

**Topology 1, Math 581, Fall 2013: Notes and homework**

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**Class of January 15:**

- Discuss the problems from the last semester's final.
- Note that the last semester's abbreviated notes are still available on my web page. We covered chapters 1-3 from the text. In particular, we covered
  - product and metric topologies;
  - connected and locally connected topological spaces;
  - compact and locally compact topological spaces;
  - countability axioms (section 30, to be reviewed again)

**To be covered this semester****Countability axioms (review)**

- A topological space  $X$  is *first countable* (or *satisfies the first countability axiom*) provided for every  $x \in X$  there exists a countable basis  $\mathcal{B}_x$  of  $X$  at  $x$ .
- A topological space  $X$  is *second countable* (or *satisfies the second countability axiom*) provided  $X$  has a countable basis.
- A topological space  $X$  is *separable* provided  $X$  has a countable dense subset  $D$ , that is, such that  $\text{cl}(D) = X$ .
- A topological space  $X$  is *Lindelöf* provided every open cover of  $X$  has a countable subcover.

**Separation axioms**

- (already seen)  $X$  is a  $T_0$  *space* provided for every distinct  $x, y \in X$  there exists an open set  $U \subset X$  such that either  $x \in U$  and  $y \notin U$  or  $y \in U$  and  $x \notin U$  (i.e., such that  $U$  contains precisely one of the points  $x$  and  $y$ ).

- (already seen)  $X$  is a  $T_1$  space provided for every distinct  $x, y \in X$  there exists an open set  $U \subset X$  such that  $x \in U$  and  $y \notin U$ .
- (already seen)  $X$  is Hausdorff (or a  $T_2$  space) provided for every distinct  $x, y \in X$  there exist disjoint open sets  $U, V \subset X$  such that  $x \in U$  and  $y \in V$ .
- (new)  $X$  is regular (or a  $T_3$  space) provided it is a  $T_1$  space and for every closed set  $K$  in  $X$  and  $x \in X \setminus K$  there exist disjoint open sets  $U, V \subset X$  such that  $x \in U$  and  $K \subset V$ .
- (new)  $X$  is normal (or a  $T_4$  space) provided it is a  $T_1$  space and for every disjoint closed sets  $K$  and  $L$  in  $X$  there exist disjoint open sets  $U, V \subset X$  such that  $K \subset U$  and  $L \subset V$ .
- (new)  $X$  is completely regular (or a  $T_{3\frac{1}{2}}$  space) provided it is a  $T_1$  space and for every closed set  $K$  in  $X$  and  $x \in X \setminus K$  there exist a continuous function  $f: X \rightarrow [0, 1]$  such that  $f(x) = 0$  and  $f[K] \subset \{1\}$ .

### Important related theorems

- *Urysohn Lemma*: Every  $T_4$  space is a  $T_{3\frac{1}{2}}$  space.
- *Tietze Extension Theorem*: If  $X$  is normal,  $K \subset X$  is closed, and  $f: K \rightarrow [0, 1]$  is continuous, then  $f$  can be extended to a continuous  $F: X \rightarrow [0, 1]$ .
- *Urysohn Metrization Theorem*: If  $X$  is regular and second countable, then it is metrizable.

### The Tychonoff Theorem

- *The Tychonoff Theorem*: Arbitrary product of compact spaces is compact.

### Some material from Chapters 6-8

	subspace	closed subspace	countable product	arbitrary product	continuous image
compact	$\mathbb{N}, \mathbb{R} \setminus [0, 1]$	Y	Y, Tych. Thm	Y, Tych. Thm	Y
connected	$\mathbb{N}, \mathbb{R} \setminus [0, 1]$	$\mathbb{N}, \mathbb{R} \setminus [0, 1]$	Y	Y	Y
Hausdorff	Y	Y	Y	Y	$\mathbb{N}$ , Ex a
2nd countable	Y	Y	Y	$\mathbb{N}, \mathbb{R}^{\text{uncountable}}$	$\mathbb{N}$ , Ex c
1st countable	Y	Y	Y	$\mathbb{N}, \mathbb{R}^{\text{uncountable}}$	$\mathbb{N}$ , Ex b,c
separable	$\mathbb{N}, L \subset (\mathbb{R}_\ell)^2$	$\mathbb{N}, L \subset (\mathbb{R}_\ell)^2$	Y, 10 p. 194	$\mathbb{N}$ , 16 p. 195	Y, 11 p. 194
Lindelöf	$\mathbb{N}, \mathbb{R}_\infty$	Y, 9 p. 194	$\mathbb{N}, (\mathbb{R}_\ell)^2$	$\mathbb{N}, (\mathbb{R}_\ell)^2$	Y, 11 p. 194
metrizable	Y	Y	Y	$\mathbb{N}, \mathbb{R}^{\text{uncountable}}$	$\mathbb{N}$ , Ex a

