

$$10 \quad \begin{array}{rcl} z_2 - z_1 = b_1 & z_1 = -b_1 - b_2 - b_3 & \\ z_3 - z_2 = b_2 & z_2 = -b_2 - b_3 & \\ 0 - z_3 = b_3 & z_3 = -b_3 & \end{array} = \begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \Delta^{-1} \mathbf{b}$$

11 The forward differences of the squares are  $(t+1)^2 - t^2 = t^2 + 2t + 1 - t^2 = 2t + 1$ . Differences of the  $n$ th power are  $(t+1)^n - t^n = t^n - t^n + nt^{n-1} + \dots$ . The leading term is the derivative  $nt^{n-1}$ . The binomial theorem gives all the terms of  $(t+1)^n$ .

12 Centered difference matrices of *even* size seem to be invertible. Look at eqns. 1 and 4:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \quad \begin{array}{l} \text{First} \\ \text{solve} \\ x_2 = b_1 \\ -x_3 = b_4 \end{array} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -b_2 - b_4 \\ b_1 \\ -b_4 \\ b_1 + b_3 \end{bmatrix}$$

13 *Odd size*: The five centered difference equations lead to  $b_1 + b_3 + b_5 = 0$ .

$$\begin{array}{rcl} x_2 & = & b_1 \\ x_3 - x_1 & = & b_2 \\ x_4 - x_2 & = & b_3 \\ x_5 - x_3 & = & b_4 \\ -x_4 & = & b_5 \end{array} \quad \begin{array}{l} \text{Add equations 1, 3, 5} \\ \text{The left side of the sum is zero} \\ \text{The right side is } b_1 + b_3 + b_5 \\ \text{There cannot be a solution unless } b_1 + b_3 + b_5 = 0. \end{array}$$

14 An example is  $(a, b) = (3, 6)$  and  $(c, d) = (1, 2)$ . The ratios  $a/c$  and  $b/d$  are equal. Then  $ad = bc$ . Then (when you divide by  $bd$ ) the ratios  $a/b$  and  $c/d$  are equal!

## Problem Set 2.1, page 40

- The columns are  $\mathbf{i} = (1, 0, 0)$  and  $\mathbf{j} = (0, 1, 0)$  and  $\mathbf{k} = (0, 0, 1)$  and  $\mathbf{b} = (2, 3, 4) = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ .
- The planes are the same:  $2x = 4$  is  $x = 2$ ,  $3y = 9$  is  $y = 3$ , and  $4z = 16$  is  $z = 4$ . The solution is the same point  $\mathbf{X} = \mathbf{x}$ . The columns are changed; but same combination.
- The solution is not changed! The second plane and row 2 of the matrix and all columns of the matrix (vectors in the column picture) are changed.
- If  $z = 2$  then  $x + y = 0$  and  $x - y = z$  give the point  $(1, -1, 2)$ . If  $z = 0$  then  $x + y = 6$  and  $x - y = 4$  produce  $(5, 1, 0)$ . Halfway between those is  $(3, 0, 1)$ .
- If  $x, y, z$  satisfy the first two equations they also satisfy the third equation. The line  $\mathbf{L}$  of solutions contains  $\mathbf{v} = (1, 1, 0)$  and  $\mathbf{w} = (\frac{1}{2}, 1, \frac{1}{2})$  and  $\mathbf{u} = \frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{w}$  and all combinations  $c\mathbf{v} + d\mathbf{w}$  with  $c + d = 1$ .
- Equation 1 + equation 2 - equation 3 is now  $0 = -4$ . Line misses plane; *no solution*.
- Column 3 = Column 1 makes the matrix singular. Solutions  $(x, y, z) = (1, 1, 0)$  or  $(0, 1, 1)$  and you can add any multiple of  $(-1, 0, 1)$ ;  $\mathbf{b} = (4, 6, c)$  needs  $c = 10$  for solvability (then  $\mathbf{b}$  lies in the plane of the columns).
- Four planes in 4-dimensional space normally meet at a *point*. The solution to  $A\mathbf{x} = (3, 3, 3, 2)$  is  $\mathbf{x} = (0, 0, 1, 2)$  if  $A$  has columns  $(1, 0, 0, 0)$ ,  $(1, 1, 0, 0)$ ,  $(1, 1, 1, 0)$ ,  $(1, 1, 1, 1)$ . The equations are  $x + y + z + t = 3$ ,  $y + z + t = 3$ ,  $z + t = 3$ ,  $t = 2$ .
- (a)  $A\mathbf{x} = (18, 5, 0)$  and (b)  $A\mathbf{x} = (3, 4, 5, 5)$ .

- 10** Multiplying as linear combinations of the columns gives the same  $A\mathbf{x}$ . By rows or by columns: **9** separate multiplications for 3 by 3.
- 11**  $A\mathbf{x}$  equals (14, 22) and (0, 0) and (9, 7).
- 12**  $A\mathbf{x}$  equals (z, y, x) and (0, 0, 0) and (3, 3, 6).
- 13** (a)  $\mathbf{x}$  has  $n$  components and  $A\mathbf{x}$  has  $m$  components (b) Planes from each equation in  $A\mathbf{x} = \mathbf{b}$  are in  $n$ -dimensional space, but the columns are in  $m$ -dimensional space.
- 14**  $2x + 3y + z + 5t = 8$  is  $A\mathbf{x} = \mathbf{b}$  with the 1 by 4 matrix  $A = [2 \ 3 \ 1 \ 5]$ . The solutions  $\mathbf{x}$  fill a 3D “plane” in 4 dimensions. It could be called a *hyperplane*.
- 15** (a)  $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  (b)  $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- 16**  $90^\circ$  rotation from  $R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ ,  $180^\circ$  rotation from  $R^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$ .
- 17**  $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$  produces (y, z, x) and  $Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$  recovers (x, y, z).  $Q$  is the inverse of  $P$ .
- 18**  $E = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$  and  $E = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  subtract the first component from the second.
- 19**  $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$  and  $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ ,  $E\mathbf{v} = (3, 4, 8)$  and  $E^{-1}E\mathbf{v}$  recovers (3, 4, 5).
- 20**  $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  projects onto the  $x$ -axis and  $P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$  projects onto the  $y$ -axis.  $\mathbf{v} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$  has  $P_1\mathbf{v} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$  and  $P_2P_1\mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ .
- 21**  $R = \frac{1}{2} \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$  rotates all vectors by  $45^\circ$ . The columns of  $R$  are the results from rotating (1, 0) and (0, 1)!
- 22** The dot product  $A\mathbf{x} = [1 \ 4 \ 5] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = (1 \text{ by } 3)(3 \text{ by } 1)$  is zero for points (x, y, z) on a plane in three dimensions. The columns of  $A$  are one-dimensional vectors.
- 23**  $A = [1 \ 2 \ ; \ 3 \ 4]$  and  $\mathbf{x} = [5 \ -2]'$  and  $\mathbf{b} = [1 \ 7]'$ .  $\mathbf{r} = \mathbf{b} - A * \mathbf{x}$  prints as zero.
- 24**  $A * \mathbf{v} = [3 \ 4 \ 5]'$  and  $\mathbf{v}' * \mathbf{v} = 50$ . But  $\mathbf{v} * A$  gives an error message from 3 by 1 times 3 by 3.
- 25**  $\mathbf{ones}(4, 4) * \mathbf{ones}(4, 1) = [4 \ 4 \ 4 \ 4]'$ ;  $B * \mathbf{w} = [10 \ 10 \ 10 \ 10]'$ .
- 26** The row picture has two lines meeting at the solution (4, 2). The column picture will have  $4(1, 1) + 2(-2, 1) = 4(\text{column } 1) + 2(\text{column } 2) = \text{right side } (0, 6)$ .
- 27** The row picture shows **2 planes in 3-dimensional space**. The column picture is in **2-dimensional space**. The solutions normally lie on a *line*.

