

- 23 As in Problem 22: Row space basis  $(3, 0, 3), (1, 1, 2)$ ; column space basis  $(1, 4, 2), (2, 5, 7)$ ; the rank of  $(3 \text{ by } 2)$  times  $(2 \text{ by } 3)$  cannot be larger than the rank of either factor, so  $\text{rank} \leq 2$  and the  $3 \text{ by } 3$  product is not invertible.
- 24  $A^T \mathbf{y} = \mathbf{d}$  puts  $\mathbf{d}$  in the row space of  $A$ ; unique solution if the left nullspace (nullspace of  $A^T$ ) contains only  $\mathbf{y} = \mathbf{0}$ .
- 25 (a) True ( $A$  and  $A^T$  have the same rank) (b) False  $A = [1 \ 0]$  and  $A^T$  have very different left nullspaces (c) False ( $A$  can be invertible and unsymmetric even if  $C(A) = C(A^T)$ ) (d) True (The subspaces for  $A$  and  $-A$  are always the same. If  $A^T = A$  or  $A^T = -A$  they are also the same for  $A^T$ )
- 26 The rows of  $C = AB$  are combinations of the rows of  $B$ . So  $\text{rank } C \leq \text{rank } B$ . Also  $\text{rank } C \leq \text{rank } A$ , because the columns of  $C$  are combinations of the columns of  $A$ .
- 27 Choose  $d = bc/a$  to make  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  a rank-1 matrix. Then the row space has basis  $(a, b)$  and the nullspace has basis  $(-b, a)$ . Those two vectors are perpendicular!
- 28  $B$  and  $C$  (checkers and chess) both have rank 2 if  $p \neq 0$ . Row 1 and 2 are a basis for the row space of  $C$ ,  $B^T \mathbf{y} = \mathbf{0}$  has 6 special solutions with  $-1$  and  $1$  separated by a zero;  $N(C^T)$  has  $(-1, 0, 0, 0, 0, 0, 1)$  and  $(0, -1, 0, 0, 0, 0, 1, 0)$  and columns 3, 4, 5, 6 of  $I$ ;  $N(C)$  is a challenge.
- 29  $a_{11} = 1, a_{12} = 0, a_{13} = 1, a_{22} = 0, a_{32} = 1, a_{31} = 0, a_{23} = 1, a_{33} = 0, a_{21} = 1$ .
- 30 The subspaces for  $A = \mathbf{u}\mathbf{v}^T$  are pairs of orthogonal lines ( $\mathbf{v}$  and  $\mathbf{v}^\perp$ ,  $\mathbf{u}$  and  $\mathbf{u}^\perp$ ). If  $B$  has those same four subspaces then  $B = cA$  with  $c \neq 0$ .
- 31 (a)  $AX = 0$  if each column of  $X$  is a multiple of  $(1, 1, 1)$ ;  $\dim(\text{nullspace}) = 3$ . (b) If  $AX = B$  then all columns of  $B$  add to zero; dimension of the  $B$ 's = 6. (c)  $3 + 6 = \dim(M^{3 \times 3}) = 9$  entries in a  $3 \text{ by } 3$  matrix.
- 32 The key is equal row spaces. First row of  $A =$  combination of the rows of  $B$ : only possible combination (notice  $I$ ) is 1 (row 1 of  $B$ ). Same for each row so  $F = G$ .

### Problem Set 4.1, page 202

- 1 Both nullspace vectors are orthogonal to the row space vector in  $\mathbf{R}^3$ . The column space is perpendicular to the nullspace of  $A^T$  (two lines in  $\mathbf{R}^2$  because  $\text{rank} = 1$ ).
- 2 The nullspace of a  $3 \text{ by } 2$  matrix with rank 2 is  $\mathbf{Z}$  (only zero vector) so  $\mathbf{x}_n = \mathbf{0}$ , and row space =  $\mathbf{R}^2$ . Column space = plane perpendicular to left nullspace = line in  $\mathbf{R}^3$ .
- 3 (a)  $\begin{bmatrix} 1 & 2 & -3 \\ 2 & -3 & 1 \\ -3 & 5 & -2 \end{bmatrix}$  (b) Impossible,  $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$  not orthogonal to  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  (c)  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  in  $C(A)$  and  $N(A^T)$  is impossible: not perpendicular (d) Need  $A^2 = 0$ ; take  $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$  (e)  $(1, 1, 1)$  in the nullspace (columns add to  $\mathbf{0}$ ) and also row space; no such matrix.
- 4 If  $AB = 0$ , the columns of  $B$  are in the nullspace of  $A$ . The rows of  $A$  are in the left nullspace of  $B$ . If  $\text{rank} = 2$ , those four subspaces would have dimension 2 which is impossible for  $3 \text{ by } 3$ .
- 5 (a) If  $A\mathbf{x} = \mathbf{b}$  has a solution and  $A^T \mathbf{y} = \mathbf{0}$ , then  $\mathbf{y}$  is perpendicular to  $\mathbf{b}$ .  $\mathbf{b}^T \mathbf{y} = (A\mathbf{x})^T \mathbf{y} = \mathbf{x}^T (A^T \mathbf{y}) = 0$ . (b) If  $A^T \mathbf{y} = (1, 1, 1)$  has a solution,  $(1, 1, 1)$  is in the row space and is orthogonal to every  $\mathbf{x}$  in the nullspace.

- 6 Multiply the equations by  $y_1, y_2, y_3 = 1, 1, -1$ . Equations add to  $0 = 1$  so no solution:  $\mathbf{y} = (1, 1, -1)$  is in the left nullspace.  $A\mathbf{x} = \mathbf{b}$  would need  $0 = (\mathbf{y}^T A)\mathbf{x} = \mathbf{y}^T \mathbf{b} = 1$ .
- 7 Multiply the 3 equations by  $\mathbf{y} = (1, 1, -1)$ . Then  $x_1 - x_2 = 1$  plus  $x_2 - x_3 = 1$  minus  $x_1 - x_3 = 1$  is  $0 = 1$ . Key point: This  $\mathbf{y}$  in  $N(A^T)$  is not orthogonal to  $\mathbf{b} = (1, 1, 1)$  so  $\mathbf{b}$  is not in the column space and  $A\mathbf{x} = \mathbf{b}$  has *no solution*.
- 8  $\mathbf{x} = \mathbf{x}_r + \mathbf{x}_n$ , where  $\mathbf{x}_r$  is in the row space and  $\mathbf{x}_n$  is in the nullspace. Then  $A\mathbf{x}_n = \mathbf{0}$  and  $A\mathbf{x} = A\mathbf{x}_r + A\mathbf{x}_n = A\mathbf{x}_r$ . All  $A\mathbf{x}$  are in  $C(A)$ .
- 9  $A\mathbf{x}$  is always in the *column space* of  $A$ . If  $A^T A\mathbf{x} = \mathbf{0}$  then  $A\mathbf{x}$  is also in the nullspace of  $A^T$ . So  $A\mathbf{x}$  is perpendicular to itself. Conclusion:  $A\mathbf{x} = \mathbf{0}$  if  $A^T A\mathbf{x} = \mathbf{0}$ .
- 10 (a) With  $A^T = A$ , the column and row spaces are the same (b)  $\mathbf{x}$  is in the nullspace and  $\mathbf{z}$  is in the column space = row space: so these “eigenvectors” have  $\mathbf{x}^T \mathbf{z} = 0$ .
- 11 **For A:** The nullspace is spanned by  $(-2, 1)$ , the row space is spanned by  $(1, 2)$ . The column space is the line through  $(1, 3)$  and  $N(A^T)$  is the perpendicular line through  $(3, -1)$ . **For B:** The nullspace of  $B$  is spanned by  $(0, 1)$ , the row space is spanned by  $(1, 0)$ . The column space and left nullspace are the same as for  $A$ .
- 12  $\mathbf{x}$  splits into  $\mathbf{x}_r + \mathbf{x}_n = (1, -1) + (1, 1) = (2, 0)$ . Notice  $N(A^T)$  is a plane  $(1, 0) = (1, 1)/2 + (1, -1)/2 = \mathbf{x}_r + \mathbf{x}_n$ .
- 13  $V^T W = \mathbf{0}$  makes each basis vector for  $V$  orthogonal to each basis vector for  $W$ . Then every  $\mathbf{v}$  in  $V$  is orthogonal to every  $\mathbf{w}$  in  $W$  (combinations of the basis vectors).
- 14  $A\mathbf{x} = B\hat{\mathbf{x}}$  means that  $\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ -\hat{\mathbf{x}} \end{bmatrix} = \mathbf{0}$ . Three homogeneous equations in four unknowns always have a nonzero solution. Here  $\mathbf{x} = (3, 1)$  and  $\hat{\mathbf{x}} = (1, 0)$  and  $A\mathbf{x} = B\hat{\mathbf{x}} = (5, 6, 5)$  is in both column spaces. Two planes in  $\mathbf{R}^3$  must share a line.
- 15 A  $p$ -dimensional and a  $q$ -dimensional subspace of  $\mathbf{R}^n$  share at least a line if  $p + q > n$ . (The  $p + q$  basis vectors of  $V$  and  $W$  cannot be independent.)
- 16  $A^T \mathbf{y} = \mathbf{0}$  leads to  $(A\mathbf{x})^T \mathbf{y} = \mathbf{x}^T A^T \mathbf{y} = 0$ . Then  $\mathbf{y} \perp A\mathbf{x}$  and  $N(A^T) \perp C(A)$ .
- 17 If  $S$  is the subspace of  $\mathbf{R}^3$  containing only the zero vector, then  $S^\perp$  is  $\mathbf{R}^3$ . If  $S$  is spanned by  $(1, 1, 1)$ , then  $S^\perp$  is the plane spanned by  $(1, -1, 0)$  and  $(1, 0, -1)$ . If  $S$  is spanned by  $(2, 0, 0)$  and  $(0, 0, 3)$ , then  $S^\perp$  is the line spanned by  $(0, 1, 0)$ .
- 18  $S^\perp$  is the nullspace of  $A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 2 & 2 \end{bmatrix}$ . Therefore  $S^\perp$  is a *subspace* even if  $S$  is not.
- 19  $L^\perp$  is the 2-dimensional subspace (a plane) in  $\mathbf{R}^3$  perpendicular to  $L$ . Then  $(L^\perp)^\perp$  is a 1-dimensional subspace (a line) perpendicular to  $L^\perp$ . In fact  $(L^\perp)^\perp$  is  $L$ .
- 20 If  $V$  is the whole space  $\mathbf{R}^4$ , then  $V^\perp$  contains only the *zero vector*. Then  $(V^\perp)^\perp = \mathbf{R}^4 = V$ .
- 21 For example  $(-5, 0, 1, 1)$  and  $(0, 1, -1, 0)$  span  $S^\perp = \text{nullspace of } A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{bmatrix}$ .
- 22  $(1, 1, 1, 1)$  is a basis for  $P^\perp$ .  $A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$  has  $P$  as its nullspace and  $P^\perp$  as row space.
- 23  $\mathbf{x}$  in  $V^\perp$  is perpendicular to any vector in  $V$ . Since  $V$  contains all the vectors in  $S$ ,  $\mathbf{x}$  is also perpendicular to any vector in  $S$ . So every  $\mathbf{x}$  in  $V^\perp$  is also in  $S^\perp$ .

