Math 441 Exam 2

1. (2.5 pts each, total 15) Answer the following simple questions, and provide a brief explanation.

a) If the vectors in the set $S = \{\bar{v}_1, .., \bar{v}_k\}$ are linearly independent, what is a basis for their span, $span(S)$? What is the dimension of $span(S)$?

A basis for $span(S)$ is $S = \{\bar{v}_1, .., \bar{v}_k\}$ itself. The set spans $span(S)$ by definition and is linearly independent by assumption. This makes it a basis.

b) If the $m \times n$ matrix A has rank m, what - explicitly - is its column space, and what - explicitly - is a basis for its column space? Where is a basis of its row space?

$rank(A) = \dim C(A) = m$. The dimension of the column space is m, that means that the column space is all of R^m. You can give $\bar{e}_1, .., \bar{e}_m$, the standard basis vectors, as a basis of the column space. Likewise $rank(A) = \dim C(A^T) = m$ is the dimension of the row space, so the m rows of A are independent and form a basis of the row space of A.

c) If the columns of a matrix A are linearly independent, what does the row reduced echelon form of A look like?

There is a pivot in each column but the number of rows is undetermined (except that there are at least as many rows as columns), so the form of the matrix is

$$R = \begin{bmatrix} I \\ \dots \\ 0 \end{bmatrix}$$

d) If $rank(A) = n$, what can we say about solutions of linear systems $A\bar{x} = \bar{b}$ that have A as the matrix of coefficients? Similarly, what can we say about solutions of $A\bar{x} = \bar{b}$ if $rank(A) < n$.

If $rank(A) = n$, we know that $A\bar{x} = \bar{b}$ has a unique solution (if there is one). The book describes this as "zero or one solution" for each $\bar{b}$. If $rank(A) < n$ then no solution is unique (zero or infinite number of solutions)

e) Can you create a 5×5 matrix with 4 (but not 5) independent columns and 3 (but not 4) independent rows? Why or why not.

No, the number of independent rows=number of independent columns=rank(A)

f) If we have a set of column vectors $S = \{\bar{v}_1, .., \bar{v}_k\}$ that spans a subspace V, explain how we computationally go about selecting a subset of the vectors in S that is a basis for V.

Form a matrix A with the vectors $\bar{v}_1, .., \bar{v}_k$ as columns. Find the row echelon form of A. The pivot columns in A are the vectors in S that form a basis for V.

2. (15 pts) Given a matrix A and its reduced row echelon form R, answer the following questions:

$$A = \begin{bmatrix} 1 & 2 & -3 & 1 & 3 \\ 1 & 1 & -1 & 1 & 2 \\ 1 & -1 & 3 & 1 & 0 \\ 2 & -1 & 4 & 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R$$

a) (6 pts) Find a basis for each of the the spaces $C(A)$, $C(A^T)$ and $N(A)$.

(We write the vectors using horizontal notation below)
Basis for $C(A)$ are the pivot columns in A :
$\{\text{col}1, \text{col}2, \text{col}4\} = \left\{\begin{pmatrix} 1 & 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}\right\}$

Basis for $C(A^T)$ are the nonzero rows in R :
$\left\{\begin{pmatrix} 1 & 0 & 1 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -2 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 & 2 \end{pmatrix}\right\}$
Basis for $N(A)$ are the special solutions of $A\bar{x} = 0$ from solving $R\bar{x} = \bar{0}$:
$\left\{\begin{pmatrix} -1 & 2 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 0 & -2 & 1 \end{pmatrix}\right\}$

b)(6 pts) Express each column in A as a linear combination of the basis vectors in $C(A)$ and the first and last rows of A as a linear combination of the basis vectors in $C(A^T)$