- 28** The row picture shows four *lines* in the 2D plane. The column picture is in *four-dimensional* space. No solution unless the right side is a combination of *the two columns*.
- 29**  $u_2 = \begin{bmatrix} .7 \\ .3 \end{bmatrix}$  and  $u_3 = \begin{bmatrix} .65 \\ .35 \end{bmatrix}$ . The components add to 1. They are always positive.  $u_7, v_7, w_7$  are all close to  $(.6, .4)$ . Their components still add to 1.
- 30**  $\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} .6 \\ .4 \end{bmatrix} = \begin{bmatrix} .6 \\ .4 \end{bmatrix} = \text{steady state } s$ . No change when multiplied by  $\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$ .
- 31**  $M = \begin{bmatrix} 8 & 3 & 4 \\ 1 & 5 & 9 \\ 6 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 5+u & 5-u+v & 5-v \\ 5-u-v & 5 & 5+u+v \\ 5+v & 5+u-v & 5-u \end{bmatrix}$ ;  $M_3(1, 1, 1) = (15, 15, 15)$ ;  
 $M_4(1, 1, 1, 1) = (34, 34, 34, 34)$  because  $1 + 2 + \cdots + 16 = 136$  which is  $4(34)$ .
- 32**  $A$  is singular when its third column  $w$  is a combination  $cu + dv$  of the first columns. A typical column picture has  $b$  outside the plane of  $u, v, w$ . A typical row picture has the intersection line of two planes parallel to the third plane. *Then no solution.*
- 33**  $w = (5, 7)$  is  $5u + 7v$ . Then  $Aw$  equals 5 times  $Au$  plus 7 times  $Av$ .
- 34**  $\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$  has the solution  $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 8 \\ 6 \end{bmatrix}$ .
- 35**  $x = (1, \dots, 1)$  gives  $Sx =$  sum of each row  $= 1 + \cdots + 9 = 45$  for Sudoku matrices. 6 row orders  $(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)$  are in Section 2.7. The same 6 permutations of *blocks* of rows produce Sudoku matrices, so  $6^4 = 1296$  orders of the 9 rows all stay Sudoku. (And also 1296 permutations of the 9 columns.)

## Problem Set 2.2, page 51

- Multiply by  $\ell_{21} = \frac{10}{2} = 5$  and subtract to find  $2x + 3y = 14$  and  $-6y = 6$ . The pivots to circle are 2 and  $-6$ .
- $-6y = 6$  gives  $y = -1$ . Then  $2x + 3y = 1$  gives  $x = 2$ . Multiplying the right side  $(1, 11)$  by 4 will multiply the solution by 4 to give the new solution  $(x, y) = (8, -4)$ .
- Subtract  $-\frac{1}{2}$  (or add  $\frac{1}{2}$ ) times equation 1. The new second equation is  $3y = 3$ . Then  $y = 1$  and  $x = 5$ . If the right side changes sign, so does the solution:  $(x, y) = (-5, -1)$ .
- Subtract  $\ell = \frac{c}{a}$  times equation 1. The new second pivot multiplying  $y$  is  $d - (cb/a)$  or  $(ad - bc)/a$ . Then  $y = (ag - cf)/(ad - bc)$ .
- $6x + 4y$  is 2 times  $3x + 2y$ . There is no solution unless the right side is  $2 \cdot 10 = 20$ . Then all the points on the line  $3x + 2y = 10$  are solutions, including  $(0, 5)$  and  $(4, -1)$ . (The two lines in the row picture are the same line, containing all solutions).
- Singular system if  $b = 4$ , because  $4x + 8y$  is 2 times  $2x + 4y$ . Then  $g = 32$  makes the lines become the *same*: infinitely many solutions like  $(8, 0)$  and  $(0, 4)$ .
- If  $a = 2$  elimination must fail (two parallel lines in the row picture). The equations have no solution. With  $a = 0$ , elimination will stop for a row exchange. Then  $3y = -3$  gives  $y = -1$  and  $4x + 6y = 6$  gives  $x = 3$ .

- 8 If  $k = 3$  elimination must fail: no solution. If  $k = -3$ , elimination gives  $0 = 0$  in equation 2: infinitely many solutions. If  $k = 0$  a row exchange is needed: one solution.
- 9 On the left side,  $6x - 4y$  is 2 times  $(3x - 2y)$ . Therefore we need  $b_2 = 2b_1$  on the right side. Then there will be infinitely many solutions (two parallel lines become one single line).
- 10 The equation  $y = 1$  comes from elimination (subtract  $x + y = 5$  from  $x + 2y = 6$ ). Then  $x = 4$  and  $5x - 4y = c = 16$ .
- 11 (a) Another solution is  $\frac{1}{2}(x + X, y + Y, z + Z)$ . (b) If 25 planes meet at two points, they meet along the whole line through those two points.
- 12 Elimination leads to an upper triangular system; then comes back substitution.
- $$\begin{array}{rcl} 2x + 3y + z = 8 & & x = 2 \\ y + 3z = 4 & \text{gives} & y = 1 \\ 8z = 8 & & z = 1 \end{array}$$
- If a zero is at the start of row 2 or 3, that avoids a row operation.
- 13 
$$\begin{array}{rcl} 2x - 3y & = & 3 \\ 4x - 5y + z = 7 & \text{gives} & y + z = 1 \\ 2x - y - 3z = 5 & & 2y + 3z = 2 \end{array}$$

$$\begin{array}{rcl} 2x - 3y = 3 & & 2x - 3y = 3 \\ y + z = 1 & \text{and} & y + z = 1 \\ 2y + 3z = 2 & & -5z = 0 \end{array}$$

$$\begin{array}{rcl} x = 3 & & x = 3 \\ y = 1 & & y = 1 \\ z = 0 & & z = 0 \end{array}$$

Subtract  $2 \times$  row 1 from row 2, subtract  $1 \times$  row 1 from row 3, subtract  $2 \times$  row 2 from row 3

14 Subtract 2 times row 1 from row 2 to reach  $(d-10)y - z = 2$ . Equation (3) is  $y - z = 3$ . If  $d = 10$  exchange rows 2 and 3. If  $d = 11$  the system becomes singular.

15 The second pivot position will contain  $-2 - b$ . If  $b = -2$  we exchange with row 3. If  $b = -1$  (singular case) the second equation is  $-y - z = 0$ . A solution is  $(1, 1, -1)$ .

16 (a) **Example of 2 exchanges** 
$$\begin{array}{l} 0x + 0y + 2z = 4 \\ x + 2y + 2z = 5 \\ 0x + 3y + 4z = 6 \end{array}$$
 (exchange 1 and 2, then 2 and 3)

(b) **Exchange but then break down** 
$$\begin{array}{l} 0x + 3y + 4z = 4 \\ x + 2y + 2z = 5 \\ 0x + 3y + 4z = 6 \end{array}$$
 (rows 1 and 3 are not consistent)

17 If row 1 = row 2, then row 2 is zero after the first step; exchange the zero row with row 3 and there is no *third* pivot. If column 2 = column 1, then column 2 has no pivot.

18 *Example*  $x + 2y + 3z = 0, 4x + 8y + 12z = 0, 5x + 10y + 15z = 0$  has 9 different coefficients but rows 2 and 3 become  $0 = 0$ : infinitely many solutions.

19 Row 2 becomes  $3y - 4z = 5$ , then row 3 becomes  $(q + 4)z = t - 5$ . If  $q = -4$  the system is singular—no third pivot. Then if  $t = 5$  the third equation is  $0 = 0$ . Choosing  $z = 1$  the equation  $3y - 4z = 5$  gives  $y = 3$  and equation 1 gives  $x = -9$ .

20 Singular if row 3 is a combination of rows 1 and 2. From the end view, the three planes form a triangle. This happens if rows  $1 + 2 =$  row 3 on the left side but not the right side:  $x + y + z = 0, x - 2y - z = 1, 2x - y = 4$ . No parallel planes but still no solution.

21 (a) Pivots  $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}$  in the equations  $2x + y = 0, \frac{3}{2}y + z = 0, \frac{4}{3}z + t = 0, \frac{5}{4}t = 5$  after elimination. Back substitution gives  $t = 4, z = -3, y = 2, x = -1$ . (b) If the off-diagonal entries change from  $+1$  to  $-1$ , the pivots are the same. The solution is  $(1, 2, 3, 4)$  instead of  $(-1, 2, -3, 4)$ .

22 The fifth pivot is  $\frac{6}{5}$  for both matrices (1's or  $-1$ 's off the diagonal). The  $n$ th pivot is  $\frac{n+1}{n}$ .