- 24  $AA^{-1} = I$ : Column 1 of  $A^{-1}$  is orthogonal to the space spanned by the 2nd, 3rd, ...,  $n$ th rows of  $A$ .
- 25 If the columns of  $A$  are unit vectors, all mutually perpendicular, then  $A^T A = I$ .
- 26  $A = \begin{bmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{bmatrix}$ , This example shows a matrix with perpendicular columns.  $A^T A = 9I$  is *diagonal*:  $(A^T A)_{ij} = (\text{column } i \text{ of } A) \cdot (\text{column } j \text{ of } A)$ . When the columns are *unit vectors*, then  $A^T A = I$ .
- 27 The lines  $3x + y = b_1$  and  $6x + 2y = b_2$  are **parallel**. They are the same line if  $b_2 = 2b_1$ . In that case  $(b_1, b_2)$  is perpendicular to  $(-2, 1)$ . The nullspace of the 2 by 2 matrix is the line  $3x + y = 0$ . One particular vector in the nullspace is  $(-1, 3)$ .
- 28 (a)  $(1, -1, 0)$  is in both planes. Normal vectors are perpendicular, but planes still intersect! (b) Need *three* orthogonal vectors to span the whole orthogonal complement. (c) Lines can meet at the zero vector without being orthogonal.
- 29  $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 0 \\ 3 & 0 & -1 \end{bmatrix}$ ;  $A$  has  $\mathbf{v} = (1, 2, 3)$  in row space and column space.  $B$  has  $\mathbf{v}$  in its column space and nullspace.  $\mathbf{v}$  **can not** be in the nullspace and row space, or in the left nullspace and column space. These spaces are orthogonal and  $\mathbf{v}^T \mathbf{v} \neq 0$ .
- 30 When  $AB = 0$ , the column space of  $B$  is contained in the nullspace of  $A$ . Therefore the dimension of  $C(B) \leq$  dimension of  $N(A)$ . This means  $\text{rank}(B) \leq 4 - \text{rank}(A)$ .
- 31  $\text{null}(N')$  produces a basis for the *row space* of  $A$  (perpendicular to  $N(A)$ ).
- 32 We need  $\mathbf{r}^T \mathbf{n} = 0$  and  $\mathbf{c}^T \boldsymbol{\ell} = 0$ . All possible examples have the form  $a\mathbf{c}\mathbf{r}^T$  with  $a \neq 0$ .
- 33 Both  $\mathbf{r}$ 's orthogonal to both  $\mathbf{n}$ 's, both  $\mathbf{c}$ 's orthogonal to both  $\boldsymbol{\ell}$ 's, each pair independent. All  $A$ 's with these subspaces have the form  $[\mathbf{c}_1 \ \mathbf{c}_2]M[\mathbf{r}_1 \ \mathbf{r}_2]^T$  for a 2 by 2 invertible  $M$ .

## Problem Set 4.2, page 214

- 1 (a)  $\mathbf{a}^T \mathbf{b} / \mathbf{a}^T \mathbf{a} = 5/3$ ;  $\mathbf{p} = 5\mathbf{a}/3$ ;  $\mathbf{e} = (-2, 1, 1)/3$  (b)  $\mathbf{a}^T \mathbf{b} / \mathbf{a}^T \mathbf{a} = -1$ ;  $\mathbf{p} = \mathbf{a}$ ;  $\mathbf{e} = \mathbf{0}$ .
- 2 (a) The projection of  $\mathbf{b} = (\cos \theta, \sin \theta)$  onto  $\mathbf{a} = (1, 0)$  is  $\mathbf{p} = (\cos \theta, 0)$   
 (b) The projection of  $\mathbf{b} = (1, 1)$  onto  $\mathbf{a} = (1, -1)$  is  $\mathbf{p} = (0, 0)$  since  $\mathbf{a}^T \mathbf{b} = 0$ .
- 3  $P_1 = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$  and  $P_1 \mathbf{b} = \frac{1}{3} \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$ .  $P_2 = \frac{1}{11} \begin{bmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{bmatrix}$  and  $P_2 \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ .
- 4  $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ ,  $P_2 = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ .  $P_1$  projects onto  $(1, 0)$ ,  $P_2$  projects onto  $(1, -1)$   
 $P_1 P_2 \neq 0$  and  $P_1 + P_2$  is not a projection matrix.
- 5  $P_1 = \frac{1}{9} \begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}$ ,  $P_2 = \frac{1}{9} \begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix}$ .  $P_1$  and  $P_2$  are the projection matrices onto the lines through  $\mathbf{a}_1 = (-1, 2, 2)$  and  $\mathbf{a}_2 = (2, 2, -1)$   $P_1 P_2 = \text{zero matrix because } \mathbf{a}_1 \perp \mathbf{a}_2$ .
- XXX Above solution does not fit in 3 lines.
- 6  $\mathbf{p}_1 = (\frac{1}{9}, -\frac{2}{9}, -\frac{2}{9})$  and  $\mathbf{p}_2 = (\frac{4}{9}, \frac{4}{9}, -\frac{2}{9})$  and  $\mathbf{p}_3 = (\frac{4}{9}, -\frac{2}{9}, \frac{4}{9})$ . So  $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = \mathbf{b}$ .

$$7 \quad P_1 + P_2 + P_3 = \frac{1}{9} \begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix} + \frac{1}{9} \begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix} + \frac{1}{9} \begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix} = I.$$

We can add projections onto *orthogonal vectors*. This is important.

8 The projections of  $(1, 1)$  onto  $(1, 0)$  and  $(1, 2)$  are  $\mathbf{p}_1 = (1, 0)$  and  $\mathbf{p}_2 = (0.6, 1.2)$ . Then  $\mathbf{p}_1 + \mathbf{p}_2 \neq \mathbf{b}$ .

9 Since  $A$  is invertible,  $P = A(A^T A)^{-1} A^T = A A^{-1} (A^T)^{-1} A^T = I$ : project on all of  $\mathbf{R}^2$ .

10  $P_2 = \begin{bmatrix} 0.2 & 0.4 \\ 0.4 & 0.8 \end{bmatrix}$ ,  $P_2 \mathbf{a}_1 = \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix}$ ,  $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ ,  $P_1 P_2 \mathbf{a}_1 = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}$ . This is not  $\mathbf{a}_1 = (1, 0)$ . No,  $P_1 P_2 \neq (P_1 P_2)^2$ .

11 (a)  $\mathbf{p} = A(A^T A)^{-1} A^T \mathbf{b} = (2, 3, 0)$ ,  $\mathbf{e} = (0, 0, 4)$ ,  $A^T \mathbf{e} = \mathbf{0}$  (b)  $\mathbf{p} = (4, 4, 6)$ ,  $\mathbf{e} = \mathbf{0}$ .

12  $P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$  = projection matrix onto the column space of  $A$  (the  $xy$  plane)

$P_2 = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  = Projection matrix onto the second column space. Certainly  $(P_2)^2 = P_2$ .

$$13 \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, P = \text{square matrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{p} = P \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}.$$

14 The projection of this  $\mathbf{b}$  onto the column space of  $A$  is  $\mathbf{b}$  itself when  $\mathbf{b}$  is in that space.

But  $P$  is not necessarily  $I$ .  $P = \frac{1}{21} \begin{bmatrix} 5 & 8 & -4 \\ 8 & 17 & 2 \\ -4 & 2 & 20 \end{bmatrix}$  and  $\mathbf{b} = P\mathbf{b} = \mathbf{p} = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$ .

15  $2A$  has the same column space as  $A$ .  $\hat{\mathbf{x}}$  for  $2A$  is *half* of  $\hat{\mathbf{x}}$  for  $A$ .

16  $\frac{1}{2}(1, 2, -1) + \frac{3}{2}(1, 0, 1) = (2, 1, 1)$ . So  $\mathbf{b}$  is in the plane. Projection shows  $P\mathbf{b} = \mathbf{b}$ .

17 If  $P^2 = P$  then  $(\mathbf{I} - P)^2 = (\mathbf{I} - P)(\mathbf{I} - P) = \mathbf{I} - P\mathbf{I} - \mathbf{I}P + P^2 = \mathbf{I} - P$ . When  $P$  projects onto the column space,  $\mathbf{I} - P$  projects onto the *left nullspace*.

18 (a)  $\mathbf{I} - P$  is the projection matrix onto  $(1, -1)$  in the perpendicular direction to  $(1, 1)$   
 (b)  $\mathbf{I} - P$  projects onto the plane  $x + y + z = 0$  perpendicular to  $(1, 1, 1)$ .

19 For any basis vectors in the plane  $x - y - 2z = 0$ , say  $(1, 1, 0)$  and  $(2, 0, 1)$ , the matrix  $P$  is  $\begin{bmatrix} 5/6 & 1/6 & 1/3 \\ 1/6 & 5/6 & -1/3 \\ 1/3 & -1/3 & 1/3 \end{bmatrix}$ .

$$20 \quad \mathbf{e} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, Q = \frac{\mathbf{e}\mathbf{e}^T}{\mathbf{e}^T \mathbf{e}} = \begin{bmatrix} 1/6 & -1/6 & -1/3 \\ -1/6 & 1/6 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}, \mathbf{I} - Q = \begin{bmatrix} 5/6 & 1/6 & 1/3 \\ 1/6 & 5/6 & -1/3 \\ 1/3 & -1/3 & 1/3 \end{bmatrix}.$$

21  $(A(A^T A)^{-1} A^T)^2 = A(A^T A)^{-1} (A^T A) (A^T A)^{-1} A^T = A(A^T A)^{-1} A^T$ . So  $P^2 = P$ .  $P\mathbf{b}$  is in the column space (where  $P$  projects). Then its projection  $P(P\mathbf{b})$  is  $P\mathbf{b}$ .

