

Topology 2, Math 681, Spring 2014: Notes and homework

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Class of January 9, 2014:

- Discuss the problems from the last semester's final.
- Note that the last semester's abbreviated notes are still available on my web page, at:

<http://www.math.wvu.edu/~kcies/teach/Fall2013/Fall2013.html>

We covered (most of) chapters 1-3 from the text. In particular, we covered

- product and metric topologies;
- connected topological spaces;
- compact topological spaces;

To be covered this semester**Review, by covering the following new sections:**

- Section 25: locally connected topological spaces
- Sections 28 and 29: limit point and local compactness

Countability axioms

- A topological space X is *first countable* (or *satisfies the first countability axiom*) provided for every $x \in X$ there exists a countable basis \mathcal{B}_x of X at x .
- A topological space X is *second countable* (or *satisfies the second countability axiom*) provided X has a countable basis.
- A topological space X is *separable* provided X has a countable dense subset D , that is, such that $\text{cl}(D) = X$.
- A topological space X is *Lindelöf* provided every open cover of X has a countable subcover.

Separation axioms

- (already seen) X is a T_0 space provided for every distinct $x, y \in X$ there exists an open set $U \subset X$ such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$ (i.e., such that U contains precisely one of the points x and y).
- (already seen) X is a T_1 space provided for every distinct $x, y \in X$ there exists an open set $U \subset X$ such that $x \in U$ and $y \notin U$.
- (already seen) X is Hausdorff (or a T_2 space) provided for every distinct $x, y \in X$ there exist disjoint open sets $U, V \subset X$ such that $x \in U$ and $y \in V$.
- (new) X is regular (or a T_3 space) provided it is a T_1 space and for every closed set K in X and $x \in X \setminus K$ there exist disjoint open sets $U, V \subset X$ such that $x \in U$ and $K \subset V$.
- (new) X is normal (or a T_4 space) provided it is a T_1 space and for every disjoint closed sets K and L in X there exist disjoint open sets $U, V \subset X$ such that $K \subset U$ and $L \subset V$.
- (new) X is completely regular (or a $T_{3\frac{1}{2}}$ space) provided it is a T_1 space and for every closed set K in X and $x \in X \setminus K$ there exist a continuous function $f: X \rightarrow [0, 1]$ such that $f(x) = 0$ and $f[K] \subset \{1\}$.

Important related theorems

- *Urysohn Lemma*: Every T_4 space is a $T_{3\frac{1}{2}}$ space.
- *Tietze Extension Theorem*: If X is normal, $K \subset X$ is closed, and $f: K \rightarrow [0, 1]$ is continuous, then f can be extended to a continuous $F: X \rightarrow [0, 1]$.
- *Urysohn Metrization Theorem*: If X is regular and second countable, then it is metrizable.

The Tychonoff Theorem

- *The Tychonoff Theorem*: Arbitrary product of compact spaces is compact.

Some material from Chapters 6-8

Go over Section 25

Review:

Define components and path components.

Go over Theorems 25.1 and 25.2.

Go over Examples 1 and 2.

New material:

Components are closed, path components need not to be closed.

Define locally connected spaces and locally path connected spaces.

Go over Example 3.

Go over Theorems 25.3 and 25.4.

Written assignment for Tuesday, January 14: Exercises 1 (easy) and 2 (more difficult), page 162.

Class of January 14, 2014:**Section 28: Limit Point Compactness**

Definition 1 A space X is *limit point compact* provided every infinite subset of X has a limit point.

Theorem 1 (Thm 28.1) *If X is compact, then X is limit point compact, but not conversely.*

Go over Examples 1 and 2.

Definition 2 A space X is *sequentially compact* provided every sequence in X has a convergent subsequence.

Theorem 2 (Thm 28.2) *For a metrizable space X , the following are equivalent:*

- (1) X is compact.
- (2) X is limit point compact.
- (3) X is sequentially compact.

Remark: “(2) implies (3)” requires only first countability of X .

Mention Example 3: the space satisfies (1)–(3), but is not first countable, so not metrizable.

Go over Exercises 1, 2 page 181. (Not covered.)

Possibly, discuss Exercise 7 pages 181–182. (Not covered.)

Class of January 16, 2014:

Proved last time: For a metrizable space X , the following are equivalent:

- (1) X is compact.
- (2) X is limit point compact.
- (3) X is sequentially compact.

Suggestion: Go over Exercises 1, 2 page 181;

look over very interesting Exercise 7 pages 181–182.