- 23 If ordinary elimination leads to  $x + y = 1$  and  $2y = 3$ , the original second equation could be  $2y + \ell(x + y) = 3 + \ell$  for any  $\ell$ . Then  $\ell$  will be the multiplier to reach  $2y = 3$ .
- 24 Elimination fails on  $\begin{bmatrix} a & 2 \\ a & a \end{bmatrix}$  if  $a = 2$  or  $a = 0$ .
- 25  $a = 2$  (equal columns),  $a = 4$  (equal rows),  $a = 0$  (zero column).
- 26 Solvable for  $s = 10$  (add the two pairs of equations to get  $a + b + c + d$  on the left sides, 12 and  $2 + s$  on the right sides). The four equations for  $a, b, c, d$  are **singular!** Two solutions are  $\begin{bmatrix} 1 & 3 \\ 1 & 7 \end{bmatrix}$  and  $\begin{bmatrix} 0 & 4 \\ 2 & 6 \end{bmatrix}$ ,  $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$  and  $U = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$ .
- 27 Elimination leaves the diagonal matrix  $\text{diag}(3, 2, 1)$  in  $3x = 3, 2y = 2, z = 4$ . Then  $x = 1, y = 1, z = 4$ .
- 28  $A(2, :) = A(2, :) - 3 * A(1, :)$  subtracts 3 times row 1 from row 2.
- 29 The average pivots for  $\text{rand}(3)$  *without* row exchanges were  $\frac{1}{2}, 5, 10$  in one experiment—but pivots 2 and 3 can be arbitrarily large. Their averages are actually infinite! *With row exchanges* in MATLAB's `lu` code, the averages .75 and .50 and .365 are much more stable (and should be predictable, also for `randn` with normal instead of uniform probability distribution).
- 30 If  $A(5, 5)$  is 7 not 11, then the last pivot will be 0 not 4.
- 31 Row  $j$  of  $U$  is a combination of rows  $1, \dots, j$  of  $A$ . If  $Ax = \mathbf{0}$  then  $Ux = \mathbf{0}$  (not true if  $b$  replaces  $\mathbf{0}$ ).  $U$  is the diagonal of  $A$  when  $A$  is *lower triangular*.
- 32 The question deals with 100 equations  $Ax = \mathbf{0}$  when  $A$  is singular.
- Some linear combination of the 100 rows is **the row of 100 zeros**.
  - Some linear combination of the 100 **columns** is **the column of zeros**.
  - A very singular matrix has all ones:  $A = \text{eye}(100)$ . A better example has 99 random rows (or the numbers  $1^i, \dots, 100^i$  in those rows). The 100th row could be the sum of the first 99 rows (or any other combination of those rows with no zeros).
  - The row picture has 100 planes **meeting along a common line through  $\mathbf{0}$** . The column picture has 100 vectors all in the same 99-dimensional hyperplane.

### Problem Set 2.3, page 63

- 1  $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ,  $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 1 \end{bmatrix}$ ,  $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ .
- 2  $E_{32}E_{21}b = (1, -5, -35)$  but  $E_{21}E_{32}b = (1, -5, 0)$ . When  $E_{32}$  comes first, row 3 feels no effect from row 1.
- 3  $\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$   $M = E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix}$ .

**4** Elimination on column 4:  $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{E_{21}} \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix} \xrightarrow{E_{31}} \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix} \xrightarrow{E_{32}} \begin{bmatrix} 1 \\ -4 \\ 10 \end{bmatrix}$ . The original  $A\mathbf{x} = \mathbf{b}$  has become  $U\mathbf{x} = \mathbf{c} = (1, -4, 10)$ . Then back substitution gives  $z = -5, y = \frac{1}{2}, x = \frac{1}{2}$ . This solves  $A\mathbf{x} = (1, 0, 0)$ .

**5** Changing  $a_{33}$  from 7 to 11 will change the third pivot from 5 to 9. Changing  $a_{33}$  from 7 to 2 will change the pivot from 5 to *no pivot*.

**6** Example:  $\begin{bmatrix} 2 & 3 & 7 \\ 2 & 3 & 7 \\ 2 & 3 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$ . If all columns are multiples of column 1, there is no second pivot.

**7** To reverse  $E_{31}$ , **add** 7 times row **1** to row **3**. The inverse of the elimination matrix

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix} \text{ is } E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{bmatrix}.$$

**8**  $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  and  $M^* = \begin{bmatrix} a & b \\ c - \ell a & d - \ell b \end{bmatrix}$ .  $\det M^* = a(d - \ell b) - b(c - \ell a)$  reduces to  $ad - bc$ !

**9**  $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$ . After the exchange, we need  $E_{31}$  (not  $E_{21}$ ) to act on the new row 3.

**10**  $E_{13} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}; E_{31}E_{13} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ . Test on the identity matrix!

**11** An example with two negative pivots is  $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ . The diagonal entries can change sign during elimination.

**12** The first product is  $\begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$  rows and also columns reversed. The second product is  $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 2 & -3 \end{bmatrix}$ .

**13** (a)  $E$  times the third column of  $B$  is the third column of  $EB$ . A column that starts at zero will stay at zero. (b)  $E$  could add row 2 to row 3 to change a zero row to a nonzero row.

**14**  $E_{21}$  has  $-\ell_{21} = \frac{1}{2}$ ,  $E_{32}$  has  $-\ell_{32} = \frac{2}{3}$ ,  $E_{43}$  has  $-\ell_{43} = \frac{3}{4}$ . Otherwise the  $E$ 's match  $I$ .

**15**  $a_{ij} = 2i - 3j$ :  $A = \begin{bmatrix} -1 & -4 & -7 \\ 1 & -2 & -5 \\ 3 & 0 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & -4 & -7 \\ 0 & -6 & -12 \\ 0 & -12 & -24 \end{bmatrix}$ . The zero became  $-12$ ,

an example of *fill-in*. To remove that  $-12$ , choose  $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ .

- 16 (a) The ages of  $X$  and  $Y$  are  $x$  and  $y$ :  $x - 2y = 0$  and  $x + y = 33$ ;  $x = 22$  and  $y = 11$  (b) The line  $y = mx + c$  contains  $x = 2, y = 5$  and  $x = 3, y = 7$  when  $2m + c = 5$  and  $3m + c = 7$ . Then  $m = 2$  is the slope.

- 17 The parabola  $y = a + bx + cx^2$  goes through the 3 given points when 
$$\begin{aligned} a + b + c &= 4 \\ a + 2b + 4c &= 8 \\ a + 3b + 9c &= 14 \end{aligned}$$
 Then  $a = 2, b = 1$ , and  $c = 1$ . This matrix with columns  $(1, 1, 1), (1, 2, 3), (1, 4, 9)$  is a "Vandermonde matrix."

18  $EF = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}, FE = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b+ac & c & 1 \end{bmatrix}, E^2 = \begin{bmatrix} 1 & 0 & 0 \\ 2a & 1 & 0 \\ 2b & 0 & 1 \end{bmatrix}, F^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3c & 1 \end{bmatrix}.$

19  $PQ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ . In the opposite order, two row exchanges give  $QP = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ ,

If  $M$  exchanges rows 2 and 3 then  $M^2 = I$  (also  $(-M)^2 = I$ ). There are many square roots of  $I$ : Any matrix  $M = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$  has  $M^2 = I$  if  $a^2 + bc = 1$ .

- 20 (a) Each column of  $EB$  is  $E$  times a column of  $B$  (b)  $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}$ . All rows of  $EB$  are multiples of  $[1 \ 2 \ 4]$ .

21 No.  $E = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$  and  $F = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$  give  $EF = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$  but  $FE = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ .

22 (a)  $\sum a_{3j}x_j$  (b)  $a_{21} - a_{11}$  (c)  $a_{21} - 2a_{11}$  (d)  $(EAx)_1 = (Ax)_1 = \sum a_{1j}x_j$ .

- 23  $E(EA)$  subtracts 4 times row 1 from row 2 ( $EEA$  does the row operation twice).  $AE$  subtracts 2 times column 2 of  $A$  from column 1 (multiplication by  $E$  on the right side acts on **columns** instead of rows).

24  $[A \ b] = \begin{bmatrix} 2 & 3 & \mathbf{1} \\ 4 & 1 & \mathbf{17} \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 3 & \mathbf{1} \\ 0 & -5 & \mathbf{15} \end{bmatrix}$ . The triangular system is 
$$\begin{aligned} 2x_1 + 3x_2 &= 1 \\ -5x_2 &= 15 \end{aligned}$$
 Back substitution gives  $x_1 = 5$  and  $x_2 = -3$ .

- 25 The last equation becomes  $0 = 3$ . If the original 6 is 3, then row 1 + row 2 = row 3.

26 (a) Add two columns  $b$  and  $b^*$   $\begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix} \rightarrow x = \begin{bmatrix} -7 \\ 2 \end{bmatrix}$   
and  $x^* = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ .

- 27 (a) No solution if  $d = 0$  and  $c \neq 0$  (b) Many solutions if  $d = 0 = c$ . No effect from  $a, b$ .

28  $A = AI = A(BC) = (AB)C = IC = C$ . That middle equation is crucial.

29  $E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$  subtracts each row from the next row. The result  $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}$  still has multipliers = 1 in a 3 by 3 Pascal matrix. The product  $M$  of all elimination matrices is  $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$ . This “alternating sign Pascal matrix” is on page 88.