22  $P^T = (A(A^T A)^{-1} A^T)^T = A((A^T A)^{-1})^T A^T = A(A^T A)^{-1} A^T = P$ . ( $A^T A$  is symmetric!)

23 If  $A$  is invertible then its column space is all of  $\mathbf{R}^n$ . So  $P = I$  and  $\mathbf{e} = \mathbf{0}$ .

24 The nullspace of  $A^T$  is *orthogonal* to the column space  $C(A)$ . So if  $A^T \mathbf{b} = \mathbf{0}$ , the projection of  $\mathbf{b}$  onto  $C(A)$  should be  $\mathbf{p} = \mathbf{0}$ . Check  $P\mathbf{b} = A(A^T A)^{-1} A^T \mathbf{b} = A(A^T A)^{-1} \mathbf{0}$ .

- 25** The column space of  $P$  will be  $S$ . Then  $r = \text{dimension of } S = n$ .
- 26**  $A^{-1}$  exists since the rank is  $r = m$ . Multiply  $A^2 = A$  by  $A^{-1}$  to get  $A = I$ .
- 27** If  $A^T Ax = \mathbf{0}$  then  $Ax$  is in the nullspace of  $A^T$ . But  $Ax$  is always in the column space of  $A$ . To be in both of those perpendicular spaces,  $Ax$  must be zero. So  $A$  and  $A^T A$  have the same nullspace.
- 28**  $P^2 = P = P^T$  give  $P^T P = P$ . Then the  $(2, 2)$  entry of  $P$  equals the  $(2, 2)$  entry of  $P^T P$  which is the length squared of column 2.
- 29**  $A = B^T$  has independent columns, so  $A^T A$  (which is  $BB^T$ ) must be invertible.
- 30** (a) The column space is the line through  $\mathbf{a} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$  so  $P_C = \frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}} = \frac{1}{25} \begin{bmatrix} 9 & 12 \\ 12 & 25 \end{bmatrix}$ .  
 (b) The row space is the line through  $\mathbf{v} = (1, 2, 2)$  and  $P_R = \mathbf{v}\mathbf{v}^T/\mathbf{v}^T\mathbf{v}$ . Always  $P_C A = A$  (columns of  $A$  project to themselves) and  $AP_R = A$ . Then  $P_C AP_R = A$ !
- 31** The error  $\mathbf{e} = \mathbf{b} - \mathbf{p}$  must be perpendicular to all the  $\mathbf{a}$ 's.
- 32** Since  $P_1\mathbf{b}$  is in  $C(A)$ ,  $P_2(P_1\mathbf{b})$  equals  $P_1\mathbf{b}$ . So  $P_2 P_1 = P_1 = \mathbf{a}\mathbf{a}^T/\mathbf{a}^T\mathbf{a}$  where  $\mathbf{a} = (1, 2, 0)$ .
- 33** If  $P_1 P_2 = P_2 P_1$  then  $S$  is contained in  $T$  or  $T$  is contained in  $S$ .
- 34**  $BB^T$  is invertible as in Problem 29. Then  $(A^T A)(BB^T) = \text{product of } r \text{ by } r \text{ invertible matrices, so rank } r$ .  $AB$  can't have rank  $< r$ , since  $A^T$  and  $B^T$  cannot increase the rank.  
*Conclusion: A ( $m$  by  $r$  of rank  $r$ ) times  $B$  ( $r$  by  $n$  of rank  $r$ ) produces  $AB$  of rank  $r$ .*

### Problem Set 4.3, page 226

$$1 \quad A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} \text{ give } A^T A = \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \text{ and } A^T \mathbf{b} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}.$$

$$A^T A \hat{\mathbf{x}} = A^T \mathbf{b} \text{ gives } \hat{\mathbf{x}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \text{ and } \mathbf{p} = A \hat{\mathbf{x}} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix} \text{ and } \mathbf{e} = \mathbf{b} - \mathbf{p} = \begin{bmatrix} -1 \\ 3 \\ -5 \\ 3 \end{bmatrix} \\ E = \|\mathbf{e}\|^2 = 44$$

$$2 \quad \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}. \text{ This } A\mathbf{x} = \mathbf{b} \text{ is unsolvable. Change } \mathbf{b} \text{ to } \mathbf{p} = P\mathbf{b} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}; \hat{\mathbf{x}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \text{ exactly solves } A\hat{\mathbf{x}} = \mathbf{p}.$$

**3** In Problem 2,  $\mathbf{p} = A(A^T A)^{-1} A^T \mathbf{b} = (1, 5, 13, 17)$  and  $\mathbf{e} = \mathbf{b} - \mathbf{p} = (-1, 3, -5, 3)$ .  $\mathbf{e}$  is perpendicular to both columns of  $A$ . This shortest distance  $\|\mathbf{e}\|$  is  $\sqrt{44}$ .

**4**  $E = (C + 0D)^2 + (C + 1D - 8)^2 + (C + 3D - 8)^2 + (C + 4D - 20)^2$ . Then  $\partial E/\partial C = 2C + 2(C + D - 8) + 2(C + 3D - 8) + 2(C + 4D - 20) = 0$  and  $\partial E/\partial D = 1 \cdot 2(C + D - 8) + 3 \cdot 2(C + 3D - 8) + 4 \cdot 2(C + 4D - 20) = 0$ . These normal equations are again  $\begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$ .

- 5  $E = (C-0)^2 + (C-8)^2 + (C-8)^2 + (C-20)^2$ .  $A^T = [1 \ 1 \ 1 \ 1]$  and  $A^T A = [4]$ .  $A^T \mathbf{b} = [36]$  and  $(A^T A)^{-1} A^T \mathbf{b} = 9 = \text{best height } C$ . Errors  $\mathbf{e} = (-9, -1, -1, 11)$ .
- 6  $\mathbf{a} = (1, 1, 1, 1)$  and  $\mathbf{b} = (0, 8, 8, 20)$  give  $\hat{x} = \mathbf{a}^T \mathbf{b} / \mathbf{a}^T \mathbf{a} = 9$  and the projection is  $\hat{\mathbf{x}} \mathbf{a} = \mathbf{p} = (9, 9, 9, 9)$ . Then  $\mathbf{e}^T \mathbf{a} = (-9, -1, -1, 11)^T (1, 1, 1, 1) = 0$  and  $\|\mathbf{e}\| = \sqrt{204}$ .
- 7  $A = [0 \ 1 \ 3 \ 4]^T$ ,  $A^T A = [26]$  and  $A^T \mathbf{b} = [112]$ . Best  $D = 112/26 = 56/13$ .
- 8  $\hat{\mathbf{x}} = 56/13$ ,  $\mathbf{p} = (56/13)(0, 1, 3, 4)$ .  $(C, D) = (9, 56/13)$  don't match  $(C, D) = (1, 4)$ . Columns of  $A$  were not perpendicular so we can't project separately to find  $C$  and  $D$ .
- 9 Parabola  
Project  $\mathbf{b}$   
4D to 3D  $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$ .  $A^T A \hat{\mathbf{x}} = \begin{bmatrix} 4 & 8 & 26 \\ 8 & 26 & 92 \\ 26 & 92 & 338 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \\ 400 \end{bmatrix}$ .
- 10  $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$ . Then  $\begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 0 \\ 47 \\ -28 \\ 5 \end{bmatrix}$ . Exact cubic so  $\mathbf{p} = \mathbf{b}$ ,  $\mathbf{e} = \mathbf{0}$ . This Vandermonde matrix gives exact interpolation by a cubic at 0, 1, 3, 4.
- 11 (a) The best line  $x = 1 + 4t$  gives the center point  $\hat{\mathbf{b}} = 9$  when  $\hat{t} = 2$ .  
(b) The first equation  $Cm + D \sum t_i = \sum b_i$  divided by  $m$  gives  $C + D\hat{t} = \hat{\mathbf{b}}$ .
- 12 (a)  $\mathbf{a} = (1, \dots, 1)$  has  $\mathbf{a}^T \mathbf{a} = m$ ,  $\mathbf{a}^T \mathbf{b} = b_1 + \dots + b_m$ . Therefore  $\hat{x} = \mathbf{a}^T \mathbf{b} / m$  is the **mean** of the  $b$ 's (b)  $\mathbf{e} = \mathbf{b} - \hat{x} \mathbf{a}$   $\mathbf{b} = (1, 2, b)$   $\|\mathbf{e}\|^2 = \sum_{i=1}^m (b_i - \hat{x})^2 = \mathbf{variance}$
- (c)  $\mathbf{p} = (3, 3, 3)$   
 $\mathbf{e} = (-2, -1, 3)$   $\mathbf{p}^T \mathbf{e} = 0$ .  $P = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ .
- 13  $(A^T A)^{-1} A^T (\mathbf{b} - A\mathbf{x}) = \hat{\mathbf{x}} - \mathbf{x}$ . When  $\mathbf{e} = \mathbf{b} - A\mathbf{x}$  averages to  $\mathbf{0}$ , so does  $\hat{\mathbf{x}} - \mathbf{x}$ .
- 14 The matrix  $(\hat{\mathbf{x}} - \mathbf{x})(\hat{\mathbf{x}} - \mathbf{x})^T$  is  $(A^T A)^{-1} A^T (\mathbf{b} - A\mathbf{x})(\mathbf{b} - A\mathbf{x})^T A (A^T A)^{-1}$ . When the average of  $(\mathbf{b} - A\mathbf{x})(\mathbf{b} - A\mathbf{x})^T$  is  $\sigma^2 I$ , the average of  $(\hat{\mathbf{x}} - \mathbf{x})(\hat{\mathbf{x}} - \mathbf{x})^T$  will be the *output covariance matrix*  $(A^T A)^{-1} A^T \sigma^2 A (A^T A)^{-1}$  which simplifies to  $\sigma^2 (A^T A)^{-1}$ .
- 15 When  $A$  has 1 column of ones, Problem 14 gives the expected error  $(\hat{\mathbf{x}} - \mathbf{x})^2$  as  $\sigma^2 (A^T A)^{-1} = \sigma^2 / m$ . By taking  $m$  measurements, the variance drops from  $\sigma^2$  to  $\sigma^2 / m$ .
- 16  $\frac{1}{10} b_{10} + \frac{9}{10} \hat{x}_9 = \frac{1}{10} (b_1 + \dots + b_{10})$ . Knowing  $\hat{x}_9$  avoids adding all  $b$ 's.
- 17  $\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}$ . The solution  $\hat{\mathbf{x}} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$  comes from  $\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 35 \\ 42 \end{bmatrix}$ .
- 18  $\mathbf{p} = A\hat{\mathbf{x}} = (5, 13, 17)$  gives the heights of the closest line. The error is  $\mathbf{b} - \mathbf{p} = (2, -6, 4)$ . This error  $\mathbf{e}$  has  $P\mathbf{e} = P\mathbf{b} - P\mathbf{p} = \mathbf{p} - \mathbf{p} = \mathbf{0}$ .
- 19 If  $\mathbf{b} = \text{error } \mathbf{e}$  then  $\mathbf{b}$  is perpendicular to the column space of  $A$ . Projection  $\mathbf{p} = \mathbf{0}$ .
- 20 If  $\mathbf{b} = A\hat{\mathbf{x}} = (5, 13, 17)$  then  $\hat{\mathbf{x}} = (9, 4)$  and  $\mathbf{e} = \mathbf{0}$  since  $\mathbf{b}$  is in the column space of  $A$ .
- 21  $\mathbf{e}$  is in  $N(A^T)$ ;  $\mathbf{p}$  is in  $C(A)$ ;  $\hat{\mathbf{x}}$  is in  $C(A^T)$ ;  $N(A) = \{\mathbf{0}\} = \text{zero vector only}$ .

