Math 441 Exam 4

Eigenvalues/eigenvectors:

1) I give you a matrix and you find the eigenvalues and eigenvectors

2) I tell you an eigenvalue and ask you to find the corresponding eigenvectors (basis of null space of $A - \lambda I$) Usually there is only one eigenvector, but if there are two, what does it tell you about the multiplicity of λ as a root of the characteristic polynomial (and what's that)?

Find the eigenvectors for $\lambda = 1$ and the matrix $A = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 1 & \cdots & 1 & 1 \end{bmatrix}$. Explain (Hint: What is the rank of $A - I$? How many eigenvectors are there? Why?)

$\lambda = 0, n$ are eigenvectors of the $n \times n$ matrix of all 1's. How do you know there are no other eigenvalues? (Hint: What are the eigenvectors)

3) Verify that a given vector is an eigenvector (how?)

4) Find the characteristic polynomial of a given matrix.

5) Explain why $AV = V\Lambda$ if V contains eigenvectors of A.

6) If I give you the eigenvectors, find a matrix S such that $S^{-1}AS = D$ diagonal. If $AS = SD$ why does S contain eigenvectors and D eigenvalues?

7) Calculate A^n for a diagonalizable matrix A. Show the connection between $A^n(c_1\bar{v}_1 + \ldots + c_n\bar{v}_n) = c_1\lambda_1^n\bar{v}_1 + \ldots + c_n\lambda_n^n\bar{v}_n$ and $A^n = V\Lambda^n V^{-1}$ where V is a matrix of eigenvectors
(Given $\bar{x}$, we have $\bar{x} = V\bar{c} = c_1\bar{v}_1 + \ldots + c_n\bar{v}_n$ if $\bar{c} = V^{-1}\bar{x}$. Next $A^k(c_1\bar{v}_1 + \ldots + c_n\bar{v}_n) = c_1\lambda_1^k\bar{v}_1 + \ldots + c_n\lambda_n^k\bar{v}_n = \begin{bmatrix} \lambda_1^k\bar{v}_1 & \lambda_n^k\bar{v}_n \end{bmatrix}\bar{c} = (V\Lambda^k)\bar{c} = V\Lambda^k V^{-1}\bar{x}$ so $A^k = V\Lambda^k V^{-1}$)

8) A^T and A have the same eigenvalues. Why?
($rank(A - \lambda I) = rank\left((A - \lambda I)^T\right) = rank(A^T - \lambda I)$ so the condition for an eigenvalue of A, $rank(A - \lambda I) < n$ - why? - applies if and only if $rank(A^T - \lambda I) < n$, i.e. λ is an eigenvalue of A^T)

9) For a symmetric matrix A, show that eigenvectors corresponding to different eigenvalues are orthogonal.

10) If A is similar to B then A^n is similar to B^n . Why?

11) Explain the decomposition of a symmetric matrix A as given in the text:
$A = \lambda_1 \bar{v}_1 \bar{v}_1^T + \ldots + \lambda_n \bar{v}_n \bar{v}_n^T$ where the $\bar{v}$ are the orthonormal eigenvectors of A.

(Method 1: $A = Q\Lambda Q^T$ so $Q\Lambda = \begin{bmatrix} \lambda_1 \bar{v}_1 & \lambda_2 \bar{v}_2 & \ldots & \lambda_n \bar{v}_n \end{bmatrix}$ and $Q^T = \begin{bmatrix} \bar{v}_1^T \\ \vdots \\ \bar{v}_n^T \end{bmatrix}$. so use block multiplication.)

(Method 2, more illuminating: Given $\bar{x}$, we have $\bar{x} = (\bar{v}_1^T \bar{x})\bar{v}_1 + \ldots + (\bar{v}_n^T \bar{x})\bar{v}_1$ by projecting $\bar{x}$ onto the orthonormal basis vectors. Now

$$A\bar{x} = A[(\bar{v}_1^T \bar{x})\bar{v}_1 + \ldots + (\bar{v}_n^T \bar{x})\bar{v}_1] = (\bar{v}_1^T \bar{x})\lambda_1 \bar{v}_1 + \ldots + (\bar{v}_n^T \bar{x})\lambda_n \bar{v}_n$$

$$= \lambda_1 \bar{v}_1 (\bar{v}_1^T \bar{x}) + \ldots + \lambda_n \bar{v}_n (\bar{v}_n^T \bar{x}) = \lambda_1 (\bar{v}_1 \bar{v}_1^T)\bar{x} + \ldots + \lambda_n (\bar{v}_n \bar{v}_n^T)\bar{x}$$ so $A = \lambda_1 \bar{v}_1 \bar{v}_1^T + \ldots + \lambda_n \bar{v}_n \bar{v}_n^T$)

12) Singular value decomposition: Given any $m \times n$ matrix A, to find an orthonormal set of vectors in R^n that is transformed by A into an orthogonal set of vectors in R^m we take the orthonormal eigenvectors of the symmetric matrix $A^T A$. Each eigenvector $\bar{v}$ corresponding to eigenvalue λ of $A^T A$ satisfies $\|A\bar{v}\|^2 = \lambda$ so $\|A\bar{v}\| = \sqrt{\lambda} = \sigma$, one of the singular values.

13) For a triangular matrix, the eigenvalues are precisely the entries on the diagonal. Why?