Here the space  $X_\infty$ , in particular  $\mathbb{R}_\infty$ , is the one point compactification of a discrete space  $X$ , that is,  $X_\infty = X \cup \{\infty\}$ , where  $\infty \notin X$ , has the topology  $\tau = \mathcal{P}(X) \cup \{X_\infty \setminus F : F \text{ is a finite subset of } X\}$ .

**Example.** For a set  $X$  let  $\tau_d$  be a discrete topology on  $X$  and  $\mathcal{T}$  an arbitrary topology on  $X$ . Then a function  $f: \langle X, \tau_d \rangle \rightarrow \langle X, \mathcal{T} \rangle$ , given by  $f(x) = x$ , is continuous bijection.

- (a) If  $\mathcal{T} = \{\emptyset, X\}$  is anti-discrete topology and  $X = \mathbb{N}$ , then domain of  $f$  is metric, while  $f[X]$  is not Hausdorff.
- (b) If  $X = \mathbb{R}^\omega$  and  $\mathcal{T}$  is a box topology, then domain of  $f$  is first countable (as metric), while  $f[X]$  is not first countable.
- (c) Let  $X = \mathbb{N}$  and  $\mathcal{T}$  be such that  $\langle X, \mathcal{T} \rangle$  is not second countable. (This will be proved together with Tychonoff's theorem.) Then domain of  $f$  is second countable, while  $f[X]$  is not (since it is not first countable).

**Class of January 17:**

Finish the discussion of the table, with the emphasize on separable, Lindelöf, and metrizable spaces. (Metrizability case not discussed on Jan 15.) In particular go over the following exercises on page 194:

- Ex 9, 10, 11, 16;
- Recall Ex 2 and 4; Go over Ex 5;
- Point to Ex 12.

**Written assignment for Tuesday, January 22:** Exercise 14, page 194.

**Class of January 22:**

Review Example 3:  $\mathbb{R}_\ell$  is Lindelöf. Main steps:

- Note, that it is enough to consider only the covers  $\mathcal{V}$  of  $\mathbb{R}_\ell$  composed of the standard basic open sets, that is,  $\mathcal{V}$  of the form  $\{[a_\xi, b_\xi)\}_{\xi \in J}$ .
- Note that  $\mathcal{U} = \{(a_\xi, b_\xi)\}_{\xi \in J}$  is an open (w.r.t. the standard topology) cover of  $C = \bigcup_{\xi \in J} (a_\xi, b_\xi)$ .
- Since  $C$  (with the standard topology) is second countable, we can find countable  $J_0 \subset J$  with  $C = \bigcup_{\xi \in J_0} (a_\xi, b_\xi)$ .
- Prove that  $\mathbb{R} \setminus C$  is countable. Then, find countable  $J_1 \subset J$  with  $\mathbb{R} \setminus C \subset \bigcup_{\xi \in J_1} [a_\xi, b_\xi)$ .
- Notice that  $\mathcal{V}_0 = \{[a_\xi, b_\xi)\}_{\xi \in J_0 \cup J_1} \subset \mathcal{V}$  is countable and covers  $\mathbb{R} = C \cup (\mathbb{R} \setminus C)$ .

**Section 31: The Separation Axioms**

- (already seen)  $X$  is a  $T_0$  space provided for every distinct  $x, y \in X$  there exists an open set  $U \subset X$  such that either  $x \in U$  and  $y \notin U$  or  $y \in U$  and  $x \notin U$  (i.e., such that  $U$  contains precisely one of the points  $x$  and  $y$ ).
- (already seen)  $X$  is a  $T_1$  space provided for every distinct  $x, y \in X$  there exists an open set  $U \subset X$  such that  $x \in U$  and  $y \notin U$ .
- (already seen)  $X$  is Hausdorff (or a  $T_2$  space) provided for every distinct  $x, y \in X$  there exist disjoint open sets  $U, V \subset X$  such that  $x \in U$  and  $y \in V$ .
- (new)  $X$  is regular (or a  $T_3$  space) provided it is a  $T_1$  space and for every closed set  $K$  in  $X$  and  $x \in X \setminus K$  there exist disjoint open sets  $U, V \subset X$  such that  $x \in U$  and  $K \subset V$ .
- (new)  $X$  is normal (or a  $T_4$  space) provided it is a  $T_1$  space and for every disjoint closed sets  $K$  and  $L$  in  $X$  there exist disjoint open sets  $U, V \subset X$  such that  $K \subset U$  and  $L \subset V$ .