22 The least squares equation is  $\begin{bmatrix} 5 & \mathbf{0} \\ \mathbf{0} & 10 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$ . Solution:  $C = 1, D = -1$ .

Line  $1 - t$ . Symmetric  $t$ 's  $\Rightarrow$  diagonal  $A^T A$

23  $e$  is orthogonal to  $p$ ; then  $\|e\|^2 = e^T(b - p) = e^T b = b^T b - b^T p$ .

24 The derivatives of  $\|Ax - b\|^2 = x^T A^T A x - 2b^T A x + b^T b$  (this term is constant) are zero when  $2A^T A x = 2A^T b$ , or  $x = (A^T A)^{-1} A^T b$ .

25 3 points on a line: *Equal slopes*  $(b_2 - b_1)/(t_2 - t_1) = (b_3 - b_2)/(t_3 - t_2)$ . Linear algebra: Orthogonal to  $(1, 1, 1)$  and  $(t_1, t_2, t_3)$  is  $y = (t_2 - t_3, t_3 - t_1, t_1 - t_2)$  in the left nullspace.  $b$  is in the column space. Then  $y^T b = 0$  is the same equal slopes condition written as  $(b_2 - b_1)(t_3 - t_2) = (b_3 - b_2)(t_2 - t_1)$ .

26  $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix}$  has  $A^T A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ ,  $A^T b = \begin{bmatrix} 8 \\ -2 \\ -3 \end{bmatrix}$ ,  $\begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -3/2 \end{bmatrix}$ . At  $x, y = 0, 0$  the best plane  $2 - x - \frac{3}{2}y$  has height  $C = 2 =$  average of  $0, 1, 3, 4$ .

27 The shortest link connecting two lines in space is *perpendicular to those lines*.

28 Only 1 plane contains  $\mathbf{0}, a_1, a_2$  unless  $a_1, a_2$  are *dependent*. Same test for  $a_1, \dots, a_n$ .

29 There is exactly one hyperplane containing the  $n$  points  $\mathbf{0}, a_1, \dots, a_{n-1}$  *When the  $n - 1$  vectors  $a_1, \dots, a_{n-1}$  are linearly independent*. (For  $n = 3$ , the vectors  $a_1$  and  $a_2$  must be independent. Then the three points  $\mathbf{0}, a_1, a_2$  determine a plane.) The equation of the plane in  $\mathbf{R}^n$  will be  $a_n^T x = 0$ . Here  $a_n$  is any nonzero vector on the line (it is only a line!) perpendicular to  $a_1, \dots, a_{n-1}$ .

## Problem Set 4.4, page 239

1 (a) *Independent* (b) *Independent and orthogonal* (c) *Independent and orthonormal*. For orthonormal vectors, (a) becomes  $(1, 0), (0, 1)$  and (b) is  $(.6, .8), (.8, -.6)$ .

2 Divide by length 3 to get  $q_1 = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}), q_2 = (-\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$ .  $Q^T Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  but  $Q Q^T = \begin{bmatrix} 5/9 & 2/9 & -4/9 \\ 2/9 & 8/9 & 2/9 \\ -4/9 & 2/9 & 5/9 \end{bmatrix}$ .

3 (a)  $A^T A$  will be  $16I$  (b)  $A^T A$  will be diagonal with entries 1, 4, 9.

4 (a)  $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ ,  $Q Q^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq I$ . Any  $Q$  with  $n < m$  has  $Q Q^T \neq I$ .

(b)  $(1, 0)$  and  $(0, 0)$  are *orthogonal*, not *independent*. Nonzero orthogonal vectors are independent. (c) Starting from  $q_1 = (1, 1, 1)/\sqrt{3}$  my favorite is  $q_2 = (1, -1, 0)/\sqrt{2}$  and  $q_3 = (1, 1, -2)/\sqrt{6}$ .

5 *Orthogonal* vectors are  $(1, -1, 0)$  and  $(1, 1, -1)$ . *Orthonormal* are  $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0), (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$ .

- 6  $Q_1 Q_2$  is orthogonal because  $(Q_1 Q_2)^T Q_1 Q_2 = Q_2^T Q_1^T Q_1 Q_2 = Q_2^T Q_2 = I$ .
- 7 When Gram-Schmidt gives  $Q$  with orthonormal columns,  $Q^T Q \hat{x} = Q^T \mathbf{b}$  becomes  $\hat{x} = Q^T \mathbf{b}$ .
- 8 If  $\mathbf{q}_1$  and  $\mathbf{q}_2$  are orthonormal vectors in  $\mathbf{R}^5$  then  $(\mathbf{q}_1^T \mathbf{b})\mathbf{q}_1 + (\mathbf{q}_2^T \mathbf{b})\mathbf{q}_2$  is closest to  $\mathbf{b}$ .
- 9 (a)  $Q = \begin{bmatrix} .8 & -.6 \\ .6 & .8 \\ 0 & 0 \end{bmatrix}$  has  $P = Q Q^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$  (b)  $(Q Q^T)(Q Q^T) = Q(Q^T Q)Q^T = Q Q^T$ .
- 10 (a) If  $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$  are orthonormal then the dot product of  $\mathbf{q}_1$  with  $c_1 \mathbf{q}_1 + c_2 \mathbf{q}_2 + c_3 \mathbf{q}_3 = \mathbf{0}$  gives  $c_1 = 0$ . Similarly  $c_2 = c_3 = 0$ . Independent  $\mathbf{q}$ 's (b)  $Q \mathbf{x} = \mathbf{0} \Rightarrow Q^T Q \mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x} = \mathbf{0}$ .
- 11 (a) Two orthonormal vectors are  $\mathbf{q}_1 = \frac{1}{10}(1, 3, 4, 5, 7)$  and  $\mathbf{q}_2 = \frac{1}{10}(-7, 3, 4, -5, 1)$   
 (b) Closest in the plane:  $\text{project } Q Q^T(1, 0, 0, 0, 0) = (0.5, -0.18, -0.24, 0.4, 0)$ .
- 12 (a) Orthonormal  $\mathbf{a}$ 's:  $\mathbf{a}_1^T \mathbf{b} = \mathbf{a}_1^T(x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + x_3 \mathbf{a}_3) = x_1(\mathbf{a}_1^T \mathbf{a}_1) = x_1$   
 (b) Orthogonal  $\mathbf{a}$ 's:  $\mathbf{a}_1^T \mathbf{b} = \mathbf{a}_1^T(x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + x_3 \mathbf{a}_3) = x_1(\mathbf{a}_1^T \mathbf{a}_1)$ . Therefore  $x_1 = \mathbf{a}_1^T \mathbf{b} / \mathbf{a}_1^T \mathbf{a}_1$   
 (c)  $x_1$  is the first component of  $A^{-1}$  times  $\mathbf{b}$ .
- 13 The multiple to subtract is  $\frac{\mathbf{a}^T \mathbf{b}}{\mathbf{a}^T \mathbf{a}}$ . Then  $\mathbf{B} = \mathbf{b} - \frac{\mathbf{a}^T \mathbf{b}}{\mathbf{a}^T \mathbf{a}} \mathbf{a} = (4, 0) - 2 \cdot (1, 1) = (2, -2)$ .
- 14  $\begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix} = [\mathbf{q}_1 \quad \mathbf{q}_2] \begin{bmatrix} \|\mathbf{a}\| & \mathbf{q}_1^T \mathbf{b} \\ 0 & \|\mathbf{B}\| \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 2\sqrt{2} \\ 0 & 2\sqrt{2} \end{bmatrix} = QR$ .
- 15 (a)  $\mathbf{q}_1 = \frac{1}{3}(1, 2, -2)$ ,  $\mathbf{q}_2 = \frac{1}{3}(2, 1, 2)$ ,  $\mathbf{q}_3 = \frac{1}{3}(2, -2, -1)$  (b) The nullspace of  $A^T$  contains  $\mathbf{q}_3$  (c)  $\hat{\mathbf{x}} = (A^T A)^{-1} A^T(1, 2, 7) = (1, 2)$ .
- 16 The projection  $\mathbf{p} = (\mathbf{a}^T \mathbf{b} / \mathbf{a}^T \mathbf{a}) \mathbf{a} = 14\mathbf{a} / 49 = 2\mathbf{a} / 7$  is closest to  $\mathbf{b}$ ;  $\mathbf{q}_1 = \mathbf{a} / \|\mathbf{a}\| = \mathbf{a} / 7$  is  $(4, 5, 2, 2) / 7$ .  $\mathbf{B} = \mathbf{b} - \mathbf{p} = (-1, 4, -4, -4) / 7$  has  $\|\mathbf{B}\| = 1$  so  $\mathbf{q}_2 = \mathbf{B}$ .
- 17  $\mathbf{p} = (\mathbf{a}^T \mathbf{b} / \mathbf{a}^T \mathbf{a}) \mathbf{a} = (3, 3, 3)$  and  $\mathbf{e} = (-2, 0, 2)$ .  $\mathbf{q}_1 = (1, 1, 1) / \sqrt{3}$  and  $\mathbf{q}_2 = (-1, 0, 1) / \sqrt{2}$ .
- 18  $\mathbf{A} = \mathbf{a} = (1, -1, 0, 0)$ ;  $\mathbf{B} = \mathbf{b} - \mathbf{p} = (\frac{1}{2}, \frac{1}{2}, -1, 0)$ ;  $\mathbf{C} = \mathbf{c} - \mathbf{p}_A - \mathbf{p}_B = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -1)$ . Notice the pattern in those orthogonal  $\mathbf{A}, \mathbf{B}, \mathbf{C}$ . In  $\mathbf{R}^5$ ,  $\mathbf{D}$  would be  $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -1)$ .
- 19 If  $A = QR$  then  $A^T A = R^T Q^T Q R = R^T R =$  lower triangular times upper triangular (this Cholesky factorization of  $A^T A$  uses the same  $R$  as Gram-Schmidt!). The example has  $A = \begin{bmatrix} -1 & 1 \\ 2 & 1 \\ 2 & 4 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -1 & 2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix} = QR$  and the same  $R$  appears in  $A^T A = \begin{bmatrix} 9 & 9 \\ 9 & 18 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix} = R^T R$ .
- 20 (a) True (b) True.  $Q \mathbf{x} = x_1 \mathbf{q}_1 + x_2 \mathbf{q}_2$ .  $\|Q \mathbf{x}\|^2 = x_1^2 + x_2^2$  because  $\mathbf{q}_1 \cdot \mathbf{q}_2 = 0$ .
- 21 The orthonormal vectors are  $\mathbf{q}_1 = (1, 1, 1, 1) / 2$  and  $\mathbf{q}_2 = (-5, -1, 1, 5) / \sqrt{52}$ . Then  $\mathbf{b} = (-4, -3, 3, 0)$  projects to  $\mathbf{p} = (-7, -3, -1, 3) / 2$ . And  $\mathbf{b} - \mathbf{p} = (-1, -3, 7, -3) / 2$  is orthogonal to both  $\mathbf{q}_1$  and  $\mathbf{q}_2$ .
- 22  $A = (1, 1, 2)$ ,  $B = (1, -1, 0)$ ,  $C = (-1, -1, 1)$ . These are not yet unit vectors.

