x} \\ (\text{row 3}) \cdot \mathbf{x} \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}.$$

2 The solutions are $y_1 = 1, y_2 = 0, y_3 = 0$ (right side = column 1) and $y_1 = 1, y_2 = 3, y_3 = 5$. That second example illustrates that the first n odd numbers add to n^2 .

$$\begin{array}{rcl} y_1 & = & B_1 \\ y_1 + y_2 & = & B_2 \\ y_1 + y_2 + y_3 & = & B_3 \end{array} \quad \text{gives} \quad \begin{array}{rcl} y_1 & = & B_1 \\ y_2 & = & -B_1 + B_2 \\ y_3 & = & -B_2 + B_3 \end{array} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

The inverse of $S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ is $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$: independent columns in A and S !

4 The combination $0\mathbf{w}_1 + 0\mathbf{w}_2 + 0\mathbf{w}_3$ always gives the zero vector, but this problem looks for other *zero* combinations (then the vectors are *dependent*, they lie in a plane): $\mathbf{w}_2 = (\mathbf{w}_1 + \mathbf{w}_3)/2$ so one combination that gives zero is $\frac{1}{2}\mathbf{w}_1 - \mathbf{w}_2 + \frac{1}{2}\mathbf{w}_3$.

5 The rows of the 3 by 3 matrix in Problem 4 must also be *dependent*: $\mathbf{r}_2 = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_3)$. The column and row combinations that produce $\mathbf{0}$ are the same: this is unusual.

$$6 \quad c = 3 \quad \begin{bmatrix} 1 & 3 & 5 \\ 1 & 2 & 4 \\ 1 & 1 & 3 \end{bmatrix} \quad \text{has column 3} = 2(\text{column 1}) + \text{column 2}$$

$$c = -1 \quad \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{has column 3} = -\text{column 1} + \text{column 2}$$

$$c = 0 \quad \begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 5 \\ 3 & 3 & 6 \end{bmatrix} \quad \text{has column 3} = 3(\text{column 1}) - \text{column 2}$$

7 All three rows are perpendicular to the solution \mathbf{x} (the three equations $\mathbf{r}_1 \cdot \mathbf{x} = 0$ and $\mathbf{r}_2 \cdot \mathbf{x} = 0$ and $\mathbf{r}_3 \cdot \mathbf{x} = 0$ tell us this). Then the whole plane of the rows is perpendicular to \mathbf{x} (the plane is also perpendicular to all multiples $c\mathbf{x}$).

$$\begin{array}{rcl} x_1 - 0 & = & b_1 \\ x_2 - x_1 & = & b_2 \\ x_3 - x_2 & = & b_3 \\ x_4 - x_3 & = & b_4 \end{array} \quad \begin{array}{rcl} x_1 & = & b_1 \\ x_2 & = & b_1 + b_2 \\ x_3 & = & b_1 + b_2 + b_3 \\ x_4 & = & b_1 + b_2 + b_3 + b_4 \end{array} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = A^{-1}\mathbf{b}$$

9 The cyclic difference matrix C has a line of solutions (in 4 dimensions) to $C\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{when } \mathbf{x} = \begin{bmatrix} c \\ c \\ c \\ c \end{bmatrix} = \text{any constant vector.}$$

$$10 \quad \begin{array}{rcl} z_2 - z_1 = b_1 & z_1 = -b_1 - b_2 - b_3 & \\ z_3 - z_2 = b_2 & z_2 = -b_2 - b_3 & \\ 0 - z_3 = b_3 & z_3 = -b_3 & \end{array} = \begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \Delta^{-1} \mathbf{b}$$

11 The forward differences of the squares are $(t+1)^2 - t^2 = t^2 + 2t + 1 - t^2 = 2t + 1$. Differences of the n th power are $(t+1)^n - t^n = t^n - t^n + nt^{n-1} + \dots$. The leading term is the derivative nt^{n-1} . The binomial theorem gives all the terms of $(t+1)^n$.

12 Centered difference matrices of *even* size seem to be invertible. Look at eqns. 1 and 4:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \quad \begin{array}{l} \text{First} \\ \text{solve} \\ x_2 = b_1 \\ -x_3 = b_4 \end{array} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -b_2 - b_4 \\ b_1 \\ -b_4 \\ b_1 + b_3 \end{bmatrix}$$

13 *Odd size*: The five centered difference equations lead to $b_1 + b_3 + b_5 = 0$.

$$\begin{array}{rcl} x_2 & = & b_1 \\ x_3 - x_1 & = & b_2 \\ x_4 - x_2 & = & b_3 \\ x_5 - x_3 & = & b_4 \\ -x_4 & = & b_5 \end{array} \quad \begin{array}{l} \text{Add equations 1, 3, 5} \\ \text{The left side of the sum is zero} \\ \text{The right side is } b_1 + b_3 + b_5 \\ \text{There cannot be a solution unless } b_1 + b_3 + b_5 = 0. \end{array}$$

14 An example is $(a, b) = (3, 6)$ and $(c, d) = (1, 2)$. The ratios a/c and b/d are equal. Then $ad = bc$. Then (when you divide by bd) the ratios a/b and c/d are equal!

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- The columns are $\mathbf{i} = (1, 0, 0)$ and $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$ and $\mathbf{b} = (2, 3, 4) = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$.
- The planes are the same: $2x = 4$ is $x = 2$, $3y = 9$ is $y = 3$, and $4z = 16$ is $z = 4$. The solution is the same point $\mathbf{X} = \mathbf{x}$. The columns are changed; but same combination.
- The solution is not changed! The second plane and row 2 of the matrix and all columns of the matrix (vectors in the column picture) are changed.
- If $z = 2$ then $x + y = 0$ and $x - y = z$ give the point $(1, -1, 2)$. If $z = 0$ then $x + y = 6$ and $x - y = 4$ produce $(5, 1, 0)$. Halfway between those is $(3, 0, 1)$.
- If x, y, z satisfy the first two equations they also satisfy the third equation. The line \mathbf{L} of solutions contains $\mathbf{v} = (1, 1, 0)$ and $\mathbf{w} = (\frac{1}{2}, 1, \frac{1}{2})$ and $\mathbf{u} = \frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{w}$ and all combinations $c\mathbf{v} + d\mathbf{w}$ with $c + d = 1$.
- Equation 1 + equation 2 - equation 3 is now $0 = -4$. Line misses plane; *no solution*.
- Column 3 = Column 1 makes the matrix singular. Solutions $(x, y, z) = (1, 1, 0)$ or $(0, 1, 1)$ and you can add any multiple of $(-1, 0, 1)$; $\mathbf{b} = (4, 6, c)$ needs $c = 10$ for solvability (then \mathbf{b} lies in the plane of the columns).
- Four planes in 4-dimensional space normally meet at a *point*. The solution to $A\mathbf{x} = (3, 3, 3, 2)$ is $\mathbf{x} = (0, 0, 1, 2)$ if A has columns $(1, 0, 0, 0)$, $(1, 1, 0, 0)$, $(1, 1, 1, 0)$, $(1, 1, 1, 1)$. The equations are $x + y + z + t = 3$, $y + z + t = 3$, $z + t = 3$, $t = 2$.
- (a) $A\mathbf{x} = (18, 5, 0)$ and (b) $A\mathbf{x} = (3, 4, 5, 5)$.