New material:

Section 29: Local Compactness

Definition 3 A space X is *locally compact* provided every $x \in X$ there is an open set $U \ni x$ such that $\text{cl}(U)$ is compact.

Compact implies locally compact.

Go over Examples 1 and 2. \mathbb{Q} is not locally compact.

State and prove Theorem 29.1.

Define *one-point compactification* of a locally compact space.

Go over Example 4.

State and prove Theorem 29.2.

State and prove Corollary 29.3.

State and prove Corollary 29.4.

Go over Exercises 1 and 3 page 186.

Class of January 23, 2014 (no class on January 21, snow day):

Section 30: The Countability Axioms

The next definition and theorem were covered last semester.

Definition 4 A topological space X is *first countable* (or *satisfies the first countability axiom*) provided for every $x \in X$ there exists a countable basis \mathcal{B}_x of X at x .

Theorem 3 Let X be a first countable topological space and let $A \subset X$. Then $x \in \text{cl}(A)$ if, and only if, there is a sequence of points of A converging to x . Moreover, the implication " \Leftarrow " does not require the assumption of first countability.

New material:

Definition 5 A topological space X is *second countable* (or *satisfies the second countability axiom*) provided X has a countable basis.

Go over Examples 1 and 2.

Go over Theorem 30.2.

Definition 6

- A subset A of a space X is *dense (in X)* provided $\text{cl}(A) = X$.
- A topological space X is *separable* provided X has a countable dense subset D , that is, such that $\text{cl}(D) = X$.
- A topological space X is *Lindelöf* provided every open cover of X has a countable subcover.

Go over Theorem 30.3.

Written assignment for Tuesday, January 28: Exercises 14, page 194.

Class of January 28:

- A topological space X is *second countable* (or *satisfies the second countability axiom*) provided X has a countable basis.
- A topological space X is *separable* provided X has a countable dense subset D , that is, such that $\text{cl}(D) = X$.
- A topological space X is *Lindelöf* provided every open cover of X has a countable subcover.

	subspace	closed subspace	countable product	arbitrary product	continuous image
compact	$\mathbb{N}, [0, 1]; [0, 1]$	Y	Y, Tych. Thm	Y, Tych. Thm	Y
connected	$\mathbb{N}, \mathbb{R} \setminus (0, 1)$	$\mathbb{N}, \mathbb{R} \setminus (0, 1)$	Y	Y	Y
Hausdorff	Y	Y	Y	Y	\mathbb{N} , Ex a
2nd count	Y	Y	Y	$\mathbb{N}, \mathbb{R}^{\text{uncount}}$ p. 133	\mathbb{N} , Ex c
1st count	Y	Y	Y	$\mathbb{N}, \mathbb{R}^{\text{uncount}}$ p. 133	\mathbb{N} , Ex b,c
separable	$\mathbb{N}, L \subset (\mathbb{R}_\ell)^2$	$\mathbb{N}, L \subset (\mathbb{R}_\ell)^2$	Y, 10 p. 194	\mathbb{N} , 16 p. 195	Y, 11 p. 194
Lindelöf	$\mathbb{N}, \mathbb{R}_\infty$	Y, 9 p. 194	$\mathbb{N}, (\mathbb{R}_\ell)^2$	$\mathbb{N}, (\mathbb{R}_\ell)^2$	Y, 11 p. 194
metrizable	Y	Y	Y	$\mathbb{N}, \mathbb{R}^{\text{uncount}}$ p. 133	\mathbb{N} , Ex a

Here the space X_∞ , in particular \mathbb{R}_∞ , is the one point compactification of a discrete space X , that is, $X_\infty = X \cup \{\infty\}$, where $\infty \notin X$, has the topology $\tau = \mathcal{P}(X) \cup \{X_\infty \setminus F : F \text{ is a finite subset of } X\}$.

Example. For a set X let τ_d be a discrete topology on X and \mathcal{T} an arbitrary topology on X . Then a function $f: \langle X, \tau_d \rangle \rightarrow \langle X, \mathcal{T} \rangle$, given by $f(x) = x$, is continuous bijection.

- If $\mathcal{T} = \{\emptyset, X\}$ is anti-discrete topology and $X = \mathbb{N}$, then domain of f is metric, while $f[X]$ is not Hausdorff.
- If $X = \mathbb{R}^\omega$ and \mathcal{T} is a box topology, then domain of f is first countable (as metric), while $f[X]$ is not first countable.
- Let $X = \mathbb{N}$ and \mathcal{T} be such that $\langle X, \mathcal{T} \rangle$ is not second countable. (This will be proved together with Tychonoff's theorem.) Then domain of f is second countable, while $f[X]$ is not (since it is not first countable).

