

MATH 251.009
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Spring 2013

SAMPLE FINAL TEST
(longer than the actual Final Test)

Solve the following exercises. **Show your work.**

Ex. 1. ST #1 Ex 3: Find the determinant of the matrix. Each time you expand the matrix, you must expand it over a row or column that has the largest number of zeros. If necessary, use the row (or column) reduction method to create additional zeros.

$$\begin{bmatrix} 1 & 2 & -11 & 1 \\ 0 & 0 & 2 & 0 \\ 7 & 1 & 0 & 2 \\ -2 & 0 & 9 & 0 \end{bmatrix}$$

Solution: Check your notes.

Ex. 2. ST #1 Ex 4: Find the inverse matrix of

$$\begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Solution: Check your notes.

Ex. 3. ST #1 part of Ex 6: Let $\mathbf{a} = \langle 0, 1, 2 \rangle$, $\mathbf{b} = \langle -1, 0, 7 \rangle$, and $\mathbf{c} = \langle 2, 3, -1 \rangle$. Evaluate $(\mathbf{a} \cdot \mathbf{b}) \cdot (\mathbf{b} \times \mathbf{c})$.

Solution: Check your notes.

Ex. 4. ST #2 Ex 1: Find a vector equation of the line that passes through the point $P(11, 13, -7)$ and is perpendicular to the plane with the equation: $x - 2z = 17$.

Solution: The direction vector \mathbf{v} of the line coincides with the normal vector of the plane: $\langle 1, 0, -2 \rangle$.

$$\text{Answer: } \langle x, y, z \rangle = \langle 11, 13, -7 \rangle + t\langle 1, 0, -2 \rangle, \text{ or } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ 13 \\ -7 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}.$$

Ex. 5. ST #2 Ex 7: Let $\mathbf{v}(t) = \mathbf{i}(t + e)^{-1} + \mathbf{k} t^3$ be a velocity of a particle. Find the acceleration vector $\mathbf{a}(t)$ of the particle and its position vector $\mathbf{r}(t)$, where its initial position was $\mathbf{r}(0) = 3\mathbf{i}$.

Solution: $\mathbf{a}(t) = \mathbf{v}'(t) = -(t + e)^{-2}\mathbf{i} + 3t^2\mathbf{k}$.

$\mathbf{r}(t) = \int \mathbf{v}(t) dt = \mathbf{i} \ln |t + e| + \mathbf{k} t^4/4 + \vec{C}$. To find \vec{C} , we calculate $\mathbf{r}(0)$:

$\mathbf{i} \ln |0 + e| + \mathbf{k} 0^4/4 + \vec{C} = 3\mathbf{i}$. Since $\ln e = 1$, we get $\mathbf{i} + \vec{C} = 3\mathbf{i}$ and $\vec{C} = 2\mathbf{i}$. Therefore

$\mathbf{r}(t) = \mathbf{i} \ln |t + e| + \mathbf{k} t^4/4 + 2\mathbf{i} = (2 + \ln |t + e|)\mathbf{i} + \frac{t^4}{4}\mathbf{k}$.

Answer: $\mathbf{a}(t) = -(t + e)^{-2}\mathbf{i} + 3t^2\mathbf{k}$ and $\mathbf{r}(t) = (2 + \ln |t + e|)\mathbf{i} + \frac{t^4}{4}\mathbf{k}$.

Ex. 6. ST #2 Ex 5(b): Describe and sketch the graph of the equation: $4y = x^2 + z^2$.

Solution: Answer: Circular paraboloid, revolving around the y -axis, opening towards the positive side of the y -axis. Sketch: check your notes.

Ex. 7. ST #3 Ex. 2: Compute the first order partial derivatives of $f(x, y, z) = ze^{x^2} \cos y$.

Solution:

$$\frac{\partial f}{\partial x} = f_x = ze^{x^2} \cos y \cdot 2x = 2xze^{x^2} \cos y$$

$$\frac{\partial f}{\partial y} = f_y = ze^{x^2} (-\sin y) = -ze^{x^2} \sin y$$

$$\frac{\partial f}{\partial z} = f_z = e^{x^2} \cos y$$

Ex. 8. ST #3 Ex. 3: Compute all second order partial derivatives of $g(s, t) = e^{5t} + t \sin(3s)$.