Go over Lemma 31.1.

Go over Exercises 1 and 2.

Go over Theorem 31.2. Note that it is false for the normal spaces. (Note, where the subspace part of the proof for the regular spaces brakes for the normal spaces.)

Go over Example 1:  $\mathbb{R}_K$  is Hausdorff but not regular.

Go over Exercise 4, Hausdorff and regular cases.

Go over Examples 2 ( $\mathbb{R}_\ell$  is normal) and 3 ( $(\mathbb{R}_\ell)^2$  is not normal).

Go over Exercise 4, normal case.

**Written assignment for Tuesday, January 29:** Exercise 5, page 199.

### Class of January 24:

Recall

- $X$  is *regular* (or a  $T_3$  space) provided it is a  $T_1$  space and for every closed set  $K$  in  $X$  and  $x \in X \setminus K$  there exist disjoint open sets  $U, V \subset X$  such that  $x \in U$  and  $K \subset V$ .
- $X$  is *normal* (or a  $T_4$  space) provided it is a  $T_1$  space and for every disjoint closed sets  $K$  and  $L$  in  $X$  there exist disjoint open sets  $U, V \subset X$  such that  $K \subset U$  and  $L \subset V$ .
- Theorem 31.2: a subspace of regular space is regular; the product of regular spaces is regular. Same for Hausdorff spaces.

Go over Examples 2 ( $\mathbb{R}_\ell$  is normal) and 3 ( $(\mathbb{R}_\ell)^2$  is not normal). Also, proof of Theorem 7.8 (which is used in Example 3).

So, product of normal spaces need not be normal. Also,  $(\mathbb{R}_\ell)^2$  is regular but not normal.

Latter we will prove that  $(\mathbb{R}_\ell)^2$  is homeomorphic to a subspace of some normal spaces. So, a subspace of normal space need not be normal.

Go over Exercise 4, normal case.

### Section 32: Normal spaces

Show that every regular Lindelöf space is normal. This is Ex 4 page 205. Proof the same as for Thm 32.1.

Corollary: the product of two Lindelöf spaces need not be Lindelöf, justified by  $(\mathbb{R}_\ell)^2$ .

Thm 32.2: Every metrizable space is normal.

Thm 32.3: Every compact Hausdorff space is normal.

### Class of January 29:

Last class we proved that:

- $\mathbb{R}_\ell$  is normal, but  $(\mathbb{R}_\ell)^2$  is not.
- Every regular Lindelöf space is normal.
- Every metrizable space is normal.
- Every compact Hausdorff space is normal.

New material:

Go over Example 1.

Go over Exercises 1 and 2.

### Section 33: The Urysohn Lemma

State and prove the Urysohn Lemma.

**Written assignment for Tuesday, February 5:** One (difficult) direction of Exercise 4, page 213: Prove that

- (a) for every closed  $G_\delta$  set  $A \subset X$  there is continuous  $f: X \rightarrow [0, 1]$  such that  $f^{-1}(0) = A$ .

Show the reverse of (a) as well as similar direction for Exercise 5, page 213.

**Class of January 31:**

Last time we proved

- Urysohn Lemma: *If  $X$  is normal and  $A$  and  $B$  are closed disjoint subsets of  $X$ , then there exists continuous  $f: X \rightarrow [0,1]$  such that  $f[A] \subseteq \{0\}$  and  $f[B] \subseteq \{1\}$ .*

Suggestion: Look over the exercises 1 and 3, page 212.

Define *completely regular* (or  $T_{3.5}$ ) spaces.

Prove Theorem 33.2.