Go over Examples 3 and 4. (Very important!)

Go over Exercises 9, 10, 11 page 194, pointing their role in the table.

Point to Exercise 16, page 194. Read the details.

Class of January 30:

Go over Exercises 2, 4, and 5, page 194.

Suggested Exercises to examine by the students: 1(a), 12, 13, and 16, page 194.

Section 31: The Separation Axioms

- (already seen) X is a T_0 space provided for every distinct $x, y \in X$ there exists an open set $U \subset X$ such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$ (i.e., such that U contains precisely one of the points x and y).
- (already seen) X is a T_1 space provided for every distinct $x, y \in X$ there exists an open set $U \subset X$ such that $x \in U$ and $y \notin U$.
- (already seen) X is Hausdorff (or a T_2 space) provided for every distinct $x, y \in X$ there exist disjoint open sets $U, V \subset X$ such that $x \in U$ and $y \in V$.
- (new) X is regular (or a T_3 space) provided it is a T_1 space and for every closed set K in X and $x \in X \setminus K$ there exist disjoint open sets $U, V \subset X$ such that $x \in U$ and $K \subset V$.
- (new) X is normal (or a T_4 space) provided it is a T_1 space and for every disjoint closed sets K and L in X there exist disjoint open sets $U, V \subset X$ such that $K \subset U$ and $L \subset V$.

Go over Lemma 31.1.

Stated Theorem 31.2: a subspace of regular space is regular; the product of regular spaces is regular; same for Hausdorff spaces.

Written assignment for Tuesday, February 4: Exercise 5, page 199.

Class of February 4:

Recall

- X is *regular* (or a T_3 space) provided it is a T_1 space and: for every closed set K in X and $x \in X \setminus K$ there exist disjoint open sets $U, V \subset X$ such that $x \in U$ and $K \subset V$ (equivalently, for every open U and $x \in U$, there exists an open V with $x \in V \subset \text{cl}(V) \subset U$).
- X is *normal* (or a T_4 space) provided it is a T_1 space and: for every disjoint closed sets K and L in X there exist disjoint open sets $U, V \subset X$ such that $K \subset U$ and $L \subset V$ (equivalently, for every open U and closed $F \subset U$, there exists an open V with $F \subset V \subset \text{cl}(V) \subset U$).

Go over Exercises 1 and 2.

Go over Theorem 31.2: a subspace of regular space is regular; the product of regular spaces is regular. (Same for Hausdorff spaces, as proved last semester.) Note that it is false for the normal spaces. (Point to where the subspace part of the proof for the regular spaces brakes for the normal spaces.)

Go over Example 1: \mathbb{R}_K is Hausdorff but not regular.

Go over Exercise 4.

Go over Examples 2 (\mathbb{R}_ℓ is normal) and 3 ($(\mathbb{R}_\ell)^2$ is not normal).

Class of February 6:

Go over Theorem 7.8 and review how it was used in Example 3: $(\mathbb{R}_\ell)^2$ is not normal.

Note that the product of normal spaces need not be normal. Also, $(\mathbb{R}_\ell)^2$ is regular but not normal.

Latter we will prove that $(\mathbb{R}_\ell)^2$ is homeomorphic to a subspace of some normal spaces. So, a subspace of normal space need not be normal.

Section 32: Normal spaces

Show that every regular Lindelöf space is normal. This is Ex 4 page 205. Proof the same as for Thm 32.1.

Corollary: the product of two Lindelöf spaces need not be Lindelöf, justified by $(\mathbb{R}_\ell)^2$.

Thm 32.2: Every metrizable space is normal.

Thm 32.3: Every compact Hausdorff space is normal.

Go over Example 1.

Go over Exercise 1.

Students, try to solve Exercise 2 (not a homework).

Written assignment for Tuesday, February 11: Exercise 3, page 205.

Class of February 11:

Section 33: The Urysohn Lemma

Prove:

Urysohn Lemma: *If X is normal and A and B are closed disjoint subsets of X , then there exists continuous $f: X \rightarrow [0, 1]$ such that $f[A] \subseteq \{0\}$ and $f[B] \subseteq \{1\}$.*

Define completely regular (or $T_{3.5}$) spaces.

Prove Theorem 33.2.