Solution:

$$g_s = 3t \cos(3s) \quad g_{ss} = -9t \sin(3s) \quad g_{st} = 3 \cos(3s)$$

$$g_t = 5e^{5t} + \sin(3s) \quad g_{ts} = 3 \cos(3s) \quad g_{tt} = 25e^{5t}$$

Ex. 9. ST #3 Ex. 4: Find an equation of the plane tangent to the surface $z = x^2 - 5y^3$ at the point $P(2, 1, -1)$.

Solution:

$$z_x = 2x; \quad z_x(P) = 2 \cdot 2 = 4;$$

$$z_y = -15y^2; \quad z_y(P) = -15 \cdot 1^2 = -15;$$

$$\text{Normal vector } \mathbf{n} = \langle z_x(P), z_y(P), -1 \rangle = \langle 4, -15, -1 \rangle.$$

$$\text{Answer: } 4(x - 2) - 15(y - 1) - 1(z + 1) = 0 \quad \text{or} \quad 4x - 15y - z + 6 = 0.$$

Ex. 10. ST #3 Ex. 5: Find the absolute maximum and the absolute minimum of the function $f(x, y) = x^3 - xy$ on the region bounded below by parabola $y = x^2 - 1$ and above by line $y = 3$. You will get credit **only** if **all** critical points are found.

Solution: The curves intersect, when $x^2 - 1 = 3$, that is, when $x = \pm 2$.

Thus, we need to consider the region above $x^2 - 1$ and below 3 for x in the interval $[-2, 2]$.

Region's interior: $f_x(x, y) = 3x^2 - y$ and $f_y(x, y) = -x$. This leads to system $3x^2 - y = 0$ and $-x = 0$, with only solution $(x, y) = (0, 0)$. This point belongs to the region. This is our first critical point.

Lower boundary: $y = x^2 - 1$ and $-2 \leq x \leq 2$. Then

$$g(x) = f(x, x^2 - 1) = x^3 - x(x^2 - 1) = x \text{ and } g'(x) = 1 \text{ is never } 0.$$

So, there are no true critical points, but we need to consider the endpoints of g , $x = \pm 2$.

This give us the critical points $(x, y) = (\pm 2, 3)$.

Upper boundary: $y = 3$ and $-2 \leq x \leq 2$. Then

$$g(x) = f(x, 3) = x^3 - 3x \text{ and } g'(x) = 3x^2 - 3, \text{ which is 0 when } x = \pm 1 \in [-2, 2].$$

This give us the critical points $(x, y) = (\pm 1, 3)$. (Plus the end points $(x, y) = (\pm 2, 3)$, considered above.)

Checking the critical points: $f(0, 0) = 0$;

$$f(2, 3) = 2^3 - 6 = 2; f(-2, 3) = (-2)^3 + 6 = -2;$$

$$f(1, 3) = 1^3 - 3 = -2; f(-1, 3) = (-1)^3 + 3 = 2;$$

Answer: f has the absolute maximum value 2, at points $(2, 3)$ and $(-1, 3)$.

f has the absolute minimum value -2 , at points $(-2, 3)$ and $(1, 3)$.

Ex. 11. ST #4 Ex. 1(a)&(c): Set up the integral formulas, **including the limits of the integrations**, for the following problems. *Do not evaluate the integrals!*

- (a) The volume of the solid bounded by $z = x^2 + y^2$, $z = 0$, $x = 0$, $y = 0$, and $x + y = 1$.

Solution: If T is a triangle bounded by $x = 0$, $y = 0$, and $x + y = 1$ (i.e., $y = 1 - x$), then $V = \int \int_E 1 dV = \int \int_T \int_0^{x^2+y^2} 1 dz dA = \int_0^1 \int_0^{1-x} \int_0^{x^2+y^2} 1 dz dy dx$

- (c) The mass of the solid T with the density $\delta(x, y, z) = x^2 + e^z$ bounded by the surfaces: $6x + 2y + z = 12$, $x = 0$, $y = 0$, and $z = 0$.

Solution: The solid is a tetrahedron with a triangular base B on the xy -plane $z = 0$ bounded by $6x + 2y = 12$, $x = 0$, $y = 0$. The upper bound of T is $z = 12 - 6x - 2y$. So, $mass = \int \int \int_T \delta(x, y, z) dV = \int \int_B \int_0^{12-6x-2y} (x^2 + e^z) dz dA$.

Since the triangle side $6x + 2y = 12$ means that $y = 6 - 3x$, which equals 0 for $x = 2$, we get $mass = \int_0^2 \int_0^{6-3x} \int_0^{12-6x-2y} (x^2 + e^z) dz dy dx$.

