

Math 441 Exam 4

1. (12 pts) Given $A = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$

a) Find the eigenvalues and eigenvectors of A

$$\det \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix} = \lambda^2 - \lambda - 2 = 0, \lambda = -1, 2$$

$\lambda = -1$:

$$A - \lambda I = \begin{bmatrix} 6 & -6 \\ 3 & -3 \end{bmatrix}, \bar{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda = 2$:

$$A - \lambda I = \begin{bmatrix} 3 & -6 \\ 3 & -6 \end{bmatrix}, \bar{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

b) Write A in diagonalized form, $A = SDS^{-1}$ where D is a diagonal matrix. Then find a formula for A^k . As $k \rightarrow \infty$, every column of $\frac{1}{2^k}A^k$ is a multiple of what vector - what is the significance of this vector?

$$A = V\Lambda V^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$A^k = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (-1)^k & 0 \\ 0 & 2^k \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2^{k+1} + (-1)^{k+1} & -2^{k+1} + 2(-1)^k \\ 2^k + 2(-1)^{k+1} & -2^k + 4(-1)^k \end{bmatrix}$$

$\frac{1}{2^k}A^k = \begin{bmatrix} 2 + 2^{-k}(-1)^{k+1} & -2 + 2^{1-k}(-1)^k \\ 1 + 2^{1-k}(-1)^{k+1} & -1 + 2^{2-k}(-1)^k \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}$, each column is a multiple of the eigenvector with the largest eigenvalue, $\lambda = 2$.

You can also calculate:

$$\begin{aligned} \frac{1}{2^k}A^k &= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (-1)^k 2^{-k} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} \end{aligned}$$

2. (12 pts) If $A = \begin{bmatrix} 3 & 2 & -1 \\ 6 & 4 & -2 \\ -3 & -2 & 1 \end{bmatrix}$

The rank of A is 1 so one eigenvalue is $\lambda = \underline{0}$. (the null space of $A - 0I$ is not just $\vec{0}$)

There must be 1 / 2 / 3 eigenvectors corresponding to this value of λ because $\dim N(A) = 3 - 1 = \underline{2}$.

Those corresponding eigenvectors are: $\vec{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$ (any two independent vectors in the null space will do)

The characteristic polynomial must have a factor of λ^2 because $\lambda = 0$ has multiplicity at least two due to two independent eigenvectors for $\lambda = 0$.

The characteristic polynomial of A is $p(\lambda) = \lambda^2(8 - \lambda)$ (calculate below).

$$\det \begin{bmatrix} 3 - \lambda & 2 & -1 \\ 6 & 4 - \lambda & -2 \\ -3 & -2 & 1 - \lambda \end{bmatrix} = 8\lambda^2 - \lambda^3$$

The remaining eigenvalue is $\lambda = \underline{8}$ and the corresponding eigenvector is

$\vec{v}_3 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$. (Show your work below)

Add the second row to the first below to aid in calculations:

$$\begin{bmatrix} -5 & 2 & -1 \\ 6 & -4 & -2 \\ -3 & -2 & -7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & -3 \\ 6 & -4 & -2 \\ -3 & -2 & -7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & -3 \\ 0 & 8 & 16 \\ 0 & -8 & -16 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

3. (6 pts)

If I tell you that the eigenvectors of $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ are $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, how can you easily check? What eigenvalues correspond to these eigenvectors?

Take each proposed eigenvector and multiply by A :

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ so } \lambda_1 = 0$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ so } \lambda_2 = 1$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ so } \lambda_3 = 2$$

What eigenvalues correspond to these eigenvectors? Given that A is symmetric, it should have a complete set of orthonormal eigenvectors? What vectors (explicitly) are those? Write A in the form $A = Q\Lambda Q^T$ where Q is an orthogonal matrix.

Divide each eigenvector by its length and create an orthogonal matrix Q out of them:

$$A = Q\Lambda Q^T = \begin{bmatrix} -\frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \\ 0 & 1 & 0 \\ \frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -\frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \\ 0 & 1 & 0 \\ \frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \end{bmatrix}^T$$

4. (4 pts) Explain the following: If $A = SDS^{-1}$ where D is diagonal and S is some nonsingular matrix, then the eigenvectors of A are the columns of S and the eigenvalues of A are the diagonal entries of D .

In this problem you don't know anything about S, D , except that $A = SDS^{-1}$. Next $AS = SD$ follows by multiplying both sides on the right by S . If we express S in terms of its columns as $S = [\bar{v}_1 \ \bar{v}_2 \ \dots \ \bar{v}_n]$ and the diagonal entries of D as d_1, d_2, \dots, d_n then by matrix multiplication we can write:

$$AS = [A\bar{v}_1 \ A\bar{v}_2 \ \dots \ A\bar{v}_n] = [d_1\bar{v}_1 \ d_2\bar{v}_2 \ \dots \ d_n\bar{v}_n]$$

and matching up the columns, we have $A\bar{v}_1 = d_1\bar{v}_1, \dots, A\bar{v}_n = d_n\bar{v}_n$ so that the columns of S contain eigenvectors of A and the diagonal entries of D are the corresponding eigenvalues.

5. (8 pts) For the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$ compute orthonormal vectors \bar{v}_1, \bar{v}_2 such that the vectors $\bar{u}_1 = A\bar{v}_1$ and $\bar{u}_2 = A\bar{v}_2$ are orthogonal. The vectors \bar{v}_1, \bar{v}_2 are the eigenvectors of what matrix?

The vectors \bar{v}_1, \bar{v}_2 must be eigenvectors of $A^T A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$. These are easy compute.

$$\lambda_1 = 1, \bar{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \lambda_2 = 5, \bar{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\text{Now } \bar{u}_1 = A\bar{v}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \text{ and } \bar{u}_2 = A\bar{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \text{ which are orthogonal.}$$

We might note that, as developed during the SVD discussion,

$$\|\bar{u}_1\| = \sqrt{2} = \sqrt{\lambda_1} \|\bar{v}_1\| = \sigma_1 \|\bar{v}_1\|$$

$$\|\bar{u}_2\| = \sqrt{10} = \sqrt{5} \|\bar{v}_2\| = \sqrt{\lambda_2} \|\bar{v}_2\| = \sigma_2 \|\bar{v}_2\|$$

For orthonormal \bar{v}'_s we would of course take $\bar{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\bar{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with the \bar{u}'_s similarly multiplied by $\frac{1}{\sqrt{2}}$ as compared with their values above.