Go over the expanded table:

	subspace	closed subspace	countable product	arbitrary product	continuous image
2nd countable	Y	Y	Y	N, $\mathbb{R}^{\text{uncountable}}$	N, Ex c
1st countable	Y	Y	Y	N, $\mathbb{R}^{\text{uncountable}}$	N, Ex b,c
separable	N, $L \subset (\mathbb{R}_\ell)^2$	N, $L \subset (\mathbb{R}_\ell)^2$	Y, 10 p. 194	N, 16 p. 195	Y, 11 p. 194
Lindelöf	N, \mathbb{R}_∞	Y	N, $(\mathbb{R}_\ell)^2$	N, $(\mathbb{R}_\ell)^2$	Y, 11 p. 194
compact	N, $[0, 1]; [0, 1]$	Y	Y, Tych. Thm	Y, Tych. Thm	Y
connected	N, $\mathbb{R} \setminus (0, 1)$	N, $\mathbb{R} \setminus (0, 1)$	Y	Y	Y
Hausdorff	Y	Y	Y	Y	N, Ex a
regular	Y	Y	Y	Y	N, Ex a
completely reg	Y	Y	Y	Y	N, Ex a
normal	N, p. 203	Y, 1 p. 205	N, $(\mathbb{R}_\ell)^2$	N, $(\mathbb{R}_\ell)^2$	N, Ex a
metrizable	Y	Y	Y	N, $\mathbb{R}^{\text{uncountable}}$	N, Ex a

Answers

Example. For a set X let τ_d be a discrete topology on X and \mathcal{T} an arbitrary topology on X . Then a function $f: \langle X, \tau_d \rangle \rightarrow \langle X, \mathcal{T} \rangle$, given by $f(x) = x$, is continuous bijection.

- (a) If $\mathcal{T} = \{\emptyset, X\}$ is anti-discrete topology and $X = \mathbb{N}$, then domain of f is metric, while $f[X]$ is not Hausdorff.
- (b) If $X = \mathbb{R}^\omega$ and \mathcal{T} is a box topology, then domain of f is first countable (as metric), while $f[X]$ is not first countable.
- (c) Let $X = \mathbb{N}$ and \mathcal{T} be such that $\langle X, \mathcal{T} \rangle$ is not second countable. (This will be proved together with Tychonoff's theorem.) Then domain of f is second countable, while $f[X]$ is not (since it is not first countable).

Class of February 13:

Recall

- **Urysohn Lemma:** *If X is normal and A and B are closed disjoint subsets of X , then there exists continuous $f: X \rightarrow [0, 1]$ such that $f[A] \subseteq \{0\}$ and $f[B] \subseteq \{1\}$.*
- X is *completely regular* (or a $T_{3.5}$ space) provided it is a T_1 space and: for every closed set K in X and $x \in X \setminus K$ there exists a continuous $f: X \rightarrow [0, 1]$ such that $f(x) = 0$ and $f[K] \subseteq \{1\}$.
- normality implies complete regularity but not conversely (ex: $(\mathbb{R}_l)^2$);
- complete regularity implies regularity but not conversely (example to be given latter);
- Thm 33.2: Subspace of a completely regular space is completely regular.

New material:

Thm 33.2: Product of completely regular spaces is completely regular.

Go over a part of Exercise 4, page 213:

- *If $f: X \rightarrow [0, 1]$ is continuous, then $A = f^{-1}(0)$ is a G_δ -set (that is, A is an intersection of countably many open sets).*

Written assignment for Tuesday, February 18: A more difficult direction of Exercise 4, page 213:

- (a) Prove that if X is normal, then for every closed G_δ set $A \subset X$ there is a continuous $f: X \rightarrow [0, 1]$ such that $f^{-1}(0) = A$.

Exercise 5, page 213 (a version of Urysohn Lemma): Let X be normal. *There exists a continuous $f: X \rightarrow [0, 1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$ if, and only if, A and B are disjoint closed G_δ sets.*

PROOF. “ \implies ” follows from •.

“ \impliedby ” By (a) there exists continuous functions $f_A, f_B: X \rightarrow [0, 1]$ with $f_A^{-1}(0) = A$ and $f_B^{-1}(0) = B$. Then $f(x) = \frac{f_A(x)}{f_A(x) + f_B(x)}$ is as needed. ■

Go over Exercise 2, page 212 and Exercise 5 page 205.

Suggestion: Look over Exercises 1 and 3, page 212; 7 and 8, page 213.