- 23 You can see why  $\mathbf{q}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ,  $\mathbf{q}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ,  $\mathbf{q}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ .  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix} = QR$ .
- 24 (a) One basis for the subspace  $\mathcal{S}$  of solutions to  $x_1 + x_2 + x_3 - x_4 = 0$  is  $\mathbf{v}_1 = (1, -1, 0, 0)$ ,  $\mathbf{v}_2 = (1, 0, -1, 0)$ ,  $\mathbf{v}_3 = (1, 0, 0, 1)$  (b) Since  $\mathcal{S}$  contains solutions to  $(1, 1, 1, -1)^T \mathbf{x} = 0$ , a basis for  $\mathcal{S}^\perp$  is  $(1, 1, 1, -1)$  (c) Split  $(1, 1, 1, 1) = \mathbf{b}_1 + \mathbf{b}_2$  by projection on  $\mathcal{S}^\perp$  and  $\mathcal{S}$ :  $\mathbf{b}_2 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$  and  $\mathbf{b}_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2})$ .
- 25 This question shows 2 by 2 formulas for  $QR$ ; breakdown  $R_{22} = 0$  when  $A$  is singular.  $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \cdot \frac{1}{\sqrt{5}} \begin{bmatrix} 5 & 3 \\ 0 & 1 \end{bmatrix}$ . Singular  $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$ . The Gram-Schmidt process breaks down when  $ad - bc = 0$ .
- 26  $(\mathbf{q}_2^T \mathbf{C}^*) \mathbf{q}_2 = \frac{\mathbf{B}^T \mathbf{c}}{\mathbf{B}^T \mathbf{B}} \mathbf{B}$  because  $\mathbf{q}_2 = \frac{\mathbf{B}}{\|\mathbf{B}\|}$  and the extra  $\mathbf{q}_1$  in  $\mathbf{C}^*$  is orthogonal to  $\mathbf{q}_2$ .
- 27 When  $a$  and  $b$  are not orthogonal, the projections onto these lines *do not add* to the projection onto the plane of  $a$  and  $b$ . We must use the orthogonal  $A$  and  $B$  (or orthonormal  $\mathbf{q}_1$  and  $\mathbf{q}_2$ ) to be allowed to add 1D projections.
- 28 There are  $mn$  multiplications in (11) and  $\frac{1}{2}m^2n$  multiplications in each part of (12).
- 29  $\mathbf{q}_1 = \frac{1}{3}(2, 2, -1)$ ,  $\mathbf{q}_2 = \frac{1}{3}(2, -1, 2)$ ,  $\mathbf{q}_3 = \frac{1}{3}(1, -2, -2)$ .
- 30 The columns of the wavelet matrix  $W$  are *orthonormal*. Then  $W^{-1} = W^T$ . See Section 7.2 for more about wavelets: a useful orthonormal basis with many zeros.
- 31 (a)  $c = \frac{1}{2}$  normalizes all the orthogonal columns to have unit length (b) The projection  $(\mathbf{a}^T \mathbf{b} / \mathbf{a}^T \mathbf{a}) \mathbf{a}$  of  $\mathbf{b} = (1, 1, 1, 1)$  onto the first column is  $\mathbf{p}_1 = \frac{1}{2}(-1, 1, 1, 1)$ . (Check  $\mathbf{e} = \mathbf{0}$ .) To project onto the plane, add  $\mathbf{p}_2 = \frac{1}{2}(1, -1, 1, 1)$  to get  $(0, 0, 1, 1)$ .
- 32  $Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  reflects across  $x$  axis,  $Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$  across plane  $y + z = 0$ .
- 33 Orthogonal and lower triangular  $\Rightarrow \pm 1$  on the main diagonal and zeros elsewhere.
- 34 (a)  $Q\mathbf{u} = (I - 2\mathbf{u}\mathbf{u}^T)\mathbf{u} = \mathbf{u} - 2\mathbf{u}\mathbf{u}^T\mathbf{u}$ . This is  $-\mathbf{u}$ , provided that  $\mathbf{u}^T\mathbf{u}$  equals 1 (b)  $Q\mathbf{v} = (I - 2\mathbf{u}\mathbf{u}^T)\mathbf{v} = \mathbf{v} - 2\mathbf{u}\mathbf{u}^T\mathbf{v} = \mathbf{v}$ , provided that  $\mathbf{u}^T\mathbf{v} = 0$ .
- 35 Starting from  $\mathbf{A} = (1, -1, 0, 0)$ , the orthogonal (not orthonormal) vectors  $\mathbf{B} = (1, 1, -2, 0)$  and  $\mathbf{C} = (1, 1, 1, -3)$  and  $\mathbf{D} = (1, 1, 1, 1)$  are in the directions of  $\mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4$ . The 4 by 4 and 5 by 5 matrices with *integer orthogonal columns* (not orthogonal rows, since not orthonormal  $Q$ !) are  $\begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & -3 & 1 \end{bmatrix}$  and
- $$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & 1 & 1 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & 0 & 0 & -4 & 1 \end{bmatrix}$$

- 36**  $[Q, R] = \mathbf{qr}(A)$  produces from  $A$  ( $m$  by  $n$  of rank  $n$ ) a “full-size” square  $Q = [Q_1 \ Q_2]$  and  $\begin{bmatrix} R \\ 0 \end{bmatrix}$ . The columns of  $Q_1$  are the orthonormal basis from Gram-Schmidt of the column space of  $A$ . The  $m - n$  columns of  $Q_2$  are an orthonormal basis for the left nullspace of  $A$ . Together the columns of  $Q = [Q_1 \ Q_2]$  are an orthonormal basis for  $\mathbf{R}^m$ .
- 37** This question describes the next  $\mathbf{q}_{n+1}$  in Gram-Schmidt using the matrix  $Q$  with the columns  $\mathbf{q}_1, \dots, \mathbf{q}_n$  (instead of using those  $\mathbf{q}$ 's separately). Start from  $\mathbf{a}$ , subtract its projection  $\mathbf{p} = Q^T \mathbf{a}$  onto the earlier  $\mathbf{q}$ 's, divide by the length of  $\mathbf{e} = \mathbf{a} - Q^T \mathbf{a}$  to get  $\mathbf{q}_{n+1} = \mathbf{e} / \|\mathbf{e}\|$ .

### Problem Set 5.1, page 251

- 1**  $\det(2A) = 8$ ;  $\det(-A) = (-1)^4 \det A = \frac{1}{2}$ ;  $\det(A^2) = \frac{1}{4}$ ;  $\det(A^{-1}) = 2 = \det(A^T)^{-1}$ .
- 2**  $\det(\frac{1}{2}A) = (\frac{1}{2})^3 \det A = -\frac{1}{8}$  and  $\det(-A) = (-1)^3 \det A = 1$ ;  $\det(A^2) = 1$ ;  $\det(A^{-1}) = -1$ .
- 3** (a) *False*:  $\det(I + I)$  is not  $1 + 1$  (b) *True*: The product rule extends to  $ABC$  (use it twice) (c) *False*:  $\det(4A)$  is  $4^n \det A$  (d) *False*:  $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ ,  $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ ,  
 $AB - BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$  is invertible.
- 4** Exchange rows 1 and 3 to show  $|J_3| = -1$ . Exchange rows 1 and 4, then 2 and 3 to show  $|J_4| = 1$ .
- 5**  $|J_5| = 1$ ,  $|J_6| = -1$ ,  $|J_7| = -1$ . Determinants 1, 1, -1, -1 repeat so  $|J_{101}| = 1$ .
- 6** To prove Rule 6, multiply the zero row by  $t = 2$ . The determinant is multiplied by 2 (Rule 3) but the matrix is the same. So  $2 \det(A) = \det(A)$  and  $\det(A) = 0$ .
- 7**  $\det(Q) = 1$  for rotation and  $\det(Q) = -1$  for reflection ( $1 - 2 \sin^2 \theta - 2 \cos^2 \theta = -1$ ).
- 8**  $Q^T Q = I \Rightarrow |Q|^2 = 1 \Rightarrow |Q| = \pm 1$ ;  $Q^n$  stays orthogonal so  $\det$  can't blow up.
- 9**  $\det A = 1$  from two row exchanges.  $\det B = 2$  (subtract rows 1 and 2 from row 3, then columns 1 and 2 from column 3).  $\det C = 0$  (equal rows) even though  $C = A + B$ !
- 10** If the entries in every row add to zero, then  $(1, 1, \dots, 1)$  is in the nullspace: singular  $A$  has  $\det = 0$ . (The columns add to the zero column so they are linearly dependent.) If every row adds to one, then rows of  $A - I$  add to zero (not necessarily  $\det A = 1$ ).
- 11**  $CD = -DC \Rightarrow \det CD = (-1)^n \det DC$  and *not*  $-\det DC$ . If  $n$  is even we can have an invertible  $CD$ .
- 12**  $\det(A^{-1})$  divides twice by  $ad - bc$  (once for each row). This gives  $\frac{ad-bc}{(ad-bc)^2} = \frac{1}{ad-bc}$ .
- 13** Pivots 1, 1, 1 give determinant = 1; pivots 1, -2, -3/2 give determinant = 3.
- 14**  $\det(A) = 36$  and the 4 by 4 second difference matrix has  $\det = 5$ .
- 15** The first determinant is 0, the second is  $1 - 2t^2 + t^4 = (1 - t^2)^2$ .