30 Given positive integers with  $ad - bc = 1$ . Certainly  $c < a$  and  $b < d$  would be impossible. Also  $c > a$  and  $b > d$  would be impossible with integers. This leaves row 1 < row 2 OR row 2 < row 1. An example is  $M = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ . Multiply by  $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$  to get  $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ , then multiply twice by  $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$  to get  $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ . This shows that  $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ .

$$31 \quad E_{21} = \begin{bmatrix} 1 & & & \\ 1/2 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_{32} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 2/3 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_{43} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 3/4 & 1 \end{bmatrix},$$

$$E_{43} E_{32} E_{21} = \begin{bmatrix} 1 & & & \\ 1/2 & 1 & & \\ 1/3 & 2/3 & 1 & \\ 1/4 & 2/4 & 3/4 & 1 \end{bmatrix}$$

### Problem Set 2.4, page 75

- 1 If all entries of  $A, B, C, D$  are 1, then  $BA = 3 \text{ ones}(5)$  is 5 by 5;  $AB = 5 \text{ ones}(3)$  is 3 by 3;  $ABD = 15 \text{ ones}(3, 1)$  is 3 by 1.  $DBA$  and  $A(B + C)$  are not defined.
- 2 (a)  $A$  (column 3 of  $B$ )      (b) (Row 1 of  $A$ )  $B$       (c) (Row 3 of  $A$ )(column 4 of  $B$ )  
(d) (Row 1 of  $C$ ) $D$ (column 1 of  $E$ ).
- 3  $AB + AC$  is the same as  $A(B + C) = \begin{bmatrix} 3 & 8 \\ 6 & 9 \end{bmatrix}$ . (*Distributive law*).
- 4  $A(BC) = (AB)C$  by the *associative law*. In this example both answers are  $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$  from column 1 of  $AB$  and row 2 of  $C$  (multiply columns times rows).
- 5 (a)  $A^2 = \begin{bmatrix} 1 & 2b \\ 0 & 1 \end{bmatrix}$  and  $A^n = \begin{bmatrix} 1 & nb \\ 0 & 1 \end{bmatrix}$ .      (b)  $A^2 = \begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix}$  and  $A^n = \begin{bmatrix} 2^n & 2^n \\ 0 & 0 \end{bmatrix}$ .
- 6  $(A + B)^2 = \begin{bmatrix} 10 & 4 \\ 6 & 6 \end{bmatrix} = A^2 + AB + BA + B^2$ . But  $A^2 + 2AB + B^2 = \begin{bmatrix} 16 & 2 \\ 3 & 0 \end{bmatrix}$ .
- 7 (a) True      (b) False      (c) True      (d) False: usually  $(AB)^2 \neq A^2B^2$ .

- 8 The rows of  $DA$  are 3 (row 1 of  $A$ ) and 5 (row 2 of  $A$ ). Both rows of  $EA$  are row 2 of  $A$ . The columns of  $AD$  are 3 (column 1 of  $A$ ) and 5 (column 2 of  $A$ ). The first column of  $AE$  is zero, the second is column 1 of  $A$  + column 2 of  $A$ .

9  $AF = \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix}$  and  $E(AF)$  equals  $(EA)F$  because matrix multiplication is associative.

10  $FA = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}$  and then  $E(FA) = \begin{bmatrix} a+c & b+d \\ a+2c & b+2d \end{bmatrix}$ .  $E(FA)$  is not the same as  $F(EA)$  because multiplication is not commutative.

11 (a)  $B = 4I$  (b)  $B = 0$  (c)  $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$  (d) Every row of  $B$  is 1, 0, 0.

12  $AB = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = BA = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$  gives  $b = c = 0$ . Then  $AC = CA$  gives  $a = d$ . The only matrices that commute with  $B$  and  $C$  (and all other matrices) are multiples of  $I$ :  $A = aI$ .

13  $(A - B)^2 = (B - A)^2 = A(A - B) - B(A - B) = A^2 - AB - BA + B^2$ . In a typical case (when  $AB \neq BA$ ) the matrix  $A^2 - 2AB + B^2$  is different from  $(A - B)^2$ .

14 (a) True ( $A^2$  is only defined when  $A$  is square) (b) False (if  $A$  is  $m$  by  $n$  and  $B$  is  $n$  by  $m$ , then  $AB$  is  $m$  by  $m$  and  $BA$  is  $n$  by  $n$ ). (c) True (d) False (take  $B = 0$ ).

15 (a)  $mn$  (use every entry of  $A$ ) (b)  $mnp = p \times$  part (a) (c)  $n^3$  ( $n^2$  dot products).

16 (a) Use only column 2 of  $B$  (b) Use only row 2 of  $A$  (c)–(d) Use row 2 of first  $A$ .

17  $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$  has  $a_{ij} = \min(i, j)$ .  $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$  has  $a_{ij} = (-1)^{i+j} =$

“alternating sign matrix”.  $A = \begin{bmatrix} 1/1 & 1/2 & 1/3 \\ 2/1 & 2/2 & 2/3 \\ 3/1 & 3/2 & 3/3 \end{bmatrix}$  has  $a_{ij} = i/j$  (this will be an example of a *rank one matrix*).

18 Diagonal matrix, lower triangular, symmetric, all rows equal. Zero matrix fits all four.

19 (a)  $a_{11}$  (b)  $\ell_{31} = a_{31}/a_{11}$  (c)  $a_{32} - (\frac{a_{31}}{a_{11}})a_{12}$  (d)  $a_{22} - (\frac{a_{21}}{a_{11}})a_{12}$ .

20  $A^2 = \begin{bmatrix} 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ,  $A^3 = \begin{bmatrix} 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ,  $A^4 =$  zero matrix for *strictly triangular*  $A$ .

Then  $A\mathbf{v} = A \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 2y \\ 2z \\ 2t \\ 0 \end{bmatrix}$ ,  $A^2\mathbf{v} = \begin{bmatrix} 4z \\ 4t \\ 0 \\ 0 \end{bmatrix}$ ,  $A^3\mathbf{v} = \begin{bmatrix} 8t \\ 0 \\ 0 \\ 0 \end{bmatrix}$ ,  $A^4\mathbf{v} = 0$ .

21  $A = A^2 = A^3 = \dots = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$  but  $AB = \begin{bmatrix} .5 & -.5 \\ .5 & -.5 \end{bmatrix}$  and  $(AB)^2 = \text{zero matrix!}$

22  $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$  has  $A^2 = -I$ ;  $BC = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ;  
 $DE = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = -ED$ . You can find more examples.

23  $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  has  $A^2 = 0$ . Note: Any matrix  $A = \text{column times row} = \mathbf{uv}^T$  will

have  $A^2 = \mathbf{uv}^T \mathbf{uv}^T = 0$  if  $\mathbf{v}^T \mathbf{u} = 0$ .  $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$  has  $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

but  $A^3 = 0$ ; strictly triangular as in Problem 20.

24  $(A_1)^n = \begin{bmatrix} 2^n & 2^n - 1 \\ 0 & 1 \end{bmatrix}$ ,  $(A_2)^n = 2^{n-1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ ,  $(A_3)^n = \begin{bmatrix} a^n & a^{n-1}b \\ 0 & 0 \end{bmatrix}$ .

25  $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a \\ d \\ g \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} b \\ e \\ h \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} c \\ f \\ i \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ .

26 **Columns of A**  $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 0 \\ 6 & 6 & 0 \\ 6 & 6 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 4 & 8 & 4 \\ 1 & 2 & 1 \end{bmatrix} =$   
 $\begin{bmatrix} 3 & 3 & 0 \\ 10 & 14 & 4 \\ 7 & 8 & 1 \end{bmatrix} = AB$ .

27 (a) (row 3 of  $A$ )  $\cdot$  (column 1 of  $B$ ) and (row 3 of  $A$ )  $\cdot$  (column 2 of  $B$ ) are both zero.

(b)  $\begin{bmatrix} x \\ x \\ 0 \end{bmatrix} \begin{bmatrix} 0 & x & x \\ 0 & x & x \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & x & x \\ 0 & x & x \\ 0 & 0 & 0 \end{bmatrix}$  and  $\begin{bmatrix} x \\ x \\ x \end{bmatrix} \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix} = \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix}$ : **both upper**.

28  $A$  times  $B$  with cuts  $A \begin{bmatrix} | & | & | & | \\ | & | & | & | \\ | & | & | & | \end{bmatrix}$ ,  $\begin{bmatrix} \text{---} \\ \text{---} \\ \text{---} \end{bmatrix} B$ ,  $\begin{bmatrix} \text{---} \\ \text{---} \\ \text{---} \end{bmatrix} \begin{bmatrix} | & | & | & | \\ | & | & | & | \\ | & | & | & | \end{bmatrix}$ ,  $\begin{bmatrix} | & | & | \\ | & | & | \\ | & | & | \end{bmatrix} \begin{bmatrix} \text{---} \\ \text{---} \\ \text{---} \end{bmatrix}$

29  $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  and  $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$  produce zeros in the 2, 1 and 3, 1 entries.

Multiply  $E$ 's to get  $E = E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$ . Then  $EA = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$  is the result of both  $E$ 's since  $(E_{31}E_{21})A = E_{31}(E_{21}A)$ .

30 In 29,  $\mathbf{c} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$ ,  $D = \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix}$ ,  $D - \mathbf{cb}/a = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$  in the lower corner of  $EA$ .

31  $\begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} A\mathbf{x} - B\mathbf{y} \\ B\mathbf{x} + A\mathbf{y} \end{bmatrix}$  real part      Complex matrix times complex vector  
 imaginary part.      needs 4 real times real multiplications.