Homework assigned on January 17, 2013

Ex. 1. If X is Lindelöf and Y is compact, then $X \times Y$ is Lindelöf.

PROOF. Let \mathcal{U} be an open cover of $X \times Y$. For every $x \in X$ the section $\{x\} \times Y$ is compact, so there is a finite family $\mathcal{U}_x \subset \mathcal{U}$ with $\{x\} \times Y \subset \bigcup \mathcal{U}_x$. By “tube lemma” there is an open $U_x \subset X$ with $\{x\} \times Y \subset U_x \times Y \subset \bigcup \mathcal{U}_x$. Then, $\{U_x : x \in X\}$ is an open cover of X . Since X is Lindelöf it has a countable subcover $\{U_{x_i} : i = 0, 1, 2, \dots\}$. Let $\mathcal{U}_0 = \bigcup_{i=0}^{\infty} \mathcal{U}_{x_i}$. It is countable, as a countable union of finite sets and $\mathcal{U}_0 \subset \mathcal{U}$. It is enough to prove that \mathcal{U}_0 covers $X \times Y$. This is so, since $X \times Y \subset \bigcup_{i=0}^{\infty} U_{x_i} \times Y \subset \bigcup_{i=0}^{\infty} \bigcup \mathcal{U}_{x_i} = \bigcup \mathcal{U}_0$.

Class of February 18:

Prove **Urysohn metrization theorem:**

Every regular second countable space X is metrizable.

1. Notice that every regular second countable space is normal (as it is regular Lindelöf), so we can use Urysohn Lemma.
2. Prove that there exists a countable family \mathcal{F} of continuous functions $f: X \rightarrow [0, 1]$ such that:
 - (*) For every open $U \subset X$ and $x \in U$ there is an $f \in \mathcal{F}$ such that $f(x) > 0$ and $f[X \setminus U] \subset \{0\}$.

A family \mathcal{F} of continuous functions $f: X \rightarrow \mathbb{R}$ satisfying (*) is said to *separate points from closed sets* in X .

3. Prove that (Thm 34.2): *For any T_1 space X family $\{f_\alpha\}_{\alpha \in J}$ separating points from closed sets in X the mapping $F: X \rightarrow \mathbb{R}^J$, $F(x)(\alpha) = f_\alpha(x)$, is an imbedding.*
4. Notice that, by 1 and 2, our regular second countable space X can be imbedded into \mathbb{R}^ω . Since \mathbb{R}^ω is metrizable, so is X .

Notice that (second version of the proof of thm 34.1):

Every regular second countable space X can be imbedded into \mathbb{R}^ω considered with the uniform topology.

PROOF. By 2, there exists a family $\{f_n\}_{n=1}^\infty$ separating points from closed sets in X , with $f_n: X \rightarrow [0, 1]$. Replacing f_n with f_n/n , if necessary, we can assume that $f_n: X \rightarrow [0, 1/n]$. Therefore, by 3, there is an imbedding F of X into $T = \prod_{n=1}^\infty [0, 1/n]$.

Then, the theorem follows from the fact that

The uniform topology on T coincides with the product topology.

State and prove Theorem 34.3.

Written assignment for Tuesday, February 25: Solve ex. 3 page 218.

Class of February 20:

Recall

- **Urysohn Lemma:** *If X is normal and A and B are closed disjoint subsets of X , then there exists continuous $f: X \rightarrow [0, 1]$ such that $f[A] \subseteq \{0\}$ and $f[B] \subseteq \{1\}$.*
- **Urysohn metrization theorem:** *Every regular second countable space X is metrizable.*
- *X is completely regular iff X is homeomorphic to a subspace of $[0, 1]^J$ for some J .*

New material:

State and prove **Tietze Extension Theorem:** *If X is normal, $K \subset X$ is closed, and $f: K \rightarrow [0, 1]$ is continuous, then f can be extended to a continuous $F: X \rightarrow [0, 1]$.*

Prove that Tietze Extension Theorem is true, when the interval $[0, 1]$ is replaced with \mathbb{R} .

Note that, for example, a circle does not have universal extension property (using Brouwer Fixed Point Theorem).

Class of February 25:

Give the individualized problem sets, from the old Topology Entrance Exams.

Mid term Quiz: next class.

Go over exercises 1, 2, 4, and 5 page 218.

Go over Exercise 5(a), page 223.