- 16** A singular rank one matrix has determinant = 0. The skew-symmetric  $K$  also  $\det K = 0$  (see #17).
- 17** Any 3 by 3 skew-symmetric  $K$  has  $\det(K^T) = \det(-K) = (-1)^3 \det(K)$ . This is  $-\det(K)$ . But always  $\det(K^T) = \det(K)$ . So we must have  $\det(K) = 0$  for 3 by 3.
- 18** 
$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = \begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix}$$
 (to reach 2 by 2, eliminate  $a$  and  $a^2$  in row 1 by column operations). Factor out  $b-a$  and  $c-a$  from the 2 by 2:  $(b-a)(c-a) \begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix} = (b-a)(c-a)(c-b)$ .
- 19** For triangular matrices, just multiply the diagonal entries:  $\det(U) = 6$ ,  $\det(U^{-1}) = \frac{1}{6}$ , and  $\det(U^2) = 36$ . 2 by 2 matrix:  $\det(U) = ad$ ,  $\det(U^2) = a^2 d^2$ . If  $ad \neq 0$  then  $\det(U^{-1}) = 1/ad$ .
- 20**  $\det \begin{bmatrix} a-Lc & b-Ld \\ c-la & d-lb \end{bmatrix}$  reduces to  $(ad-bc)(1-L\ell)$ . The determinant changes if you do two row operations at once.
- 21** Rules 5 and 3 give Rule 2. (Since Rules 4 and 3 give 5, they also give Rule 2.)
- 22**  $\det(A) = 3$ ,  $\det(A^{-1}) = \frac{1}{3}$ ,  $\det(A - \lambda I) = \lambda^2 - 4\lambda + 3$ . The numbers  $\lambda = 1$  and  $\lambda = 3$  give  $\det(A - \lambda I) = 0$ . *Note to instructor:* If you discuss this exercise, you can explain that this is the reason determinants come before eigenvalues. Identify  $\lambda = 1$  and  $\lambda = 3$  as the eigenvalues of  $A$ .
- 23**  $\det(A) = 10$ ,  $A^2 = \begin{bmatrix} 18 & 7 \\ 14 & 11 \end{bmatrix}$ ,  $\det(A^2) = 100$ ,  $A^{-1} = \frac{1}{10} \begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix}$  has  $\det \frac{1}{10}$ .  $\det(A - \lambda I) = \lambda^2 - 7\lambda + 10 = 0$  when  $\lambda = 2$  or  $\lambda = 5$ ; those are eigenvalues.
- 24** Here  $A = LU$  with  $\det(L) = 1$  and  $\det(U) = -6$  product of pivots, so also  $\det(A) = -6$ .  $\det(U^{-1}L^{-1}) = -\frac{1}{6} = 1/\det(A)$  and  $\det(U^{-1}L^{-1}A)$  is  $\det I = 1$ .
- 25** When the  $i, j$  entry is  $ij$ , row 2 = 2 times row 1 so  $\det A = 0$ .
- 26** When the  $ij$  entry is  $i + j$ , row 3 - row 2 = row 2 - row 1 so  $A$  is singular:  $\det A = 0$ .
- 27**  $\det A = abc$ ,  $\det B = -abcd$ ,  $\det C = a(b-a)(c-b)$  by doing elimination.
- 28** (a) *True*:  $\det(AB) = \det(A)\det(B) = 0$  (b) *False*: A row exchange gives  $-\det =$  product of pivots. (c) *False*:  $A = 2I$  and  $B = I$  have  $A - B = I$  but the determinants have  $2^n - 1 \neq 1$  (d) *True*:  $\det(AB) = \det(A)\det(B) = \det(BA)$ .
- 29**  $A$  is rectangular so  $\det(A^T A) \neq (\det A^T)(\det A)$ : these determinants are not defined.
- 30** Derivatives of  $f = \ln(ad - bc)$ : 
$$\begin{bmatrix} \partial f / \partial a & \partial f / \partial c \\ \partial f / \partial b & \partial f / \partial d \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = A^{-1}.$$
- 31** The Hilbert determinants are  $1, 8 \times 10^{-2}, 4.6 \times 10^{-4}, 1.6 \times 10^{-7}, 3.7 \times 10^{-12}, 5.4 \times 10^{-18}, 4.8 \times 10^{-25}, 2.7 \times 10^{-33}, 9.7 \times 10^{-43}, 2.2 \times 10^{-53}$ . Pivots are ratios of determinants so the 10th pivot is near  $10^{-10}$ . The Hilbert matrix is numerically difficult (*ill-conditioned*).

- 32** Typical determinants of  $\text{rand}(n)$  are  $10^6, 10^{25}, 10^{79}, 10^{218}$  for  $n = 50, 100, 200, 400$ .  $\text{randn}(n)$  with normal distribution gives  $10^{31}, 10^{78}, 10^{186}$ , Inf which means  $\geq 2^{1024}$ . MATLAB allows  $1.999999999999999 \times 2^{1023} \approx 1.8 \times 10^{308}$  but one more 9 gives Inf!
- 33** I now know that maximizing the determinant for  $1, -1$  matrices is **Hadamard's problem** (1893): see Brenner in American Math. Monthly volume 79 (1972) 626-630. Neil Sloane's wonderful On-Line Encyclopedia of Integer Sequences ([research.att.com/~njas](http://research.att.com/~njas)) includes the solution for small  $n$  (and more references) when the problem is changed to  $0, 1$  matrices. That sequence A003432 starts from  $n = 0$  with 1, 1, 1, 2, 3, 5, 9. Then the  $1, -1$  maximum for size  $n$  is  $2^{n-1}$  times the  $0, 1$  maximum for size  $n-1$  (so  $(32)(5) = 160$  for  $n = 6$  in sequence **A003433**).

To reduce the  $1, -1$  problem from 6 by 6 to the  $0, 1$  problem for 5 by 5, multiply the six rows by  $\pm 1$  to put  $+1$  in column 1. Then subtract row 1 from rows 2 to 6 to get a 5 by 5 submatrix  $S$  of  $-2, 0$  and divide  $S$  by  $-2$ .

Here is an advanced MATLAB code and a  $1, -1$  matrix with largest  $\det A = 48$  for  $n = 5$ :

```
n = 5; p = (n - 1)^2; A0 = ones(n); maxdet = 0;
for k = 0 : 2^p - 1
    Asub = rem(floor(k. * 2.^(-p + 1 : 0)), 2); A = A0; A(2 : n, 2 : n) = 1 - 2*
    reshape(Asub, n - 1, n - 1);
    if abs(det(A)) > maxdet, maxdet = abs(det(A)); maxA = A;
end
end
```

```
Output: maxA =   1   1   1   1   1   maxdet = 48.
                1   1   1  -1  -1
                1   1  -1   1  -1
                1  -1   1   1  -1
                1  -1  -1  -1   1
```

- 34** Reduce  $B$  by row operations to [row 3; row 2; row 1]. Then  $\det B = -6$  (odd permutation).

## Problem Set 5.2, page 263

- $\det A = 1 + 18 + 12 - 9 - 4 - 6 = 12$ , rows are independent;  $\det B = 0$ , row 1 + row 2 = row 3;  $\det C = -1$ , independent rows ( $\det C$  has one term, odd permutation)
- $\det A = -2$ , independent;  $\det B = 0$ , dependent;  $\det C = -1$ , independent.
- All cofactors of row 1 are zero.  $A$  has rank  $\leq 2$ . Each of the 6 terms in  $\det A$  is zero. Column 2 has no pivot.
- $a_{11}a_{23}a_{32}a_{44}$  gives  $-1$ , because  $2 \leftrightarrow 3$ ,  $a_{14}a_{23}a_{32}a_{41}$  gives  $+1$ ,  $\det A = 1 - 1 = 0$ ;  $\det B = 2 \cdot 4 \cdot 4 \cdot 2 - 1 \cdot 4 \cdot 4 \cdot 1 = 64 - 16 = 48$ .
- Four zeros in the same row guarantee  $\det = 0$ .  $A = I$  has 12 zeros (maximum with  $\det \neq 0$ ).
- (a) If  $a_{11} = a_{22} = a_{33} = 0$  then 4 terms are sure zeros (b) 15 terms must be zero.