**32**  $A$  times  $X = [\mathbf{x}_1 \ \mathbf{x}_2 \ \mathbf{x}_3]$  will be the identity matrix  $I = [A\mathbf{x}_1 \ A\mathbf{x}_2 \ A\mathbf{x}_3]$ .

**33**  $\mathbf{b} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}$  gives  $\mathbf{x} = 3\mathbf{x}_1 + 5\mathbf{x}_2 + 8\mathbf{x}_3 = \begin{bmatrix} 3 \\ 8 \\ 16 \end{bmatrix}$ ;  $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$  will have those  $\mathbf{x}_1 = (1, 1, 1)$ ,  $\mathbf{x}_2 = (0, 1, 1)$ ,  $\mathbf{x}_3 = (0, 0, 1)$  as columns of its “inverse”  $A^{-1}$ .

**34**  $A * \mathbf{ones} = \begin{bmatrix} a+b & a+b \\ c+d & c+d \end{bmatrix}$  agrees with  $\mathbf{ones} * A = \begin{bmatrix} a+c & b+b \\ a+c & b+d \end{bmatrix}$  when  $b = c$  and  $a = d$ .  
Then  $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ .

**35**  $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ ,  $A^2 = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}$ , **aba, ada cba, cda** These show  
**bab, bcb dab, dcb** 16 2-step  
**abc, adc cbc, cdc** paths in  
**bad, bcd dad, dcd** the graph

**36** Multiplying  $AB = (m \text{ by } n)(n \text{ by } p)$  needs  $mnp$  multiplications. Then  $(AB)C$  needs  $mpq$  more. Multiply  $BC = (n \text{ by } p)(p \text{ by } q)$  needs  $npq$  and then  $A(BC)$  needs  $mnq$ .

(a) If  $m, n, p, q$  are 2, 4, 7, 10 we compare  $(2)(4)(7) + (2)(7)(10) = \mathbf{196}$  with the larger number  $(2)(4)(10) + (4)(7)(10) = \mathbf{360}$ . So  $AB$  first is better, so that we multiply that 7 by 10 matrix by as few rows as possible.

(b) If  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  are  $N$  by 1, then  $(\mathbf{u}^T \mathbf{v}) \mathbf{w}^T$  needs  $2N$  multiplications but  $\mathbf{u}^T (\mathbf{v} \mathbf{w}^T)$  needs  $N^2$  to find  $\mathbf{v} \mathbf{w}^T$  and  $N^2$  more to multiply by the row vector  $\mathbf{u}^T$ . Apologies to use the transpose symbol so early.

(c) We are comparing  $mnp + mpq$  with  $mnq + npq$ . Divide all terms by  $mnpq$ : Now we are comparing  $q^{-1} + n^{-1}$  with  $p^{-1} + m^{-1}$ . This yields a simple important rule. If matrices  $A$  and  $B$  are multiplying  $\mathbf{v}$  for  $AB\mathbf{v}$ , **don't multiply the matrices first**.

**37** The proof of  $(AB)\mathbf{c} = A(B\mathbf{c})$  used the column rule for matrix multiplication—this rule is clearly linear, column by column.

Even for nonlinear transformations,  $A(B(\mathbf{c}))$  would be the “composition” of  $A$  with  $B$  (applying  $B$  then  $A$ ). This composition  $A \circ B$  is just  $AB$  for matrices.

One of many uses for the associative law: The left-inverse  $B =$  right-inverse  $C$  from  $B = B(AC) = (BA)C = C$ .

## Problem Set 2.5, page 89

**1**  $A^{-1} = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{3} & 0 \end{bmatrix}$  and  $B^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ -1 & \frac{1}{2} \end{bmatrix}$  and  $C^{-1} = \begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix}$ .

**2** A simple row exchange has  $P^2 = I$  so  $P^{-1} = P$ . Here  $P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ . Always

$P^{-1} =$  “transpose” of  $P$ , coming in Section 2.7.

- 3  $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} .5 \\ -.2 \end{bmatrix}$  and  $\begin{bmatrix} t \\ z \end{bmatrix} = \begin{bmatrix} -.2 \\ .1 \end{bmatrix}$  so  $A^{-1} = \frac{1}{10} \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$ . This question solved  $AA^{-1} = I$  column by column, the main idea of Gauss-Jordan elimination.
- 4 The equations are  $x + 2y = 1$  and  $3x + 6y = 0$ . No solution because 3 times equation 1 gives  $3x + 6y = 3$ .
- 5 An upper triangular  $U$  with  $U^2 = I$  is  $U = \begin{bmatrix} 1 & a \\ 0 & -1 \end{bmatrix}$  for any  $a$ . And also  $-U$ .
- 6 (a) Multiply  $AB = AC$  by  $A^{-1}$  to find  $B = C$  (since  $A$  is invertible) (b) As long as  $B - C$  has the form  $\begin{bmatrix} x & y \\ -x & -y \end{bmatrix}$ , we have  $AB = AC$  for  $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ .
- 7 (a) In  $A\mathbf{x} = (1, 0, 0)$ , equation 1 + equation 2 - equation 3 is  $0 = 1$  (b) Right sides must satisfy  $b_1 + b_2 = b_3$  (c) Row 3 becomes a row of zeros—no third pivot.
- 8 (a) The vector  $\mathbf{x} = (1, 1, -1)$  solves  $A\mathbf{x} = \mathbf{0}$  (b) After elimination, columns 1 and 2 end in zeros. Then so does column 3 = column 1 + 2: no third pivot.
- 9 If you exchange rows 1 and 2 of  $A$  to reach  $B$ , you exchange **columns** 1 and 2 of  $A^{-1}$  to reach  $B^{-1}$ . In matrix notation,  $B = PA$  has  $B^{-1} = A^{-1}P^{-1} = A^{-1}P$  for this  $P$ .
- 10  $A^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1/5 \\ 0 & 0 & 1/4 & 0 \\ 0 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{bmatrix}$  and  $B^{-1} = \begin{bmatrix} 3 & -2 & 0 & 0 \\ -4 & 3 & 0 & 0 \\ 0 & 0 & 6 & -5 \\ 0 & 0 & -7 & 6 \end{bmatrix}$  (invert each block of  $B$ ).
- 11 (a) If  $B = -A$  then certainly  $A + B =$  zero matrix is not invertible. (b)  $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  and  $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$  are both singular but  $A + B = I$  is invertible.
- 12 Multiply  $C = AB$  on the left by  $A^{-1}$  and on the right by  $C^{-1}$ . Then  $A^{-1} = BC^{-1}$ .
- 13  $M^{-1} = C^{-1}B^{-1}A^{-1}$  so multiply on the left by  $C$  and the right by  $A$ :  $B^{-1} = CM^{-1}A$ .
- 14  $B^{-1} = A^{-1} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = A^{-1} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ : subtract column 2 of  $A^{-1}$  from column 1.
- 15 If  $A$  has a column of zeros, so does  $BA$ . Then  $BA = I$  is impossible. There is no  $A^{-1}$ .
- 16  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}$ . The inverse of each matrix is the other divided by  $ad - bc$ .
- 17  $E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -1 & 1 & \\ 0 & -1 & 1 \end{bmatrix} = E$ .  
Reverse the order and change  $-1$  to  $+1$  to get inverses  $E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 1 & 1 \end{bmatrix} = L = E^{-1}$ . Notice the 1's unchanged by multiplying in this order.
- 18  $A^2B = I$  can also be written as  $A(AB) = I$ . Therefore  $A^{-1}$  is  $AB$ .

19 The (1, 1) entry requires  $4a - 3b = 1$ ; the (1, 2) entry requires  $2b - a = 0$ . Then  $b = \frac{1}{5}$  and  $a = \frac{2}{5}$ . For the 5 by 5 case  $5a - 4b = 1$  and  $2b = a$  give  $b = \frac{1}{6}$  and  $a = \frac{2}{6}$ .

20  $A * \text{ones}(4, 1)$  is the zero vector so  $A$  cannot be invertible.