Solve the following problems from old Topology Entrance Exams, TEE:

Ex. 2. (TEE Fall 2012) Consider the following subsets, \vdash and \models , of \mathbb{R}^2 , where \mathbb{R}^2 is endowed with the standard topology:

$$\vdash = (\{0\} \times [-2, 2]) \cup ([0, 2] \times \{0\}) \quad \& \quad \models = (\{0\} \times [-2, 2]) \cup ([0, 2] \times \{-1, 1\}).$$

Prove, or disprove the following:

- (i) There exists a continuous function from \vdash onto \models .
- (ii) There exists a continuous function from \models onto \vdash .

Your argument must be precise, but no great details are necessary.

Comment: The solution for Ex 2 was only sketched.

Ex. 3. (TEE Spring 2013) Consider the line \mathbb{R} with the topology τ generated by the family:

$$\mathcal{B} = \{[a, b) : a \in \mathbb{Q} \ \& \ b \in \mathbb{R} \ \& \ a < b\},$$

where \mathbb{Q} stands for the set of rational numbers. Prove or disprove that $X = \langle \mathbb{R}, \tau \rangle$ is normal.

SOLUTION. First notice that every set in \mathcal{B} is a union of sets from the family $\mathcal{B}_0 = \{[a, b) : a, b \in \mathbb{Q} \ \& \ a < b\}$, so \mathcal{B}_0 is a basis for τ . In particular, X is second countable. It is regular, since every set in \mathcal{B} is clopen. Now, normality of X follows either from the Uryshon's metrization theorem, or from the fact that every regular Lindelöf space is normal.

Ex. 4. (TEE Fall 2012) For the topologies τ and σ on \mathbb{R} let symbol $C(\tau, \sigma)$ stand for the family of all continuous functions from $\langle \mathbb{R}, \tau \rangle$ into $\langle \mathbb{R}, \sigma \rangle$.

Let \mathcal{T}_s be the standard topology on \mathbb{R} and let \mathcal{T}_1 and \mathcal{T}_2 be two topologies on \mathbb{R} such that $C(\mathcal{T}_1, \mathcal{T}_2) = C(\mathcal{T}_s, \mathcal{T}_s)$. Show that:

- (i) $\mathcal{T}_2 \subseteq \mathcal{T}_1$, that is, \mathcal{T}_1 is finer than \mathcal{T}_2 .
- (ii) $\mathcal{T}_2 \neq \{\emptyset, \mathbb{R}\}$, that is, \mathcal{T}_2 is not trivial.
- (iii) $\langle \mathbb{R}, \mathcal{T}_1 \rangle$ is connected.

(Notice that $C(\mathcal{T}_1, \mathcal{T}_2) = C(\mathcal{T}_s, \mathcal{T}_s)$ does not imply that either of the topologies \mathcal{T}_1 and \mathcal{T}_2 must be equal to the standard topology \mathcal{T}_s .)

Comment: Ex 4 was not solved in class. Try to solve it at home.

Class of February 27:Administer *Mid term Quiz*

Students presentations of the solutions of some of the exercises from the old Topology Entrance Exams assigned to them. (If there are volunteers.)

Go over incomplete proof of the Tychonoff Theorem, page 231.

Definition of a *filter* on a set X : a non-empty family $\mathcal{F} \subset \mathcal{P}(X)$ (i.e., of subsets of X) such that:

- if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$;
- if $A \in \mathcal{F}$ and $A \subset C \subset X$, then $C \in \mathcal{F}$.

Filter is *proper* when $\emptyset \notin \mathcal{F}$, that is, when $\mathcal{F} \neq \mathcal{P}(X)$. Note that the filter is proper if, and only if, it has the finite intersection property, *fi*p. We will consider only proper filters.

Give an intuitive argument that any family having the finite intersection property can be extended to a maximal family having the finite intersection property. (Formal prove will be deduced from Zorn's Lemma.)

Class of March 4:

Students presentations of the solutions of some of the exercises from the old Topology Entrance Exams assigned to them.

Solved:

Ex. 5. (TEE Fall 2012) For the topologies τ and σ on \mathbb{R} let symbol $C(\tau, \sigma)$ stand for the family of all continuous functions from $\langle \mathbb{R}, \tau \rangle$ into $\langle \mathbb{R}, \sigma \rangle$.