Go over the expanded table:

	subspace	closed subspace	countable product	arbitrary product	continuous image
2nd countable	Y	Y	Y	N, $\mathbb{R}^{\text{uncountable}}$	N, Ex c
1st countable	Y	Y	Y	N, $\mathbb{R}^{\text{uncountable}}$	N, Ex b,c
separable	N, $L \subset (\mathbb{R}_\ell)^2$	N, $L \subset (\mathbb{R}_\ell)^2$	Y, 10 p. 194	N, 16 p. 195	Y, 11 p. 194
Lindelöf	N, $\mathbb{R}_\infty$	Y	N, $(\mathbb{R}_\ell)^2$	N, $(\mathbb{R}_\ell)^2$	Y, 11 p. 194
compact	n	Y	Y, Tych. Thm	Y, Tych. Thm	Y
connected	n	n	Y	Y	Y
Hausdorff	Y	Y	Y	Y	N, Ex a
regular	Y	Y	Y	Y	N, Ex a
completely reg	Y	Y	Y	Y	N, Ex a
normal	N, p. 203	Y, 1 p. 205	N, $(\mathbb{R}_\ell)^2$	N, $(\mathbb{R}_\ell)^2$	N, Ex a
metrizable	Y	Y	Y	N, $\mathbb{R}^{\text{uncountable}}$	N, Ex a

Answers

**Example.** For a set  $X$  let  $\tau_d$  be a discrete topology on  $X$  and  $\mathcal{T}$  an arbitrary topology on  $X$ . Then a function  $f: \langle X, \tau_d \rangle \rightarrow \langle X, \mathcal{T} \rangle$ , given by  $f(x) = x$ , is continuous bijection.

- If  $\mathcal{T} = \{\emptyset, X\}$  is anti-discrete topology and  $X = \mathbb{N}$ , then domain of  $f$  is metric, while  $f[X]$  is not Hausdorff.
- If  $X = \mathbb{R}^\omega$  and  $\mathcal{T}$  is a box topology, then domain of  $f$  is first countable (as metric), while  $f[X]$  is not first countable.
- Let  $X = \mathbb{N}$  and  $\mathcal{T}$  be such that  $\langle X, \mathcal{T} \rangle$  is not second countable. (This will be proved together with Tychonoff's theorem.) Then domain of  $f$  is second countable, while  $f[X]$  is not (since it is not first countable).

Go over Exercise 2, page 212.

Go over Exercise 5, page 213. Here is the difficult direction argument.

**Exercise 5, page 213 (one direction):** If  $X$  is normal, then

- (d) for every closed disjoint  $G_\delta$  subsets  $A$  and  $B$  of  $X$  there is continuous  $f: X \rightarrow [0, 1]$  such that  $f^{-1}(0) = A$  and  $f^{-1}(1) = B$ .

PROOF. We will use homework assignment property

- (a) for every closed  $G_\delta$  set  $A \subset X$  there is continuous  $f_A: X \rightarrow [0, 1]$  such that  $f_A^{-1}(0) = A$ .

**Easier version of the proof, provided by Adam Anderson:** Define  $f(x) = \frac{f_A(x) + f_B(x)}{f_B(x)}$ . Notice that it works.

**More difficult version of the proof:** First note that, by (a),

- (b) there is continuous  $g: X \rightarrow [0, 1]$  such that  $g^{-1}(0) = A \cup B$

Indeed, if  $f_A$  and  $f_B$  are the functions as in (a) for  $A$  and  $B$ , respectively, then  $g = f_A \cdot f_B$  is as desired.

Next note that

- (c) there is continuous function  $h_{AB}: X \rightarrow [0, 1]$  such that  $h_{AB}^{-1}(0) = A$  and  $h_{AB}^{-1}(1) \supset B$ .

Indeed, if  $f_0: X \rightarrow [0, 1]$  is a function from Urysohn Lemma (i.e., such that  $f_0^{-1}(0) \supset A$  and  $f_0^{-1}(1) \supset B$ ),  $g$  is as in (b), and we put  $h = f_0 + g$ , then  $h^{-1}(0) = A$  and  $h^{-1}(1) \supset B$ . Thus, function  $h_{AB} = \min\{1, h\}$  is as desired.

Finally, put  $f = \frac{1}{2}[h_{AB} + (1 - h_{BA})]$ . Then function  $f$  satisfies (d), as  $(1 - h_{AB}): X \rightarrow [0, 1]$ ,  $(1 - h_{AB})^{-1}(0) \supset A$ , and  $(1 - h_{AB})^{-1}(1) = B$ . ■

Go over Exercises 3 and 5 page 205. Solve Exercise 3: (1) directly; (2) using one point compactification.