21 Six of the sixteen 0 – 1 matrices are invertible, including all four with three 1's.

$$22 \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{bmatrix} = [I \ A^{-1}];$$

$$\begin{bmatrix} 1 & 4 & 1 & 0 \\ 3 & 9 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & -3 & -3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -3 & 4/3 \\ 0 & 1 & 1 & -1/3 \end{bmatrix} = [I \ A^{-1}].$$

$$23 [A \ I] = \left[ \begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & -1/2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array} \right] \rightarrow$$

$$\left[ \begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & -1/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1/3 & -2/3 & 1 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 0 & -3/4 & 3/2 & -3/4 \\ 0 & 0 & 4/3 & 1/3 & -2/3 & 1 \end{array} \right] \rightarrow$$

$$\left[ \begin{array}{ccc|ccc} 2 & 0 & 0 & 3/2 & -1 & 1/2 \\ 0 & 3/2 & 0 & -3/4 & 3/2 & -3/4 \\ 0 & 0 & 4/3 & 1/3 & -2/3 & 1 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & -1/2 & 1/4 \\ 0 & 1 & 0 & -1/2 & 1 & -1/2 \\ 0 & 0 & 1 & 1/4 & -1/2 & 3/4 \end{array} \right] =$$

$$[I \ A^{-1}].$$

$$24 \begin{bmatrix} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & a & 0 & 1 & 0 & -b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & -a & ac - b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

$$25 \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}^{-1} = \frac{1}{4} \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}; \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ so } B^{-1} \text{ does not exist.}$$

$$26 E_{21}A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}. E_{12}E_{21}A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Multiply by  $D = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$  to reach  $DE_{12}E_{21}A = I$ . Then  $A^{-1} = DE_{12}E_{21} = \frac{1}{2} \begin{bmatrix} 6 & -2 \\ -2 & 1 \end{bmatrix}$ .

$$27 A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \text{ (notice the pattern); } A^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

$$28 \begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 2 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & -1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1/2 & 1/2 \\ 0 & 1 & 1/2 & 0 \end{bmatrix}.$$

This is  $[I \ A^{-1}]$ : row exchanges are certainly allowed in Gauss-Jordan.

29 (a) True (If  $A$  has a row of zeros, then every  $AB$  has too, and  $AB = I$  is impossible)  
 (b) False (the matrix of all ones is singular even with diagonal 1's: *ones* (3) has 3 equal rows)  
 (c) True (the inverse of  $A^{-1}$  is  $A$  and the inverse of  $A^2$  is  $(A^{-1})^2$ ).

30 This  $A$  is not invertible for  $c = 7$  (equal columns),  $c = 2$  (equal rows),  $c = 0$  (zero column).

31 Elimination produces the pivots  $a$  and  $a-b$  and  $a-b$ .  $A^{-1} = \frac{1}{a(a-b)} \begin{bmatrix} a & 0 & -b \\ -a & a & 0 \\ 0 & -a & a \end{bmatrix}$ .

32  $A^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ . When the triangular  $A$  alternates 1 and  $-1$  on its diagonal,

$A^{-1}$  is *bidiagonal* with 1's on the diagonal and first superdiagonal.

33  $\mathbf{x} = (1, 1, \dots, 1)$  has  $P\mathbf{x} = Q\mathbf{x}$  so  $(P - Q)\mathbf{x} = \mathbf{0}$ .

34  $\begin{bmatrix} I & 0 \\ -C & I \end{bmatrix}$  and  $\begin{bmatrix} A^{-1} & 0 \\ -D^{-1}CA^{-1} & D^{-1} \end{bmatrix}$  and  $\begin{bmatrix} -D & I \\ I & 0 \end{bmatrix}$ .

35  $A$  can be invertible with diagonal zeros.  $B$  is singular because each row adds to zero.

36 The equation  $LDLD = I$  says that  $LD = \text{pascal}(4, 1)$  is its own inverse.

37  $\text{hilb}(6)$  is not the exact Hilbert matrix because fractions are rounded off. So  $\text{inv}(\text{hilb}(6))$  is not the exact either.

38 The three Pascal matrices have  $P = LU = LL^T$  and then  $\text{inv}(P) = \text{inv}(L^T)\text{inv}(L)$ .

39  $A\mathbf{x} = \mathbf{b}$  has many solutions when  $A = \text{ones}(4, 4)$  = singular matrix and  $\mathbf{b} = \text{ones}(4, 1)$ .  $A \setminus \mathbf{b}$  in MATLAB will pick the shortest solution  $\mathbf{x} = (1, 1, 1, 1)/4$ . This is the only solution that is combination of the rows of  $A$  (later it comes from the "pseudoinverse"  $A^+ = \text{pinv}(A)$  which replaces  $A^{-1}$  when  $A$  is singular). Any vector that solves  $A\mathbf{x} = \mathbf{0}$  could be added to this particular solution  $\mathbf{x}$ .

40 The inverse of  $A = \begin{bmatrix} 1 & -a & 0 & 0 \\ 0 & 1 & -b & 0 \\ 0 & 0 & 1 & -c \\ 0 & 0 & 0 & 1 \end{bmatrix}$  is  $A^{-1} = \begin{bmatrix} 1 & a & ab & abc \\ 0 & 1 & b & bc \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$ . (This

would be a good example for the cofactor formula  $A^{-1} = C^T / \det A$  in Section 5.3)

41 The product  $\begin{bmatrix} 1 & & & \\ a & 1 & & \\ b & 0 & 1 & \\ c & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & d & 1 & \\ 0 & e & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & f & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ a & 1 & & \\ b & d & 1 & \\ c & e & f & 1 \end{bmatrix}$

that in this order the multipliers shows  $a, b, c, d, e, f$  are unchanged in the product (**important for  $A = LU$  in Section 2.6**).

42  $MM^{-1} = (I_n - UV)(I_n + U(I_m - VU)^{-1}V)$  (this is testing formula 3)  
 $= I_n - UV + U(I_m - VU)^{-1}V - UVU(I_m - VU)^{-1}V$  (keep simplifying)  
 $= I_n - UV + U(I_m - VU)(I_m - VU)^{-1}V = I_n$  (formulas 1, 2, 4 are similar)

43 4 by 4 still with  $T_{11} = 1$  has pivots 1, 1, 1, 1; reversing to  $T^* = UL$  makes  $T_{44}^* = 1$ .

44 Add the equations  $C\mathbf{x} = \mathbf{b}$  to find  $0 = b_1 + b_2 + b_3 + b_4$ . Same for  $F\mathbf{x} = \mathbf{b}$ .

45 The block pivots are  $A$  and  $S = D - CA^{-1}B$  (and  $d - cb/a$  is the correct second pivot of an ordinary 2 by 2 matrix). The example problem has

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 \\ 4 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 3 & 3 \end{bmatrix} = \begin{bmatrix} -5 & -6 \\ -6 & -5 \end{bmatrix}.$$

- 46** Inverting the identity  $A(I + BA) = (I + AB)A$  gives  $(I + BA)^{-1}A^{-1} = A^{-1}(I + AB)^{-1}$ . So  $I + BA$  and  $I + AB$  are both invertible or both singular when  $A$  is invertible. (This remains true also when  $A$  is singular: Problem 6.6.19 will show that  $AB$  and  $BA$  have the same nonzero eigenvalues, and we are looking here at  $\lambda = -1$ .)

### Problem Set 2.6, page 102

- 1**  $\ell_{21} = 1$  multiplied row 1;  $L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$  times  $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \mathbf{c}$  is  $A\mathbf{x} = \mathbf{b}$ :  

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}.$$
- 2**  $L\mathbf{c} = \mathbf{b}$  is  $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ , solved by  $\mathbf{c} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$  as elimination goes forward.  
 $U\mathbf{x} = \mathbf{c}$  is  $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ , solved by  $\mathbf{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$  in back substitution.
- 3**  $\ell_{31} = 1$  and  $\ell_{32} = 2$  (and  $\ell_{33} = 1$ ): reverse steps to get  $A\mathbf{u} = \mathbf{b}$  from  $U\mathbf{x} = \mathbf{c}$ :  
 1 times  $(x + y + z = 5)$  + 2 times  $(y + 2z = 2)$  + 1 times  $(z = 2)$  gives  $x + 3y + 6z = 11$ .
- 4**  $L\mathbf{c} = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 11 \end{bmatrix}$ ;  $U\mathbf{x} = \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 2 \\ & & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix}$ ;  $\mathbf{x} = \begin{bmatrix} 5 \\ -2 \\ 2 \end{bmatrix}$ .
- 5**  $EA = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix} = U$ . With  $E^{-1}$  as  $L$ ,  $A = LU =$   

$$\begin{bmatrix} 1 & & \\ 0 & 1 & \\ 3 & 0 & 1 \end{bmatrix} U.$$
- 6**  $\begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -2 & 1 & \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -6 \end{bmatrix} = U$ . Then  $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$   $U$  is  
 the same as  $E_{21}^{-1}E_{32}^{-1}U = LU$ . The multipliers  $\ell_{21}, \ell_{32} = 2$  fall into place in  $L$ .
- 7**  $E_{32}E_{31}E_{21} A = \begin{bmatrix} 1 & & \\ & 1 & \\ & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ -3 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -2 & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}$ . This is  

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = U$$
. Put those multipliers 2, 3, 2 into  $L$ . Then  $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} U = LU$ .
- 8**  $E = E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -c & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ -b & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -a & 1 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -a & 1 & \\ ac - b & -c & 1 \end{bmatrix}$ .  
 The multipliers are just  $a, b, c$  and the upper triangular  $U$  is  $I$ . In this case  $A = L$  and its inverse is that matrix  $E = L^{-1}$ .
- 9** 2 by 2:  $d = 0$  not allowed;  $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ l & 1 & \\ m & n & 1 \end{bmatrix} \begin{bmatrix} d & e & g \\ f & h \\ i \end{bmatrix}$   $d = 1, e = 1$ , then  $l = 1$   
 $f = 0$  is not allowed  
**no pivot in row 2**

- 10**  $c = 2$  leads to zero in the second pivot position: exchange rows and not singular.  
 $c = 1$  leads to zero in the third pivot position. In this case the matrix is *singular*.