- 7  $5!/2 = 60$  permutation matrices have  $\det = +1$ . Move row 5 of  $I$  to the top; starting from  $(5, 1, 2, 3, 4)$  elimination will do four row exchanges.
- 8 Some term  $a_{1\alpha}a_{2\beta} \cdots a_{n\omega}$  in the big formula is not zero! Move rows  $1, 2, \dots, n$  into rows  $\alpha, \beta, \dots, \omega$ . Then these nonzero  $a$ 's will be on the main diagonal.
- 9 To get  $+1$  for the even permutations, the matrix needs an *even* number of  $-1$ 's. To get  $+1$  for the odd  $P$ 's, the matrix needs an *odd* number of  $-1$ 's. So all six terms  $= +1$  in the big formula and  $\det = 6$  are impossible:  $\max(\det) = 4$ .
- 10 The  $4!/2 = 12$  even permutations are  $(1, 2, 3, 4), (2, 1, 4, 3), (3, 1, 4, 2), (4, 3, 2, 1)$ , and 8  $P$ 's with one number in place and even permutation of the other three numbers.  $\det(I + P_{\text{even}}) = 16$  or  $4$  or  $0$  (16 comes from  $I + I$ ).
- 11  $C = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ .  $D = \begin{bmatrix} 0 & 42 & -35 \\ 0 & -21 & 14 \\ -3 & 6 & -3 \end{bmatrix}$ .  $\det B = 1(0) + 2(42) + 3(-35) = -21$ . Puzzle:  $\det D = 441 = (-21)^2$ . Why?
- 12  $C = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$  and  $AC^T = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ . Therefore  $A^{-1} = \frac{1}{4}C^T = C^T / \det A$ .
- 13 (a)  $C_1 = 0, C_2 = -1, C_3 = 0, C_4 = 1$  (b)  $C_n = -C_{n-2}$  by cofactors of row 1 then cofactors of column 1. Therefore  $C_{10} = -C_8 = C_6 = -C_4 = C_2 = -1$ .
- 14 We must choose 1's from column 2 then column 1, column 4 then column 3, and so on. Therefore  $n$  must be even to have  $\det A_n \neq 0$ . The number of row exchanges is  $n/2$  so  $C_n = (-1)^{n/2}$ .
- 15 The 1, 1 cofactor of the  $n$  by  $n$  matrix is  $E_{n-1}$ . The 1, 2 cofactor has a single 1 in its first column, with cofactor  $E_{n-2}$ : sign gives  $-E_{n-2}$ . So  $E_n = E_{n-1} - E_{n-2}$ . Then  $E_1$  to  $E_6$  is  $1, 0, -1, -1, 0, 1$  and this cycle of six will repeat:  $E_{100} = E_4 = -1$ .
- 16 The 1, 1 cofactor of the  $n$  by  $n$  matrix is  $F_{n-1}$ . The 1, 2 cofactor has a 1 in column 1, with cofactor  $F_{n-2}$ . Multiply by  $(-1)^{1+2}$  and also  $(-1)$  from the 1, 2 entry to find  $F_n = F_{n-1} + F_{n-2}$  (so these determinants are Fibonacci numbers).
- 17  $|B_4| = 2 \det \begin{bmatrix} 1 & -1 \\ -1 & 2 & -1 \\ -1 & 2 \end{bmatrix} + \det \begin{bmatrix} 1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 \end{bmatrix} = 2|B_3| - \det \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = 2|B_3| - |B_2|$ .  $|B_3|$  and  $-|B_2|$  are cofactors of row 4 of  $B_4$ .
- 18 Rule 3 (linearity in row 1) gives  $|B_n| = |A_n| - |A_{n-1}| = (n+1) - n = 1$ .
- 19 Since  $x, x^2, x^3$  are all in the same row, they are never multiplied in  $\det V_4$ . The determinant is zero at  $x = a$  or  $b$  or  $c$ , so  $\det V$  has factors  $(x-a)(x-b)(x-c)$ . Multiply by the cofactor  $V_3$ . The Vandermonde matrix  $V_{ij} = (x_j)^{j-1}$  is for fitting a polynomial  $p(x) = b$  at the points  $x_i$ . It has  $\det V = \text{product of all } x_k - x_m \text{ for } k > m$ .
- 20  $G_2 = -1, G_3 = 2, G_4 = -3$ , and  $G_n = (-1)^{n-1}(n-1) = (\text{product of the } \lambda\text{'s})$ .
- 21  $S_1 = 3, S_2 = 8, S_3 = 21$ . The rule looks like every second number in Fibonacci's sequence  $\dots 3, 5, 8, 13, 21, 34, 55, \dots$  so the guess is  $S_4 = 55$ . Following the solution to Problem 30 with 3's instead of 2's confirms  $S_4 = 81 + 1 - 9 - 9 - 9 = 55$ . Problem 33 directly proves  $S_n = F_{2n+2}$ .
- 22 Changing 3 to 2 in the corner reduces the determinant  $F_{2n+2}$  by 1 times the cofactor of that corner entry. This cofactor is the determinant of  $S_{n-1}$  (one size smaller) which is  $F_{2n}$ . Therefore changing 3 to 2 changes the determinant to  $F_{2n+2} - F_{2n}$  which is  $F_{2n+1}$ .

- 23** (a) If we choose an entry from  $B$  we must choose an entry from the zero block; result zero. This leaves entries from  $A$  times entries from  $D$  leading to  $(\det A)(\det D)$   
 (b) and (c) Take  $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ ,  $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ ,  $C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ ,  $D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ . See #25.
- 24** (a) All  $L$ 's have  $\det = 1$ ;  $\det U_k = \det A_k = 2, 6, -6$  for  $k = 1, 2, 3$  (b) Pivots  $2, \frac{3}{2}, \frac{-1}{3}$ .
- 25** Problem 23 gives  $\det \begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} = 1$  and  $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = |A|$  times  $|D - CA^{-1}B|$  which is  $|AD - ACA^{-1}B|$ . If  $AC = CA$  this is  $|AD - CAA^{-1}B| = \det(AD - CB)$ .
- 26** If  $A$  is a row and  $B$  is a column then  $\det M = \det AB = \text{dot product of } A \text{ and } B$ . If  $A$  is a column and  $B$  is a row then  $AB$  has rank 1 and  $\det M = \det AB = 0$  (unless  $m = n = 1$ ). This block matrix is invertible when  $AB$  is invertible which certainly requires  $m \leq n$ .
- 27** (a)  $\det A = a_{11}C_{11} + \dots + a_{1n}C_{1n}$ . Derivative with respect to  $a_{11} = \text{cofactor } C_{11}$ .
- 28** Row 1  $- 2$  row 2  $+ 3$  row 3  $= 0$  so this matrix is singular.
- 29** There are five nonzero products, all 1's with a plus or minus sign. Here are the (row, column) numbers and the signs:  $+(1, 1)(2, 2)(3, 3)(4, 4) + (1, 2)(2, 1)(3, 4)(4, 3) - (1, 2)(2, 1)(3, 3)(4, 4) - (1, 1)(2, 2)(3, 4)(4, 3) - (1, 1)(2, 3)(3, 2)(4, 4)$ . Total  $-1$ .
- 30** The 5 products in solution 29 change to  $16 + 1 - 4 - 4 - 4$  since  $A$  has 2's and -1's:  
 $(2)(2)(2)(2) + (-1)(-1)(-1)(-1) - (-1)(-1)(2)(2) - (2)(2)(-1)(-1) - (2)(-1)(-1)(2)$ .
- 31**  $\det P = -1$  because the cofactor of  $P_{14} = 1$  in row one has sign  $(-1)^{1+4}$ . The big formula for  $\det P$  has only one term  $(1 \cdot 1 \cdot 1 \cdot 1)$  with minus sign because three exchanges take 4, 1, 2, 3 into 1, 2, 3, 4;  $\det(P^2) = (\det P)(\det P) = +1$  so  $\det \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  is *not right*.
- 32** The problem is to show that  $F_{2n+2} = 3F_{2n} - F_{2n-2}$ . Keep using Fibonacci's rule:  
 $F_{2n+2} = F_{2n+1} + F_{2n} = F_{2n} + F_{2n-1} + F_{2n} = 2F_{2n} + (F_{2n} - F_{2n-2}) = 3F_{2n} - F_{2n-2}$ .
- 33** The difference from 20 to 19 multiplies its 3 by 3 cofactor  $= 1$ : then  $\det$  drops by 1.
- 34** (a) The last three rows must be dependent (b) In each of the 120 terms: Choices from the last 3 rows must use 3 columns; at least one of those choices will be zero.
- 35** Subtracting 1 from the  $n, n$  entry subtracts its cofactor  $C_{nn}$  from the determinant. That cofactor is  $C_{nn} = 1$  (smaller Pascal matrix). Subtracting 1 from 1 leaves 0.

### Problem Set 5.3, page 279

- 1** (a)  $\begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix} = 3$ ,  $\begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix} = 6$ ,  $\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$  so  $x_1 = -6/3 = -2$  and  $x_2 = 3/3 = 1$  (b)  $|A| = 4$ ,  $|B_1| = 3$ ,  $|B_2| = 2$ ,  $|B_3| = 1$ . Therefore  $x_1 = 3/4$  and  $x_2 = -1/2$  and  $x_3 = 1/4$ .

- 2 (a)  $y = \begin{vmatrix} a & 1 \\ c & 0 \end{vmatrix} / \begin{vmatrix} a & b \\ c & d \end{vmatrix} = c/(ad - bc)$  (b)  $y = \det B_2 / \det A = (fg - id)/D$ .
- 3 (a)  $x_1 = 3/0$  and  $x_2 = -2/0$ : no solution (b)  $x_1 = x_2 = \mathbf{0}/\mathbf{0}$ : undetermined.
- 4 (a)  $x_1 = \det([\mathbf{b} \ \mathbf{a}_2 \ \mathbf{a}_3]) / \det A$ , if  $\det A \neq 0$  (b) The determinant is linear in its first column so  $x_1|\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3| + x_2|\mathbf{a}_2 \ \mathbf{a}_2 \ \mathbf{a}_3| + x_3|\mathbf{a}_3 \ \mathbf{a}_2 \ \mathbf{a}_3|$ . The last two determinants are zero because of repeated columns, leaving  $x_1|\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3|$  which is  $x_1 \det A$ .
- 5 If the first column in  $A$  is also the right side  $\mathbf{b}$  then  $\det A = \det B_1$ . Both  $B_2$  and  $B_3$  are singular since a column is repeated. Therefore  $x_1 = |B_1|/|A| = 1$  and  $x_2 = x_3 = 0$ .
- 6 (a)  $\begin{bmatrix} 1 & -\frac{2}{3} & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & -\frac{7}{3} & 1 \end{bmatrix}$  (b)  $\frac{1}{4} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ . An invertible symmetric matrix has a symmetric inverse.
- 7 If all cofactors = 0 then  $A^{-1}$  would be the zero matrix if it existed; cannot exist. (And the cofactor formula gives  $\det A = 0$ .)  $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$  has no zero cofactors but it is not invertible.
- 8  $C = \begin{bmatrix} 6 & -3 & 0 \\ 3 & 1 & -1 \\ -6 & 2 & 1 \end{bmatrix}$  and  $AC^T = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ . This is  $(\det A)I$  and  $\det A = 3$ . The 1, 3 cofactor of  $A$  is 0. Multiplying by 4 or 100: no change.
- 9 If we know the cofactors and  $\det A = 1$ , then  $C^T = A^{-1}$  and also  $\det A^{-1} = 1$ . Now  $A$  is the inverse of  $C^T$ , so  $A$  can be found from the cofactor matrix for  $C$ .
- 10 Take the determinant of  $AC^T = (\det A)I$ . The left side gives  $\det AC^T = (\det A)(\det C)$  while the right side gives  $(\det A)^n$ . Divide by  $\det A$  to reach  $\det C = (\det A)^{n-1}$ .
- 11 The cofactors of  $A$  are integers. Division by  $\det A = \pm 1$  gives integer entries in  $A^{-1}$ .
- 12 Both  $\det A$  and  $\det A^{-1}$  are integers since the matrices contain only integers. But  $\det A^{-1} = 1/\det A$  so  $\det A$  must be 1 or  $-1$ .
- 13  $A = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$  has cofactor matrix  $C = \begin{bmatrix} -1 & 2 & 1 \\ 3 & -6 & 2 \\ 1 & 3 & -1 \end{bmatrix}$  and  $A^{-1} = \frac{1}{5}C^T$ .
- 14 (a) Lower triangular  $L$  has cofactors  $C_{21} = C_{31} = C_{32} = 0$  (b)  $C_{12} = C_{21}$ ,  $C_{31} = C_{13}$ ,  $C_{32} = C_{23}$  make  $S^{-1}$  symmetric. (c) Orthogonal  $Q$  has cofactor matrix  $C = (\det Q)(Q^{-1})^T = \pm Q$  also orthogonal. Note  $\det Q = 1$  or  $-1$ .
- 15 For  $n = 5$ ,  $C$  contains 25 cofactors and each 4 by 4 cofactor has 24 terms. Each term needs 3 multiplications: total 1800 multiplications vs. 125 for Gauss-Jordan.
- 16 (a) Area  $|\begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix}| = 10$  (b) and (c) Area  $10/2 = 5$ , these triangles are half of the parallelogram in (a).
- 17 Volume =  $|\begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{vmatrix}| = 20$ . Area of faces = length of cross product =  $|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix}| = \frac{-2\mathbf{i} - 2\mathbf{j} + 8\mathbf{k}}{\text{length} = 6\sqrt{2}}$
- 18 (a) Area  $\frac{1}{2} |\begin{vmatrix} 2 & 1 & 1 \\ 3 & 4 & 1 \\ 0 & 5 & 1 \end{vmatrix}| = 5$  (b)  $5 +$  new triangle area  $\frac{1}{2} |\begin{vmatrix} 2 & 1 & 1 \\ 0 & 5 & 1 \\ -1 & 0 & 1 \end{vmatrix}| = 5 + 7 = 12$ .
- 19  $|\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix}| = 4 = |\begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix}|$  because the transpose has the same determinant. See #22.