Suggestion: Look over the exercises 7 and 8, page 213.

## Homework assigned on January 17, 2013

**Ex. 1.** If  $X$  is Lindelöf and  $Y$  is compact, then  $X \times Y$  is Lindelöf.

**PROOF.** Let  $\mathcal{U}$  be an open cover of  $X \times Y$ . For every  $x \in X$  the section  $\{x\} \times Y$  is compact, so there is a finite family  $\mathcal{U}_x \subset \mathcal{U}$  with  $\{x\} \times Y \subset \bigcup \mathcal{U}_x$ . By “tube lemma” there is an open  $U_x \subset X$  with  $\{x\} \times Y \subset U_x \times Y \subset \bigcup \mathcal{U}_x$ . Then,  $\{U_x : x \in X\}$  is an open cover of  $X$ . Since  $X$  is Lindelöf it has a countable subcover  $\{U_{x_i} : i = 0, 1, 2, \dots\}$ . Let  $\mathcal{U}_0 = \bigcup_{i=0}^{\infty} \mathcal{U}_{x_i}$ . It is countable, as a countable union of finite sets and  $\mathcal{U}_0 \subset \mathcal{U}$ . It is enough to prove that  $\mathcal{U}_0$  covers  $X \times Y$ . This is so, since  $X \times Y \subset \bigcup_{i=0}^{\infty} U_{x_i} \times Y \subset \bigcup_{i=0}^{\infty} \bigcup \mathcal{U}_{x_i} = \bigcup \mathcal{U}_0$ .

### Class of February 5:

Prove **Urysohn metrization theorem**:

*Every regular second countable space is metrizable.*

(Covered only the first version of the proof.)

Go over theorems 34.2.

### Class of February 7:

Restate **Urysohn metrization theorem**:

*Every regular second countable space  $X$  is metrizable.*

Sketch the the first version of the proof. Go over the second version of its proof. That is show that

$X$  can be embedded to  $[0, 1]^\omega$ , where  $[0, 1]^\omega$  is considered with the sup metric:  $\rho(x, y) = \sup_i |x(i) - y(i)|$ .

Prove this by showing that

1. the embedding can be actually done into  $T = \prod_{n=1}^{\infty} [0, 1/n]$ ,
2. the topology of  $\langle T, \rho \rangle$  coincides with that of the subspace of  $\mathbb{R}^\omega$ , when  $\mathbb{R}^\omega$  is considered with the product topology.

Restate Theorem 34.2.

Restate Theorem 34.3.

Go over exercises 1, 2, 3, 4, and 5 page 218.

State Tietze Extension Theorem.

**Class of February 12:**

Check (above) an updated proof of Exercise 5, page 213, due to Adam Anderson.

State, again, Tietze Extension Theorem.

Prove Tietze Extension Theorem.

Go over incomplete proof of the Tychonoff Theorem, page 231.

**Class of February 14:**

Quiz #1: long, fill the table.

Give the individualized problem sets, from the old Topology Entrance Exams.

Solve problems 5, 4, and 3 from the Topology Entrance Exam August 2012.

**Class of February 19:**

Students presentations of the solutions of some of the exercises from the old Topology Entrance Exams assigned to them.

Definition of a *filter* on a set  $X$ : a non-empty family  $\mathcal{F} \subset \mathcal{P}(X)$  (i.e., of subsets of  $X$ ) such that:

- if  $A, B \in \mathcal{F}$ , then  $A \cap B \in \mathcal{F}$ ;
- if  $A \in \mathcal{F}$  and  $A \subset C \subset X$ , then  $C \in \mathcal{F}$ .

Filter is proper when  $\emptyset \notin \mathcal{F}$ , that is, when  $\mathcal{F} \neq \mathcal{P}(X)$ .

Intuitive argument that any family having the finite intersection property can be extended to a maximal family having the finite intersection property.

**Class of February 21:**

Students presentations of the solutions of some of the exercises from the old Topology Entrance Exams assigned to them.

Prove Lemma 37.2. Notice also that every maximal family having the finite intersection property is a filter. (Restate definition of a filter.)

Proof of the Tychonoff Theorem.

**Class of February 26:**

Students presentations of the solutions of some of the exercises from the old Topology Entrance Exams assigned to them.