Ex. 12. ST #4 Ex. 2: Evaluate the integrals:

(a) $\int_0^1 \int_0^\pi \frac{1}{x+1} + \sin y dy dx =$

Solution: $int = \int_0^1 \left[\frac{1}{x+1} y - \cos y \right]_0^\pi dx = \int_0^1 \left(\frac{1}{x+1} \pi - (\cos \pi - \cos 0) \right) dx$. So

$$int = \int_0^1 \left(\frac{1}{x+1} \pi - (-1 - 1) \right) dx = [\pi \ln |x+1| + 2x]_0^1 = \pi(\ln 2 - \ln 1) + 2 = \pi \ln 2 + 2$$

(b) $\int_{-2}^0 \int_0^y (x + 2y^2) dx dy =$

Solution: $int = \int_{-2}^0 \left[\frac{1}{2} x^2 + 2y^2 x \right]_{x=0}^{x=y} dy = \int_{-2}^0 \left(\frac{1}{2} y^2 + 2y^3 \right) dy = \left[\frac{1}{6} y^3 + \frac{1}{2} y^4 \right]_{-2}^0 = 0 - \left(\frac{1}{6}(-8) + \frac{1}{2}16 \right) = \frac{4}{3} - 8 = -6\frac{2}{3}$

(c) $\int \int_R \frac{dy dx}{\sqrt{9-x^2-y^2}}$, where R is the *second quadrant* region bounded by $x^2 + y^2 = 4$.

Solution: We use the polar coordinates, in which the region R is given as $0 \leq r \leq 2$ and $\pi/2 \leq \theta \leq \pi$. So, in the second equation using substitution $u = 9 - r^2$,

$$int = \int_{\pi/2}^\pi \int_0^2 (9 - r^2)^{-1/2} r dr d\theta = \int_{\pi/2}^\pi \left[-(9 - r^2)^{1/2} \right]_0^2 d\theta = \int_{\pi/2}^\pi \left[- \left((9 - 4)^{1/2} - 9^{1/2} \right) \right] d\theta = \left[3 - \sqrt{5} \right]_{\pi/2}^\pi = \frac{3 - \sqrt{5}}{2} \pi.$$

Ex. 13. ST #4 Ex. 3: Find the mass of the solid bounded by the hemisphere $x^2 + y^2 + z^2 \leq R^2$, $z \geq 0$, with the density $\delta(x, y, z) = x^2 + y^2 + z^2$.

Solution: We use the spherical coordinates. Since the solid, T , is the upper hemisphere, we get

$$\begin{aligned} \text{mass} &= \int \int \int_T \delta(x, y, z) dV = \int \int \int_T (x^2 + y^2 + z^2) dV = \int_0^{\pi/2} \int_0^{2\pi} \int_0^R (\rho^2) \rho^2 \sin \phi d\rho d\theta d\phi = \\ &= \int_0^{\pi/2} \int_0^{2\pi} \left[\frac{1}{5} \rho^5 \sin \phi \right]_0^R d\theta d\phi = \int_0^{\pi/2} \int_0^{2\pi} \frac{1}{5} R^5 \sin \phi d\theta d\phi = \int_0^{\pi/2} \left[\left(\frac{1}{5} R^5 \sin \phi \right) \theta \right]_0^{2\pi} d\phi = \\ &= \int_0^{\pi/2} \frac{2}{5} \pi R^5 \sin \phi d\phi = \left[\frac{2}{5} \pi R^5 (-\cos \phi) \right]_0^{\pi/2} = -\frac{2}{5} \pi R^5 (\cos(\pi/2) - \cos 0) = -\frac{2}{5} \pi R^5 (0 - 1) = \\ &= \frac{2}{5} \pi R^5 \end{aligned}$$

Ex. 14. ST #4 Ex. 4: Find the mass of the plane lamina bounded by $x = 0$ and $x = 9 - y^2$ with density $\delta(x, y) = x^2$.