Columns 1,2,4 appear in the basis. The dependencies/relationships among columns are the same as those in R :
$\text{col}3 = \text{col}1 - 2\,\text{col}2$, $\text{col}5 = (-1)\,\text{col}1 + \text{col}2 + (2)\,\text{col}4$
The linear combinations of the row space basis can be observed by looking in columns 1,2,4 as below:

$$\begin{aligned} \text{row}1 &= \begin{pmatrix} 1 & 2 & -3 & 1 & 3 \end{pmatrix} \\ &= (1)\begin{pmatrix} 1 & 0 & 1 & 0 & -1 \end{pmatrix} \\ &\quad +(2)\begin{pmatrix} 0 & 1 & -2 & 0 & 1 \end{pmatrix} \\ &\quad +(1)\begin{pmatrix} 0 & 0 & 0 & 1 & 2 \end{pmatrix} \end{aligned} \qquad \begin{aligned} \text{row}4 &= \begin{pmatrix} 2 & -1 & 4 & 1 & -1 \end{pmatrix} \\ &= (2)\begin{pmatrix} 1 & 0 & 1 & 0 & -1 \end{pmatrix} \\ &\quad -\begin{pmatrix} 0 & 1 & -2 & 0 & 1 \end{pmatrix} \\ &\quad +\begin{pmatrix} 0 & 0 & 0 & 1 & 2 \end{pmatrix} \end{aligned}$$

c) (3 pts) Find a vector in the null space of A with all nonzero components. Explain.
Any linear combination of our null space basis vectors is also in the null space, so we just take a combination that gives all nonzero components (that is, in general, not always possible, but it is in this case):
$2\begin{pmatrix} -1 & 2 & 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & -1 & 0 & -2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 3 & 2 & -2 & 1 \end{pmatrix}$

3. (12 pts) If $A = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 2 \\ 2 & -1 & 4 & 1 & -1 \end{bmatrix}$ find the most general solution of $A\bar{x} = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$.

We reduce to find the reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 1 & 2 & 2 \\ 2 & -1 & 4 & 1 & -1 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 & 1 & 2 \\ 0 & -1 & 2 & -1 & -3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 & -2 & -1 \end{bmatrix}$$

$\to \begin{bmatrix} 1 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{bmatrix}$. We can "read off" the solutions for $\bar{x} = \bar{x}_p + \bar{x}_n$

$$\bar{x} = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

4.(2.5 pts each, total 10 pts) If $A = \begin{bmatrix} 1 & -2 & 2 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$ answer the following without writing down any calculations.

a) What is the rank of A? Why?

The two rows are independent, so $rank(A) = 2$

b)Provide a basis of the column space of A and of the row space of A. Explain

The first two columns are independent and there can only be $rank(A) = 2$ independent columns at most, so the first two columns of A are a basis of $C(A)$. The two rows are a basis of the row space.

c) The vectors $(1,1,1,1)$ and $(2,1,-1,-2)$ are clearly in the null space of A. Argue why they must be a basis of the null space.

The dimension of the null space is $\dim(N(A)) = n - r = 4 - 2 = 2$ and we have two independent vectors in the null space supplied to us, so these must be a basis.

d) What vectors are in the left null space of A? Why?

The dimension of the left null space is $\dim(N(A^T)) = m - r = 2 - 2 = 0$ so $N(A^T)$ contains on the zero vector. (Or note that the only combination of rows that gives zero is the zero linear combination, since the rows are independent.)

5. (8 pts) Let A be an $m \times n$ matrix. Group together the statements below that are equivalent to each other.

a)$rank(A) = m$ b) $rank(A) = n$ c) $rank(A) < m$ d) $rank(A) < n$

e) $A\bar{x} = \bar{b}$ always has a solution f) $A\bar{x} = \bar{0}$ has only one solution

g) $A\bar{x} = \bar{b}$ either has no solutions, or an infinite number, but never only one.

h)The columns of A are independent i)The rows of A are independent

j) The left null space is only the zero vector. k) The null space is only the zero vector.

l) $C(A) = R^m$ m) $C(A^T) = R^n$

a) goes with e),i),j),l)
b) goes with f),h),k),m)
c) is alone
d) goes with g)