Let \mathcal{T}_s be the standard topology on \mathbb{R} and let \mathcal{T}_1 and \mathcal{T}_2 be two topologies on \mathbb{R} such that $C(\mathcal{T}_1, \mathcal{T}_2) = C(\mathcal{T}_s, \mathcal{T}_s)$. Show that:

- $\mathcal{T}_2 \subseteq \mathcal{T}_1$, that is, \mathcal{T}_1 is finer than \mathcal{T}_2 .
- $\mathcal{T}_2 \neq \{\emptyset, \mathbb{R}\}$, that is, \mathcal{T}_2 is not trivial.
- $\langle \mathbb{R}, \mathcal{T}_1 \rangle$ is connected.

(Notice that $C(\mathcal{T}_1, \mathcal{T}_2) = C(\mathcal{T}_s, \mathcal{T}_s)$ does not imply that either of the topologies \mathcal{T}_1 and \mathcal{T}_2 must be equal to the standard topology \mathcal{T}_s .)

Class of March 6:

Students presentations of the solutions of some of the exercises from the old Topology Entrance Exams assigned to them.

Class of March 18:

Student presentation of the solutions of some of the exercises from the old Topology Entrance Exams assigned to them.

Loose discussion of Peano, Space-Filling Curve, based on Section 44.

Class of March 20:

Hand take home test, due on Tuesday, March 25.

Students presentations of the solutions of some of the exercises from the old Topology Entrance Exams assigned to them.

State Lemma 37.2. and the result that any any family having the finite intersection property can be extended to a maximal family having the finite intersection property.

Use the above results to prove Tychonoff Theorem.

Class of March 25:

Collect take home test.

Prove Lemma 37.2. Notice also that every maximal family having the finite intersection property is a proper filter.

State Zorn Lemma and Hausdorff Maximal Principle, section 11.

Prove, using Hausdorff Maximal Principle, that every family having the finite intersection property can be extended to a maximal family having the finite intersection property.

Class of March 27:

Students presentations of the solutions of some of the exercises from the old Topology Entrance Exams assigned to them.

Jump to

Section 43: Complete Metric Spaces

Definition of *Cauchy sequence* and *complete metric space*.

Lemma 43.1.

Theorem 43.2.

Lemma 43.3.

Class of April 1:

Possible students presentations of the solutions of some of the exercises from the old Topology Entrance Exams assigned to them.

Theorem 43.4.

Go over Examples 1 and 2.

Definition of *uniform metric*.

Theorem 43.5.

Define $\mathcal{C}(X, Y)$ and $\mathcal{B}(X, Y)$.

Theorem 43.6.

Definition of *sup metric* on the class $\mathcal{B}(X, Y)$ of all bounded functions from X into Y .

Class of April 3:

Students presentations of the solutions of some of the exercises from the old Topology Entrance Exams assigned to them.

Recall that for a metric space $\langle Y, d \rangle$:

$\mathcal{B}(X, Y) = \{f \in Y^X : f[X] \text{ is bounded in } \langle Y, d \rangle\}$,

$\mathcal{B}(X, Y)$ is closed in Y^X considered with the *sup metric*:

$\rho(f, g) = \sup\{d(f(x), g(x)) : x \in X\}$, equivalent to the uniform metric:

$\bar{\rho}(f, g) = \sup\{\bar{d}(f(x), g(x)) : x \in X\}$.

New material:

Go over Theorem 43.7 on the completion of a metric space. See also Exercise 9 for a different proof.

Class of April 8:

Students presentations of the solutions of some of the exercises from the old Topology Entrance Exams assigned to them.

Go over exercise 4, page 270.

Section 48: Baire spaces

Define Baire space in terms of countable union of nowhere dense sets.

Go over Example 2.

State and discuss Baire category theorem.

Class of April 10:

Students presentations of the solutions of some of the exercises from the old Topology Entrance Exams assigned to them.

Restate Theorem 48.2, Baire category theorem, and prove this.

Note that Lemma 48.3 is the exercise 4, page 270, we solved.

Go over Exercise 5 page 299. Note that it implies Lemma 48.4 (under the assumptions of Theorem 48.2); however the proof of the lemma in the text is simpler.

Go over Lemma 48.1: definition Baire space in terms of countable intersections of open dense sets.

Class of April 15:

Go over Theorem 48.5: *If X is a Baire space and Y is metric, then pointwise limit of $f_n: X \rightarrow Y$ is continuous on a dense G_δ set.*

If $X = [0, 1]$, the theorem resembles Egoroff's Theorem (its corollary): Under the same assumptions, *for every $\eta > 0$ there exists a closed set $A \subset [0, 1]$ of measure $\geq 1 - \eta$ such that $f \upharpoonright A$ is continuous.* Note that f need not be continuous at any $a \in A$!