Solution: Notice that $x = 0$ and $x = 9 - y^2$ when $9 - y^2 = 0$ that is, when $y = \pm 3$.

$$\begin{aligned} \text{mass} &= \int \int_R \delta(x, y) dA = \int_{-3}^3 \int_0^{9-y^2} x^2 dx dy = \int_{-3}^3 \left[\frac{1}{3} x^3 \right]_0^{9-y^2} dy = \int_{-3}^3 \frac{1}{3} (9 - y^2)^3 dy = \\ &= \int_{-3}^3 \frac{1}{3} (9^3 - 3 \cdot 9^2 (y^2) + 3 \cdot 9 (y^2)^2 - (y^2)^3) dy = \int_{-3}^3 (3^5 - 3^4 y^2 + 3^2 y^4 - \frac{1}{3} y^6) dy = \\ &= \left[3^5 y - 3^3 y^3 + \frac{3^2}{5} y^5 - \frac{1}{21} y^7 \right]_{-3}^3 = 3^5 (3 + 3) - 3^3 (3^3 + 3^3) + \frac{3^2}{5} (3^5 + 3^5) - \frac{1}{21} (3^7 + 3^7) = \\ &= 2 \cdot 3^6 - 2 \cdot 3^6 + \frac{2}{5} 3^7 - \frac{2}{21} 3^7 = 2 \left(\frac{1}{5} - \frac{1}{21} \right) 3^7 = 2 \frac{21-5}{105} 3^7 = 2 \frac{16}{35} 3^6 = \frac{32}{35} 3^6 \end{aligned}$$

Ex. 15. ST #4 Ex. 6: Evaluate the integral, where C is the graph of $y = x^3$ from $(-1, -1)$ to $(1, 1)$

$$\int_C y^2 dx + x dy =$$

Solution: Clearly x changes from -1 to 1 . Put $x = t$. Then $y(t) = t^3$ and $-1 \leq t \leq 1$ and

$$\begin{aligned} \int_C y^2 dx + x dy &= \int_{-1}^1 (y(t))^2 x'(t) dt + x(t) y'(t) dt = \int_{-1}^1 [(t^3)^2 1 + t(3t^2)] dt = \int_{-1}^1 (t^6 + 3t^3) dt = \\ &= \left[\frac{1}{7} t^7 + \frac{3}{4} t^4 \right]_{-1}^1 = \frac{1}{7} (1 + 1) + \frac{3}{4} (1 - 1) = \frac{2}{7} \end{aligned}$$

Ex. 16. ST #4 Ex. 8: Find a potential function of the vector field and use the fundamental theorem for line integrals to evaluate

$$\int_{(\pi/2, \pi/2)}^{(\pi, \pi)} (\sin y + y \cos x) dx + (\sin x + x \cos y) dy =$$

Solution: We have $P = \sin y + y \cos x$ and $Q = \sin x + x \cos y$. It is easy to see that $\frac{\partial P}{\partial y} = \cos y + \cos x = \frac{\partial Q}{\partial x}$ so indeed we can find the potential function $f(x, y)$. We have

$$f(x, y) = \int P dx = \int \sin y + y \cos x dx = x \sin y + y \sin x + K(y).$$

Taking partial derivative, in terms of y , of both side we get

$$x \cos y + \sin x + K'(y) = \frac{\partial f}{\partial y} = Q = \sin x + x \cos y, \text{ so that } K'(y) = 0 \text{ and } K(y) = C.$$

So, the potential function $f(x, y) = x \sin y + y \sin x + C$ and

$$\begin{aligned} \text{int} &= [f(x, y)]_{(\pi/2, \pi/2)}^{(\pi, \pi)} = [x \sin y + y \sin x]_{(\pi/2, \pi/2)}^{(\pi, \pi)} = (\pi \sin \pi + \pi \sin \pi) - \left(\frac{\pi}{2} \sin \frac{\pi}{2} + \frac{\pi}{2} \sin \frac{\pi}{2} \right) = \\ &= (0 + 0) - \left(\frac{\pi}{2} + \frac{\pi}{2} \right) = -\pi \end{aligned}$$

Ex. 17. ST #4 Ex. 9: Apply Green's theorem to evaluate the following integral, where the simple closed curve C , with counter clockwise direction, is the boundary of the circle $x^2 + y^2 = 1$.

$$\oint_C (\sin x - x^2y) dx + xy^2 dy =$$

Solution: Let D denoted the disk $x^2 + y^2 \leq 1$.

By Green's theorem $\text{int} = \int \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$, where $P = \sin x - x^2y$ and $Q = xy^2$. So,

$$\text{int} = \int \int_D (y^2 - (-x^2)) dA = \int \int_D (x^2 + y^2) dA$$

Changing to the polar coordinates, we get

$$\text{int} = \int_0^{2\pi} \int_0^1 r^2 r dr d\theta = \int_0^{2\pi} \left[\frac{1}{4} r^4 \right]_0^1 d\theta = \int_0^{2\pi} \frac{1}{4} d\theta = \left[\frac{1}{4} \theta \right]_0^{2\pi} = \frac{1}{2} \pi$$