- 20** The edges of the hypercube have length  $\sqrt{1+1+1+1} = 2$ . The volume  $\det H$  is  $2^4 = 16$ . ( $H/2$  has orthonormal columns. Then  $\det(H/2) = 1$  leads again to  $\det H = 16$ .)
- 21** The maximum volume  $L_1 L_2 L_3 L_4$  is reached when the edges are orthogonal in  $\mathbf{R}^4$ . With entries 1 and  $-1$  all lengths are  $\sqrt{4} = 2$ . The maximum determinant is  $2^4 = 16$ , achieved in Problem 20. For a 3 by 3 matrix,  $\det A = (\sqrt{3})^3$  can't be achieved by  $\pm 1$ .
- 22** This question is still waiting for a solution! An 18.06 student showed me how to transform the parallelogram for  $A$  to the parallelogram for  $A^T$ , without changing its area. (Edges slide along themselves, so no change in baselength or height or area.)
- 23**  $A^T A = \begin{bmatrix} \mathbf{a}^T \\ \mathbf{b}^T \\ \mathbf{c}^T \end{bmatrix} [\mathbf{a} \ \mathbf{b} \ \mathbf{c}] = \begin{bmatrix} \mathbf{a}^T \mathbf{a} & 0 & 0 \\ 0 & \mathbf{b}^T \mathbf{b} & 0 \\ 0 & 0 & \mathbf{c}^T \mathbf{c} \end{bmatrix}$  has  $\det A^T A = (\|\mathbf{a}\| \|\mathbf{b}\| \|\mathbf{c}\|)^2$   
 $\det A = \pm \|\mathbf{a}\| \|\mathbf{b}\| \|\mathbf{c}\|$
- 24** The box has height 4 and volume  $= \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix} = 4$ .  $\mathbf{i} \times \mathbf{j} = \mathbf{k}$  and  $(\mathbf{k} \cdot \mathbf{w}) = 4$ .
- 25** The  $n$ -dimensional cube has  $2^n$  corners,  $n2^{n-1}$  edges and  $2n(n-1)$ -dimensional faces. Coefficients from  $(2+x)^n$  in Worked Example 2.4A. Cube from  $2I$  has volume  $2^n$ .
- 26** The pyramid has volume  $\frac{1}{6}$ . The 4-dimensional pyramid has volume  $\frac{1}{24}$  (and  $\frac{1}{n!}$  in  $\mathbf{R}^n$ )
- 27**  $x = r \cos \theta$ ,  $y = r \sin \theta$  give  $J = r$ . The columns are orthogonal and their lengths are 1 and  $r$ .
- 28**  $J = \begin{vmatrix} \sin \varphi \cos \theta & \rho \cos \varphi \sin \theta & -\rho \sin \varphi \sin \theta \\ \sin \varphi \sin \theta & \rho \cos \varphi \cos \theta & \rho \sin \varphi \cos \theta \\ \cos \varphi & -\rho \sin \varphi & \theta \end{vmatrix} = \rho^2 \sin \varphi$ . This Jacobian is needed for triple integrals inside spheres.
- 29** From  $x, y$  to  $r, \theta$ :  $\begin{vmatrix} \partial r / \partial x & \partial r / \partial y \\ \partial \theta / \partial x & \partial \theta / \partial y \end{vmatrix} = \begin{vmatrix} x/r & y/r \\ -y/r^2 & x/r^2 \end{vmatrix} = \begin{vmatrix} \cos \theta & \sin \theta \\ (-\sin \theta)/r & (\cos \theta)/r \end{vmatrix}$   
 $= \frac{1}{r} = \frac{1}{\text{Jacobian in 27}}$ .
- 30** The triangle with corners  $(0, 0)$ ,  $(6, 0)$ ,  $(1, 4)$  has area 24. Rotated by  $\theta = 60^\circ$  the area is *unchanged*. The determinant of the rotation matrix is  $J = \begin{vmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{vmatrix} = \begin{vmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{vmatrix} = 1$ .
- 31** Base area 10, height 2, volume 20.
- 32** The volume of the box is  $\det \begin{bmatrix} 2 & 4 & 0 \\ -1 & 3 & 0 \\ 1 & 2 & 2 \end{bmatrix} = 20$ .
- 33**  $\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$ . This is  $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ .
- 34**  $(\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ : Even permutation of  $(\mathbf{u}, \mathbf{v}, \mathbf{w})$  keeps the same determinant. Odd permutations reverse the sign.

- 35  $S = (2, 1, -1)$ , area  $\|PQ \times PS\| = \|(-2, -2, -1)\| = 3$ . The other four corners can be  $(0, 0, 0)$ ,  $(0, 0, 2)$ ,  $(1, 2, 2)$ ,  $(1, 1, 0)$ . The volume of the tilted box is  $|\det| = 1$ .
- 36 If  $(1, 1, 0)$ ,  $(1, 2, 1)$ ,  $(x, y, z)$  are in a plane the volume is  $\det \begin{bmatrix} x & y & z \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} = x - y + z = 0$ . The “box” with those edges is flattened to zero height.
- 37  $\det \begin{bmatrix} x & y & z \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix} = 7x - 5y + z$  will be zero when  $(x, y, z)$  is a combination of  $(2, 3, 1)$  and  $(1, 2, 3)$ . The plane containing those two vectors has equation  $7x - 5y + z = 0$ .
- 38 Doubling each row multiplies the volume by  $2^n$ . Then  $2 \det A = \det(2A)$  only if  $n = 1$ .
- 39  $AC^T = (\det A)I$  gives  $(\det A)(\det C) = (\det A)^n$ . Then  $\det A = (\det C)^{1/3}$  with  $n = 4$ . With  $\det A^{-1} = 1/\det A$ , construct  $A^{-1}$  using the cofactors. *Invert to find A.*
- 40 The cofactor formula adds 1 by 1 determinants (which are just entries) times their cofactors of size  $n - 1$ . Jacobi discovered that this formula can be generalized. For  $n = 5$ , Jacobi multiplied each 2 by 2 determinant from rows 1-2 (with columns  $a < b$ ) times a 3 by 3 determinant from rows 3-5 (using the remaining columns  $c < d < e$ ).  
The key question is + or - sign (as for cofactors). The product is given a + sign when  $a, b, c, d, e$  is an even permutation of 1, 2, 3, 4, 5. This gives the correct determinant +1 for that permutation matrix. More than that, all other  $P$  that permute  $a, b$  and separately  $c, d, e$  will come out with the correct sign when the 2 by 2 determinant for columns  $a, b$  multiplies the 3 by 3 determinant for columns  $c, d, e$ .
- 41 The Cauchy-Binet formula gives the determinant of a square matrix  $AB$  (and  $AA^T$  in particular) when the factors  $A, B$  are rectangular. For (2 by 3) times (3 by 2) there are 3 products of 2 by 2 determinants from  $A$  and  $B$  (printed in boldface):

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} g & j \\ h & k \\ i & \ell \end{bmatrix} \quad \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} g & j \\ h & k \\ i & \ell \end{bmatrix} \quad \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} g & j \\ h & k \\ i & \ell \end{bmatrix}$$

$$\text{Check } A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 7 \end{bmatrix} \quad AB = \begin{bmatrix} 14 & 30 \\ 30 & 66 \end{bmatrix}$$

$$\text{Cauchy-Binet: } (4 - 2)(4 - 2) + (7 - 3)(7 - 3) + (14 - 12)(14 - 12) = \mathbf{24}$$

$$(14)(66) - (30)(30) = \mathbf{24}$$

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- The eigenvalues are 1 and 0.5 for  $A$ , 1 and 0.25 for  $A^2$ , 1 and 0 for  $A^\infty$ . Exchanging the rows of  $A$  changes the eigenvalues to 1 and  $-0.5$  (the trace is now  $0.2 + 0.3$ ). Singular matrices stay singular during elimination, so  $\lambda = 0$  does not change.
- $A$  has  $\lambda_1 = -1$  and  $\lambda_2 = 5$  with eigenvectors  $x_1 = (-2, 1)$  and  $x_2 = (1, 1)$ . The matrix  $A + I$  has the same eigenvectors, with eigenvalues increased by 1 to  $\mathbf{0}$  and  $\mathbf{6}$ . That zero eigenvalue correctly indicates that  $A + I$  is singular.
- $A$  has  $\lambda_1 = 2$  and  $\lambda_2 = -1$  (check trace and determinant) with  $x_1 = (1, 1)$  and  $x_2 = (2, -1)$ .  $A^{-1}$  has the same eigenvectors, with eigenvalues  $1/\lambda = \frac{1}{2}$  and  $-1$ .