Example. Let $A \subset [0, 1]$ be a closed set of measure $\geq 1 - \eta$. Define $f: [0, 1] \rightarrow \mathbb{R}$ as: $f(x) = 0$ for $x \in A$ and $f(x) = \sin(1/\text{dist}(x, A))$ for $x \notin A$. It is not difficult to find continuous f_n 's converging pointwise to f .

Then Egoroff's Theorem is satisfied by A , while Theorem 48.5 by its complement!

Fill the gaps: Construct closed $A \subset [0, 1]$ of measure $\geq 1 - \eta$: Describe continuous f_n 's converging to f : $f_n(x) = 0$ whenever $\text{dist}(x, A) \leq 1/(2\pi n)$ and $f_n(x) = \sin(1/\text{dist}(x, A))$ otherwise.

Class of April 17:

Section 49: Continuous nowhere-differentiable functions via Baire Category Theorem.

Class of April 21:

Exercise 7b, page 182-183: Let $\langle X, d \rangle$ be compact metric and let $f: X \rightarrow X$ be such that $d(f(x), f(y)) < d(x, y)$ for all distinct $x, y \in X$. Show that f has a fix point, that is, that there is an $x \in X$ with $f(x) = x$.

PROOF. Define, by induction on n , a sequence $\langle A_n \rangle_n$ by putting $A_0 = X$ and $A_{n+1} = f[A_n]$. It is easy to prove, by induction on n , that each A_n is compact (as continuous image of a compact set is compact) and that $A_{n+1} \subset A_n$ (since $A_1 = f[A_0] = f[X] \subset X = A_0$ and $B \subset C$ implies that $f[B] \subset f[C]$). It is also clear that all sets A_n are non-empty. Thus, $A_0 \supset A_1 \supset A_2 \supset \dots$ and the set $A = \bigcap_n A_n$ is compact non-empty.

Next, note that $f[A] = A$. Clearly, $f[A] \subset A$, since if $y \in f[A]$, then $y = f(x)$ for some $x \in \bigcap_n A_n$. Thus, for every n , $y = f(x) \in f[A_n] = A_{n+1}$, so $y \in \bigcap_n A_{n+1} = A$. To see $A \subset f[A]$ fix a $y \in A$. Then, for every n we have $y \in A_{n+1} = f[A_n]$, so there exists an $x_n \in A_n$ with $y = f[x_n]$. Since X is compact, there exists a subsequence $\langle x_{n_k} \rangle_k$ converging to an $x \in X$. Note that for every n all but finitely many terms of $\langle x_{n_k} \rangle_k$ belong to the closed set, so $x \in A_n$. Thus, $x \in \bigcap_n A_n = A$ and $y = \lim_k f(x_{n_k}) = f(\lim_k x_{n_k}) = f(x) \in f[A]$. So, indeed, $f[A] = A$.

To finish the proof, it is enough to show that A is a singleton, say $A = \{x\}$, since then $\{x\} = A = f[A] = \{f(x)\}$, so $f(x) = x$. So, by way of contradiction, assume that $A = f[A]$ has more than one element. Then, $\text{diam}(f[A]) > 0$. Since A is compact, there are $x_1, y_1 \in A = f[A]$ with $\text{diam}(A) = d(x_1, y_1)$. Let $x, y \in A$ be such that $x_1 = f(x)$ and $y_1 = f(y)$. Then, $\text{diam}(A) = d(x_1, y_1) = d(f(x), f(y)) < d(x, y)$, contradicting the definition of the diameter.

Remark Note that the result is false for complete metric spaces, as shown by $X = [1, \infty)$ and a function $f(x) = x + x^{-1}$. Indeed, f clearly has no fix point and for $x < y$ from X we have $d(f(x), f(y)) = (y - x) + y^{-1} - x^{-1} < |x - y|$.

On the other hand, if $\langle X, d \rangle$ is a complete metric and $f: X \rightarrow X$ is such that there exists an $\alpha \in [0, 1)$ for which $d(f(x), f(y)) \leq \alpha d(x, y)$ for all $x, y \in X$, then f has a fix point, since for every $x_0 \in X$ a sequence $x_0, f(x_0), f(f(x_0)), f(f(f(x_0))), \dots$ is Cauchy, and its limit is a fix point of f .

Go over Exercises 1, 2, 3, 6, 7, 8, 9, and 11 page 299.