**11**  $A = \begin{bmatrix} 2 & 4 & 8 \\ 0 & 3 & 9 \\ 0 & 0 & 7 \end{bmatrix}$  has  $L = I$  ( $A$  is already upper triangular) and  $D = \begin{bmatrix} 2 & & \\ & 3 & \\ & & 7 \end{bmatrix}$ ;

$A = LU$  has  $U = A$ ;  $A = LDU$  has  $U = D^{-1}A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$  with 1's on the diagonal.

**12**  $A = \begin{bmatrix} 2 & 4 \\ 4 & 11 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = LDU$ ;  $U$  is  $L^T$ .

$$\begin{bmatrix} 1 & & \\ 4 & 1 & \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 0 & -4 & 4 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 4 & 1 & \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & -4 & \\ & & 4 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = LDL^T.$$

**13**  $\begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ & b-a & b-a & b-a \\ & & c-b & c-b \\ & & & d-c \end{bmatrix}$ . Need  $\begin{matrix} a \neq 0 \\ b \neq a \\ c \neq b \\ d \neq c \end{matrix}$  All of the multipliers are  $\ell_{ij} = 1$  for this  $A$ .

**14**  $\begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & r & r & r \\ & b-r & s-r & s-r \\ & & c-s & t-s \\ & & & d-t \end{bmatrix}$ . Need  $\begin{matrix} a \neq 0 \\ b \neq r \\ c \neq s \\ d \neq t \end{matrix}$

**15**  $\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} c = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$  gives  $c = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ . Then  $\begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} x = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$  gives  $x = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$ .

$Ax = b$  is  $LUx = \begin{bmatrix} 2 & 4 \\ 8 & 17 \end{bmatrix} x = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$ . Forward to  $\begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} x = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = c$ .

**16**  $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} c = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$  gives  $c = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$ . Then  $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$  gives  $x = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ .

Those are the forward elimination and back substitution steps for

$$Ax = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix} x = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}.$$

- 17** (a)  $L$  goes to  $I$  (b)  $I$  goes to  $L^{-1}$  (c)  $LU$  goes to  $U$ . Elimination multiply by  $L^{-1}$ !

- 18** (a) Multiply  $LDU = L_1 D_1 U_1$  by inverses to get  $L_1^{-1} L D = D_1 U_1 U^{-1}$ . The left side is lower triangular, the right side is upper triangular  $\Rightarrow$  both sides are diagonal.

(b)  $L, U, L_1, U_1$  have diagonal 1's so  $D = D_1$ . Then  $L_1^{-1} L$  and  $U_1 U^{-1}$  are both  $I$ .

**19**  $\begin{bmatrix} 1 & & \\ 1 & 1 & \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ & 1 & 1 \\ & & 1 \end{bmatrix} = LIU$ ;  $\begin{bmatrix} a & a & 0 \\ a & a+b & b \\ 0 & b & b+c \end{bmatrix} = (\text{same } L) \begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix}$

(same  $U$ ). A tridiagonal matrix  $A$  has **bidiagonal factors**  $L$  and  $U$ .

- 20** A tridiagonal  $T$  has 2 nonzeros in the pivot row and only one nonzero below the pivot (one operation to find  $\ell$  and then one for the new pivot!).  $T =$  bidiagonal  $L$  times bidiagonal  $U$ .

- 21** For the first matrix  $A$ ,  $L$  keeps the 3 lower zeros at the start of rows. But  $U$  may not have the upper zero where  $A_{24} = 0$ . For the second matrix  $B$ ,  $L$  keeps the bottom left zero at the start of row 4.  $U$  keeps the upper right zero at the start of column 4. One zero in  $A$  and two zeros in  $B$  are filled in.
- 22** Eliminating upwards,  $\begin{bmatrix} 5 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 2 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} = L$ . We reach a lower triangular  $L$ , and the multipliers are in an upper triangular  $U$ .  $A = UL$  with  $U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ .
- 23** The 2 by 2 upper submatrix  $A_2$  has the first two pivots 5, 9. Reason: Elimination on  $A$  starts in the upper left corner with elimination on  $A_2$ .
- 24** The upper left blocks all factor at the same time as  $A$ :  $A_k$  is  $L_k U_k$ .
- 25** The  $i, j$  entry of  $L^{-1}$  is  $j/i$  for  $i \geq j$ . And  $L_{i, i-1}$  is  $(1-i)/i$  below the diagonal
- 26**  $(K^{-1})_{ij} = j(n-i+1)/(n+1)$  for  $i \geq j$  (and symmetric):  $(n+1)K^{-1}$  looks good.

### Problem Set 2.7, page 115

- 1**  $A = \begin{bmatrix} 1 & 0 \\ 9 & 3 \end{bmatrix}$  has  $A^T = \begin{bmatrix} 1 & 9 \\ 0 & 3 \end{bmatrix}$ ,  $A^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1/3 \end{bmatrix}$ ,  $(A^{-1})^T = (A^T)^{-1} = \begin{bmatrix} 1 & -3 \\ 0 & 1/3 \end{bmatrix}$ ;  
 $A = \begin{bmatrix} 1 & c \\ c & 0 \end{bmatrix}$  has  $A^T = A$  and  $A^{-1} = \frac{1}{c^2} \begin{bmatrix} 0 & c \\ c & -1 \end{bmatrix} = (A^{-1})^T$ .
- 2**  $(AB)^T$  is not  $A^T B^T$  except when  $AB = BA$ . Transpose that to find:  $B^T A^T = A^T B^T$ .
- 3** (a)  $((AB)^{-1})^T = (B^{-1} A^{-1})^T = (A^{-1})^T (B^{-1})^T$ . This is also  $(A^T)^{-1} (B^T)^{-1}$ .  
 (b) If  $U$  is upper triangular, so is  $U^{-1}$ : then  $(U^{-1})^T$  is lower triangular.
- 4**  $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  has  $A^2 = 0$ . The diagonal of  $A^T A$  has dot products of columns of  $A$  with themselves. If  $A^T A = 0$ , zero dot products  $\Rightarrow$  zero columns  $\Rightarrow A =$  zero matrix.
- 5** (a)  $\mathbf{x}^T A \mathbf{y} = [0 \ 1] \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 5$  (b)  $\mathbf{x}^T A = [4 \ 5 \ 6]$  (c)  $A \mathbf{y} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ .
- 6**  $M^T = \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix}$ ;  $M^T = M$  needs  $A^T = A$  and  $B^T = C$  and  $D^T = D$ .
- 7** (a) False:  $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$  is symmetric only if  $A = A^T$ . (b) False: The transpose of  $AB$  is  $B^T A^T = BA$  when  $A$  and  $B$  are symmetric  $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$  transposes to  $\begin{bmatrix} 0 & A^T \\ A^T & 0 \end{bmatrix}$ . So  $(AB)^T = AB$  needs  $BA = AB$ . (c) True: Invertible symmetric matrices have symmetric in verses! Easiest proof is to transpose  $AA^{-1} = I$ . (d) True:  $(ABC)^T$  is  $C^T B^T A^T (= CBA$  for symmetric matrices  $A, B,$  and  $C)$ .
- 8** The 1 in row 1 has  $n$  choices; then the 1 in row 2 has  $n-1$  choices  $\dots (n!$  overall).

$$\mathbf{9} \quad P_1 P_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ but } P_2 P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If  $P_3$  and  $P_4$  exchange *different* pairs of rows,  $P_3 P_4 = P_4 P_3$  does both exchanges.

- 10** (3, 1, 2, 4) and (2, 3, 1, 4) keep 4 in place; 6 more even  $P$ 's keep 1 or 2 or 3 in place; (2, 1, 4, 3) and (3, 4, 1, 2) exchange 2 pairs. (1, 2, 3, 4), (4, 3, 2, 1) make 12 even  $P$ 's.

$$\mathbf{11} \quad PA = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \text{ is upper triangular. Multiplying on}$$

the right by a permutation matrix  $P_2$  exchanges the columns. To make this  $A$  lower tri-

angular, we also need  $P_1$  to exchange rows 2 and 3:  $P_1 A P_2 = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$

$$A \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix}.$$

- 12**  $(Px)^T(Py) = x^T P^T P y = x^T y$  since  $P^T P = I$ . In general  $Px \cdot y = x \cdot P^T y \neq x \cdot P y$ :

$$\text{Non-equality where } P \neq P^T: \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

- 13** A cyclic  $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$  or its transpose will have  $P^3 = I : (1, 2, 3) \rightarrow (2, 3, 1) \rightarrow$

$(3, 1, 2) \rightarrow (1, 2, 3)$ .  $\hat{P} = \begin{bmatrix} 1 & 0 \\ 0 & P \end{bmatrix}$  for the same  $P$  has  $\hat{P}^4 = \hat{P} \neq I$ .