State Zorn Lemma and Hausdorff Maximal Principle, section 11.

Prove, using Hausdorff Maximal Principle, that every family having the finite intersection property can be extended to a maximal family having the finite intersection property.

Jump to

**Section 43: Complete Metric Spaces**

Definition of *Cauchy sequence* and *complete metric space*.

Lemma 43.1.

Theorem 43.2.

Lemma 43.3.

Theorem 43.4.

**Class of February 28:**

Students presentations of the solutions of some of the exercises from the old Topology Entrance Exams assigned to them.

Definition of *uniform metric*.

Theorem 43.5.

Define  $\mathcal{C}(X, Y)$  and  $\mathcal{B}(X, Y)$ .

Theorem 43.6, proved only for the case of  $\mathcal{C}(X, Y)$ .

**Class of March 5:**

Students presentations of the solutions of some of the exercises from the old Topology Entrance Exams assigned to them.

Recall definition of  $\mathcal{C}(X, Y)$  and  $\mathcal{B}(X, Y)$ .

Theorem 43.6, prove for the case of  $\mathcal{B}(X, Y)$ .

Definition of *sup metric* on the class  $\mathcal{B}(X, Y)$  of all bounded functions from  $X$  into  $Y$ .

Theorem 43.7.

Go over exercise 4, page 270.

**Class of March 5:**

Students presentations of the solutions of some of the exercises from the old Topology Entrance Exams assigned to them.

Go over exercises 2 and 5 page 270.

**Class of March 12:**

Students presentations of the solutions of some of the exercises from the old Topology Entrance Exams assigned to them.

**Section 44: Peano curve**

Construct Peano curve.

Go over exercises 1 and 2 page 274.

Many comments on Peano curve current research.

**Class of March 14:**

Students presentations of the solutions of some of the exercises from the old Topology Entrance Exams assigned to them.

**Section 48: Baire spaces**

Define Baire space in terms of countable union of nowhere dense sets.

Go over Lemma 48.1 Definition Baire space in terms of countable intersections of open dense sets.

Go over Theorem 48.2: Baire category theorem.

Go over Lemma 48.3

Go over Lemma 48.4

**Class of March 19:**

Administer **Extensive quiz: theorems, definitions.**

Go, once more, over Lemma 48.4

Go over Theorem 48.5: *If  $X$  is a Baire space and  $Y$  is metric then pointwise limit of  $f_n: X \rightarrow Y$  is continuous on a dense  $G_\delta$  set.*

If  $X = [0, 1]$ , the theorem resembles Egoroff's Theorem (its corollary): Under the same assumptions, *for every  $\eta > 0$  there exists a closed set  $A \subset [0, 1]$  of measure  $\geq 1 - \eta$  such that  $f \upharpoonright A$  is continuous.* Note that  $f$  need not be continuous at any  $a \in A$ !

**Example.** Let  $A \subset [0, 1]$  be a closed set of measure  $\geq 1 - \eta$ . Define  $f: [0, 1] \rightarrow \mathbb{R}$  as:  $f(x) = 0$  for  $x \in A$  and  $f(x) = \sin(1/\text{dist}(x, A))$  for  $x \notin A$ . It is not difficult to find continuous  $f_n$ 's converging pointwise to  $f$ .

Then Egoroff's Theorem is satisfied by  $A$ , while Theorem 48.5 by its complement!

Fill the gaps: Construct closed  $A \subset [0, 1]$  of measure  $\geq 1 - \eta$ .

Describe continuous  $f_n$ 's converging to  $f$ :  $f_n(x) = 0$  whenever  $\text{dist}(x, A) \leq 1/(2\pi n)$  and  $f_n(x) = \sin(1/\text{dist}(x, A))$  otherwise.

**Class of March 21:**

Students presentations of the solutions of some of the exercises from the old Topology Entrance Exams assigned to them.

Go over Exercises 1, 2, 3, 5, and 6, 7, 8, 9, 11 page 299.

**Class of April 2:**

**Section 49:** Continuous nowhere-differentiable functions via Baire Category Theorem.

**Written assignment (mini-project) for Tuesday, April 23:**

Each part will be graded as an independent exercise, worth 10 points each. *I will not accept the solutions after the deadline.* I like to encourage you to submit the solution(s) (possibly, in separate parts) earlier, so you could make corrections, if necessary, prior the deadline.

**Exercise 6 p. 270.** A space  $X$  is said to be *topological complete* if there exist a metric space for the topology of  $X$  relative to which  $X$  is complete.

(a) Show that a closed subspace of a topological complete space is topological complete.

(b) Show that a countable product of topological complete spaces is topological complete (in the product topology).

(c) Show that an open subspace of topological complete space is topological complete.

(d) Show that if  $A$  is a  $G_\delta$  set in a topological complete space, then  $A$  is topological complete. Conclude, that the irrationals are topologically complete.

**Class of April 4:**

Three student presentations of the solutions of some of the exercises from the old Topology Entrance Exams assigned to them.

**Chapter 6:** Metrization Theorems **Section 39:** Local Finiteness

Let  $X$  be a topological space. A family  $\mathcal{A}$  of subsets of  $X$  is:

- *locally finite* provided every  $x \in X$  has an open neighborhood  $U$  such that  $U$  intersects only finitely many sets  $A$  from  $\mathcal{A}$ ;
- *discrete* provided every  $x \in X$  has an open neighborhood  $U$  such that  $U$  intersects at most one  $A$  from  $\mathcal{A}$ .

Clearly every discrete family is locally finite.

State Nagata-Smirnov Metrization Theorem based on these notions.

**Class of April 9:**

Recall that for a topological space  $X$  a family  $\mathcal{A}$  of subsets of  $X$  is:

- *locally finite* provided every  $x \in X$  has an open neighborhood  $U$  such that  $U$  intersects only finitely many sets  $A$  from  $\mathcal{A}$ ;
- *locally discrete* provided every  $x \in X$  has an open neighborhood  $U$  such that  $U$  intersects at most one  $A$  from  $\mathcal{A}$ .

Clearly every discrete family is locally finite.

Define a *countable locally finite* (or  $\sigma$ -*locally finite*) family of subsets of  $X$ . Also,  $\sigma$ -*locally discrete* family of subsets of  $X$ .

**Theorem 1 [Bing-Nagata-Smirnov Metrization Theorem]** For regular topological spaces the following conditions are equivalent.

- (1)  $X$  is metrizable.
- (2)  $X$  has a  $\sigma$ -locally discrete basis.
- (3)  $X$  has a  $\sigma$ -locally finite basis.

Prove Lemma 39.1.

Define a *refinement* of a family of subsets of  $X$ .

Prove Lemma 39.2. Notice that the refinement is, in fact,  $\sigma$ -discrete.

Prove that (1) implies (2) in the theorem.

**Class of April 11:**

On the theorem

**Theorem 2 [Bing-Nagata-Smirnov Metrization Theorem]** *For regular topological spaces the following conditions are equivalent.*

- (1) *X is metrizable.*
- (2) *X has a  $\sigma$ -locally discrete basis.*
- (3) *X has a  $\sigma$ -locally finite basis.*

we proved that (1) implies (2); clearly (2) implies (3).

Need to prove that (3) implies (1):

Prove Lemma 40.1

Prove Lemma 40.2

Prove Theorem 40.3

Go over Exercises 2, 3, and 4, page 252.

**Class of April 16:**

Start section 41: Paracompactness.

Goal: show that both metrizability and compactness imply paracompactness; and that paracompact Hausdorff space is normal.

Definition.

Obviously every compact space is paracompact.

Example 1:  $\mathbb{R}^n$  is paracompact.

Solve Exercise 1 page 260.

Theorem 41.1: Every paracompact Hausdorff space is normal.

Theorem 41.2: Every closed subspace of a paracompact space is paracompact.

Go over Example 2 and 3.

Solve Exercise 3 page 260.

State Lemma 41.3.

Use Lemma 41.3 to prove:

- Theorem 41.4: Every metrizable space is paracompact;
- Theorem 41.5: Every regular Lindelöf space is paracompact.

**Class of April 18:**

Recall definition of paracompactness.

We are proving Theorwm 41.4: Every metrizable space is paracompact.

State and prove Lemma 41.3.

Solve Exercise 2(a) page 260.

**Class of April 23:**

Restate Exercise 2(a) page 260. Use is to solve Exercise 2(b) page 260.

Solve Exercises 3 and 4 page 260.

Define partition of unity.

Prove Lemma 41.6.

Prove Theorem 41.7.

Prove Theorem 41.8.

Solve Exercises 6 and 7 page 260.