- 14** The "reverse identity"  $P$  takes  $(1, \dots, n)$  into  $(n, \dots, 1)$ . When rows and also columns are reversed,  $(PAP)_{ij}$  is  $(A)_{n-i+1, n-j+1}$ . In particular  $(PAP)_{11}$  is  $A_{nn}$ .

- 15** (a) If  $P$  sends row 1 to row 4, then  $P^T$  sends row 4 to row 1 (b)  $P = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} = P^T$  with  $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  moves all rows: 1 and 2 are exchanged, 3 and 4 are exchanged.

- 16**  $A^2 - B^2$  (but not  $(A + B)(A - B)$ , this is different) and also  $ABA$  are symmetric if  $A$  and  $B$  are symmetric.

- 17** (a)  $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = A^T$  is not invertible (b)  $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$  needs row exchange

$$(c) \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \text{ has } D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

- 18** (a)  $5 + 4 + 3 + 2 + 1 = 15$  independent entries if  $A = A^T$  (b)  $L$  has 10 and  $D$  has 5; total 15 in  $LDL^T$  (c) Zero diagonal if  $A^T = -A$ , leaving  $4 + 3 + 2 + 1 = 10$  choices.

- 19** (a) The transpose of  $R^T A R$  is  $R^T A^T R^T = R^T A R = n$  by  $n$  when  $A^T = A$  (any  $m$  by  $n$  matrix  $R$ ) (b)  $(R^T R)_{jj} = (\text{column } j \text{ of } R) \cdot (\text{column } j \text{ of } R) = (\text{length squared of column } j) \geq 0$ .

$$20 \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -7 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}; \quad \begin{bmatrix} 1 & b \\ b & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & c - b^2 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -\frac{1}{2} & 1 & \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & & \\ \frac{3}{2} & & \\ \frac{4}{3} & & \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ & 1 & -\frac{2}{3} \\ & & 1 \end{bmatrix} = LDL^T.$$

21 Elimination on a symmetric 3 by 3 matrix leaves a symmetric lower right 2 by 2 matrix.

The examples  $\begin{bmatrix} 2 & 4 & 8 \\ 4 & 3 & 9 \\ 8 & 9 & 0 \end{bmatrix}$  and  $\begin{bmatrix} 1 & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$  lead to  $\begin{bmatrix} -5 & -7 \\ -7 & -32 \end{bmatrix}$  and  $\begin{bmatrix} d - b^2 & e - bc \\ e - bc & f - c^2 \end{bmatrix}$ .

$$22 \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} A = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ & 1 & 1 \\ & & -1 \end{bmatrix}; \quad \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} A = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ & -1 & 1 \\ & & 1 \end{bmatrix}$$

$$23 A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = P \text{ and } L = U = I. \quad \text{This cyclic } P \text{ exchanges rows 1-2 then rows 2-3 then rows 3-4.}$$

$$24 PA = LU \text{ is } \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 8 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & 1/3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 3 & 8 & \\ & & -2/3 \end{bmatrix}. \text{ If we wait}$$

to exchange and  $a_{12}$  is the pivot,  $A = L_1 P_1 U_1 = \begin{bmatrix} 1 & & \\ 3 & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ .

25 The **splu** code will not end when  $\mathbf{abs}(A(k, k)) < \text{tol}$  line 4 of the **slu** code on page 100. Instead **splu** looks for a nonzero entry below the diagonal in the current column  $k$ , and executes a row exchange. The 4 lines to exchange row  $k$  with row  $r$  are at the end of Section 2.7 (page 113). To *find* that nonzero entry  $A(r, k)$ , follow  $\mathbf{abs}(A(k, k)) < \text{tol}$  by locating the first nonzero (or the largest  $A(r, k)$ ) out of  $r = k + 1, \dots, n$ .

26 One way to decide even vs. odd is to count all pairs that  $P$  has in the wrong order. Then  $P$  is even or odd when that count is even or odd. Hard step: Show that an exchange always switches that count! Then 3 or 5 exchanges will leave that count odd.

$$27 \text{ (a) } E_{21} = \begin{bmatrix} 1 & & \\ -3 & 1 & \\ & & 1 \end{bmatrix} \text{ puts 0 in the 2, 1 entry of } E_{21}A. \text{ Then } E_{21}AE_{21}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 9 \end{bmatrix}$$

is still symmetric, with zero also in its 1, 2 entry. (b) Now use  $E_{32} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -4 & 1 \end{bmatrix}$

to make the 3, 2 entry zero and  $E_{32}E_{21}AE_{21}^TE_{32}^T = D$  also has zero in its 2, 3 entry. Key point: Elimination from both sides gives the symmetric  $LDL^T$  directly.

$$28 A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{bmatrix} = A^T \text{ has 0, 1, 2, 3 in every row. (I don't know any rules for a symmetric construction like this)}$$

29 Reordering the rows and/or the columns of  $\begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix}$  will move the entry  $\mathbf{a}$ . So the result cannot be the transpose (which doesn't move  $\mathbf{a}$ ).

30 (a) Total currents are  $A^T \mathbf{y} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} y_{BC} \\ y_{CS} \\ y_{BS} \end{bmatrix} = \begin{bmatrix} y_{BC} + y_{BS} \\ -y_{BC} + y_{CS} \\ -y_{CS} - y_{BS} \end{bmatrix}$ .

(b) Either way  $(A\mathbf{x})^T \mathbf{y} = \mathbf{x}^T (A^T \mathbf{y}) = x_B y_{BC} + x_B y_{BS} - x_C y_{BC} + x_C y_{CS} - x_S y_{CS} - x_S y_{BS}$ .

31  $\begin{bmatrix} 1 & 50 \\ 40 & 1000 \\ 2 & 50 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\mathbf{x}$ ;  $A^T \mathbf{y} = \begin{bmatrix} 1 & 40 & 2 \\ 50 & 1000 & 50 \end{bmatrix} \begin{bmatrix} 700 \\ 3 \\ 3000 \end{bmatrix} = \begin{bmatrix} 6820 \\ 188000 \end{bmatrix}$  1 truck  
1 plane

32  $A\mathbf{x} \cdot \mathbf{y}$  is the *cost* of inputs while  $\mathbf{x} \cdot A^T \mathbf{y}$  is the *value* of outputs.

33  $P^3 = I$  so three rotations for  $360^\circ$ ;  $P$  rotates around  $(1, 1, 1)$  by  $120^\circ$ .

34  $\begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = EH$  = (elementary matrix) times (symmetric matrix).

35  $L(U^T)^{-1}$  is lower triangular times lower triangular, so lower triangular. The transpose of  $U^T D U$  is  $U^T D^T U^{TT} = U^T D U$  again, so  $U^T D U$  is symmetric. The factorization multiplies lower triangular by symmetric to get  $LDU$  which is  $A$ .

36 These are groups: Lower triangular with diagonal 1's, diagonal invertible  $D$ , permutations  $P$ , orthogonal matrices with  $Q^T = Q^{-1}$ .

37 Certainly  $B^T$  is northwest.  $B^2$  is a full matrix!  $B^{-1}$  is southeast:  $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ . The rows of  $B$  are in reverse order from a lower triangular  $L$ , so  $B = PL$ . Then  $B^{-1} = L^{-1}P^{-1}$  has the *columns* in reverse order from  $L^{-1}$ . So  $B^{-1}$  is *southeast*. Northwest  $B = PL$  times southeast  $PU$  is  $(PLP)U =$  upper triangular.

38 There are  $n!$  permutation matrices of order  $n$ . Eventually *two powers of  $P$  must be the same*: If  $P^r = P^s$  then  $P^{r-s} = I$ . Certainly  $r - s \leq n!$

$P = \begin{bmatrix} P_2 & \\ & P_3 \end{bmatrix}$  is 5 by 5 with  $P_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  and  $P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$  and  $P^6 = I$ .

39 To split  $A$  into (symmetric  $B$ ) + (anti-symmetric  $C$ ), the only choice is  $B = \frac{1}{2}(A + A^T)$  and  $C = \frac{1}{2}(A - A^T)$ .

40 Start from  $Q^T Q = I$ , as in  $\begin{bmatrix} \mathbf{q}_1^T \\ \mathbf{q}_2^T \end{bmatrix} \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(a) The diagonal entries give  $\mathbf{q}_1^T \mathbf{q}_1 = 1$  and  $\mathbf{q}_2^T \mathbf{q}_2 = 1$ : *unit vectors*

(b) The off-diagonal entry is  $\mathbf{q}_1^T \mathbf{q}_2 = 0$  (and in general  $\mathbf{q}_i^T \mathbf{q}_j = 0$ )

(c) The leading example for  $Q$  is the rotation matrix  $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ .