Topology 2, Math 681, Spring 2015: Notes

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Class of January 12:

- Note that the last semester's abbreviated notes are still available on my web page, at:

  http://www.math.wvu.edu/~kcies/teach/Fall2014/Fall2014.html
- Next class, *January 14, I will give extended quiz*, to check the background of everybody. I will ask for definitions and, possibly, statements of some fundamental theorems.

Written assignment for Wednesday, January 21: *Only for the students that did not take the class last semester!* Solve the exercises from last semester final test.

Quick Review (Also definition of topological space and continuous maps)

- X is *Hausdorff* (or a T_2 *space*) provided for every distinct $x, y \in X$ there exists disjoint open sets $U, V \subset X$ such that $x \in U$ and $y \in V$.
- If X is a Hausdorff topological space, then any sequence $\langle x_n \rangle_{n=1}^{\infty}$ of points of X *converges* to at most one point in X.
- The product of two Hausdorff topological spaces is a Hausdorff space. A subspace of a Hausdorff topological space is a Hausdorff space.
- A function $f \colon X \to Y$ is *continuous* provided $f^{-1}(V)$ is open in X for every open subset V of Y.
- If $\mathcal{B}$ a basis for Y, then $f \colon X \to Y$ is continuous if, and only if, $f^{-1}(B)$ is open in X for every $B \in \mathcal{B}$.
- *Product topology* $\mathcal{T}_{prod}$ on $X = \prod_{\alpha \in J} X_\alpha$ is generated by subbasis $\mathcal{S}_{box} = \{\pi_\beta^{-1}(U_\beta)$ for all $\beta \in J$ and open subsets U_β of $X_\beta\}$
- *If $f_\alpha \colon A \to X_\alpha$ and $f \colon A \to X$ is given by $f(a)(\alpha) = f_\alpha(a)$, then*

  continuity of f implies the continuity of each f_α;

  continuity of all f_α's implies the continuity of $f \colon A \to \langle X, \mathcal{T}_{prod} \rangle$;

- A *metric space* is a pair $\langle X, d\rangle$, where d is a metric on X.
  $\mathcal{B}_d = \{B(x,\varepsilon)\colon x \in X \ \&\ \varepsilon > 0\}$ is a basis for a topology on X.
- $\mathcal{T}(\mathcal{B}_d)$ is the metric topology on X (for metric d).
- Subspace of a metric space is metric.
- Every metrizable space is Hausdorff.
- Finite and countable product of metric spaces is metrizable.
- A topological space $\langle X, \mathcal{T}\rangle$ is *first countable* provided for every $x \in X$ there exists a countable basis $\mathcal{B}_x$ of X at x;
- *Let X be a first countable space and let $A \subset X$. Then $x \in \mathrm{cl}(A)$ if, and only if, there is a sequence of points of A converging to x.*

Class of January 14:
Administer Review Quiz #0. *(Postponed till the next class.)*

Expect other quiz soon, possibly the next class. It may include the same questions as in Quiz #0. It will also concern connectivity, to be reviewed today.

Review, continuation.

- A topological space X is *connected* provided it **does not** exist a pair U, V of open, non-empty disjoint sets with $X = U \cup V$.
- **(Star Lemma)** Let $\{A_\alpha\}_{\alpha \in J}$ be a family of connected subspaces of X. If $\bigcap_{\alpha \in J} A_\alpha \neq \emptyset$, then $\bigcup_{\alpha \in J} A_\alpha$ is connected.

  Solve exercise 3, page 152.
- A closure of a connected space is connected.
- Finite product of connected spaces is connected: sketch proof.
- $\mathbb{R}^\omega$ with the product topology is connected: sketch proof.
- Continuous image of connected space is connected: sketch proof.
- $A \subset \mathbb{R}$ is connected if, and only if, A is convex (an interval).
- Intermediate Value Theorem.
- Definition of *path connectedness*.
- *Topologists sine curve*: it is connected but not path connected.

Show that the continuous function f from $[0, 1]$ to $[0, 1]$ has a fixed point.

Show that the shapes given by the characters A, I, and T are not homeomorphic to each other. What about the characters K and L, with respect to A, I, and T?

Next class:
what will be covered this semester;
new material: *compactness*.

Class of January 21:

Administer Review Quiz #0.

Sections 26 and 27 (with mixed order): compactness

Definition 1 Let Y be a subset of a topological space X. A family $\mathcal{U}$ of subsets of X is a *covering* of Y provided $Y \subset \bigcup \mathcal{U}$. A covering $\mathcal{U}$ of Y is an *open covering* of Y provided every $U \in \mathcal{U}$ is open in X.

Definition 2 A topological space X is *compact* provided for every open cover $\mathcal{U}$ of X there exists a finite subfamily $\mathcal{U}_0$ of $\mathcal{U}$ that covers X (i.e., $\mathcal{U}_0 \subset \mathcal{U}$ is finite and $X = \bigcup \mathcal{U}_0$). Such a family $\mathcal{U}_0$ will be referred to as a (finite) *subcover* of $\mathcal{U}$.

Note: Although subcover $\mathcal{U}_0$ of $\mathcal{U}$ is defined in term of a union $\bigcup \mathcal{U}_0$, this union usually does not belong to $\mathcal{U}$!

Go over Examples 1 and 4: Neither $\mathbb{R}$ nor $(0, 1]$ are compact.

Go over Examples 2 and 3: Every finite space X is compact. So is $X = \{L\} \cup \{a_n : n = 1, 2, 3, \ldots\} \subset \mathbb{R}$, provided $\lim_n a_n = L$.

Lemma 1 (Lemma 26.1)

Theorem 2 (Theorem 26.2) *Closed subspace of compact space is compact.*

Theorem 3 (Theorem 26.3) *Every compact subspace of a Hausdorff space is closed.*

Go over Example 6. Proof of the theorem is based on:

Lemma 4 (Lemma 26.4) *Let X be Hausdorff. For every compact subspace Y of X and avery $x \in X \setminus Y$ there exists disjoint open sets U and V in X such that $x \in U$ and $Y \subset V$.*

Stated only:

(Theorem 27.1) Every closed interval $[a, b]$ in $\mathbb{R}$ is compact.

(Corollary 27.3 for $\mathbb{R}$) A subspace X of $\mathbb{R}$ is compact if, and only if, it is closed and bounded.

To be covered this semester

Compactness

Countability axioms

- A topological space X is *first countable* (or *satisfies the first countability axiom*) provided for every $x \in X$ there exists a countable basis $\mathcal{B}_x$ of X at x.

- A topological space X is *second countable* (or *satisfies the second countability axiom*) provided X has a countable basis.

- A topological space X is *separable* provided X has a countable dense subset D, that is, such that $\mathrm{cl}(D) = X$.

- A topological space X is *Lindelöf* provided every open cover of X has a countable subcover.

Separation axioms

- (already seen) X is a T_0 *space* provided for every distinct $x, y \in X$ there exists an open set $U \subset X$ such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$ (i.e., such that U contains precisely one of the points x and y).

- (already seen) X is a T_1 *space* provided for every distinct $x, y \in X$ there exists an open set $U \subset X$ such that $x \in U$ and $y \notin U$.

- (already seen) X is *Hausdorff* (or a T_2 *space*) provided for every distinct $x, y \in X$ there exist disjoint open sets $U, V \subset X$ such that $x \in U$ and $y \in V$.

- (new) X is *regular* (or a T_3 *space*) provided it is a T_1 space and for every closed set K in X and $x \in X \setminus K$ there exist disjoint open sets $U, V \subset X$ such that $x \in U$ and $K \subset V$.

- (new) X is *normal* (or a T_4 *space*) provided it is a T_1 space and for every disjoint closed sets K and L in X there exist disjoint open sets $U, V \subset X$ such that $K \subset U$ and $L \subset V$.

- (new) X is *completely regular* (or a $T_{3\frac{1}{2}}$ *space*) provided it is a T_1 space and for every closed set K in X and $x \in X \setminus K$ there exist a continuous function $f\colon X \to [0,1]$ such that $f(x) = 0$ and $f[K] \subset \{1\}$.

Important related theorems

- *Urysohn Lemma*: Every T_4 space is a $T_{3\frac{1}{2}}$ space.
- *Tietze Extension Theorem*: If X is normal, $K \subset X$ is closed, and $f\colon K \to [0,1]$ is continuous, then f can be extended to a continuous $F\colon X \to [0,1]$.
- *Urysohn Metrization Theorem*: If X is regular and second countable, then it is metrizable.

The Tychonoff Theorem

- *Tychonoff Theorem*: Arbitrary product of compact spaces is compact.

Some material from Chapters 6-8

Class of January 26:

Recall that:

- X is *compact* provided every open cover of X has a finite subcover.
- For a subspace Y of a compact Hausdorff space
  Y is compact if, and only if, Y is closed in X.

New material:

Theorem 5 (Theorem 27.1) *Every closed interval $[a, b]$ in $\mathbb{R}$ is compact.*

Corollary 6 (Corollary 27.3 for $\mathbb{R}$) *A subspace X of $\mathbb{R}$ is compact if, and only if, it is closed and bounded.*

Theorem 7 (Thm 26.5) *Continuous image of a compact space is compact.*

Corollary 8 (Thm 27.4: Extreme Value Theorem for Intervals) *For every continuous function $f\colon [a, b] \to \mathbb{R}$ there exist $c, d \in [a, b]$ such that $f(c) \leq f(x) \leq f(d)$ for every $x \in [a, b]$.*

Remark: The conclusion is true when $[a, b]$ is replaced by any compact space.

Theorem 9 (Thm 26.7) *Finite product of compact spaces is compact.*

Remark: Actually, arbitrary product of compact spaces is compact. This is Tychonoff Theorem. But its proof is more difficult.

Proof of Theorem 9 based on **very important**

Lemma 10 (Lem 26.8: The Tube Lemma) *Let Y be compact and $x \in X$. If an open set W of $X \times Y$ contains $\{x\} \times Y$, then there is an open set U in X such that $\{x\} \times Y \subset U \times Y \subset W$.*

Corollary 11 (Corollary 27.3 for $\mathbb{R}^n$) *A subspace X of $\mathbb{R}^n$ is compact if, and only if, it is closed and bounded.*

Corollary 12 (Extreme Value Theorem for $\mathbb{R}^n$) *If R is a closed bounded subset of $\mathbb{R}^n$, then for every continuous function $f\colon R \to \mathbb{R}$ there exist $c, d \in R$ such that $f(c) \leq f(x) \leq f(d)$ for every $x \in R$.*

Written assignment for Monday, February 2: Ex 1 and 9, p. 170/171.

Class of January 28:

Recall that:

- Every closed interval $[a, b]$ in $\mathbb{R}$ is compact.
- A continuous image of a compact space is compact.
- **The Tube Lemma:** Let Y be compact and $x \in X$. If an open set W of $X \times Y$ contains $\{x\} \times Y$, then there is an open set U in X such that $\{x\} \times Y \subset U \times Y \subset W$.
- Finite product of compact spaces is compact.
- A subspace X of $\mathbb{R}^n$ is compact if, and only if, it is closed and bounded.
- **(Extreme Value Theorem for $\mathbb{R}^n$)** If R is a closed bounded subset of $\mathbb{R}^n$, then for every continuous function $f\colon R \to \mathbb{R}$ there exist $c, d \in R$ such that $f(c) \leq f(x) \leq f(d)$ for every $x \in R$.

Briefly reviewed the proofs of the results above.

New material:

Go over Exercises 2, 3, 4, and 5 section 26.

Went over the solutions of the exercises from the final test from Fall, 2014.

Class of February 2:

Theorem 13 (Theorem 26.6) *If $f: X \to Y$ is a continuous bijection, X is compact, and Y is Hausdorff, then f is a homeomorphism.*

Definition 3 A collection $\mathcal{C}$ of subsets of X has *finite intersection property, fip,* provided $\bigcap \mathcal{C}_0 \neq \emptyset$ for every finite $\mathcal{C}_0 \subset \mathcal{C}$.

Theorem 14 (Thm 26.9) *X is compact if, and only if, $\bigcap \mathcal{C} \neq \emptyset$ for every family $\mathcal{C}$ of closed subsets of X having fip.*

Definition 4 A point x in a topological space X is an *isolated point* provided $\{x\}$ is open in X.

Theorem 15 (Thm 27.7) *Let X be a compact Hausdorff space. If X has no isolated points, then X is uncountable.*

Written assignment for Monday, February 9: Ex 4 p. 178. (It has a short, easy solution.)

Class of February 4:

Recall that:

- If $f\colon X \to Y$ is a continuous bijection, X is compact, and Y is Hausdorff, then f is a homeomorphism.
- X is compact if, and only if, $\bigcap \mathcal{C} \neq \emptyset$ for every family $\mathcal{C}$ of closed subsets of X having fip.
- Every compact Hausdorff with no isolated points is uncountable.

New material:

Go over Exercise 5 page 178.

Go over Exercises 6 and 7 from section 26.

Definition 5 A function f from a metric space $\langle X, d\rangle$ into a metric space $\langle Y, \rho\rangle$ is said to be *uniformly continuous* provided for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $x_1, x_2 \in X$

$$d(x_0, x_1) < \delta \Longrightarrow \rho(f(x_0), f(x_1)) < \varepsilon.$$

Prove that $d(\cdot, A)\colon X \to \mathbb{R}$ defined as $d(x, A) = \inf_{a\in A} d(x, a)$ is continuous for every non-empty $A \subset X$.

Theorem 16 *Let f be a continuous function from a compact metric space $\langle X, d\rangle$ into a metric space $\langle Y, \rho\rangle$. Then f is uniformly continuous.*

Prove using the Lebesgue number lemma, where a *diameter* of a subset D of a metric space $\langle X, d\rangle$ is defined as $\mathrm{diam}(D) = \sup\{d(x, y)\colon x, y \in D\}$.

Lemma 17 (Lem 27.3) *Let $\mathcal{A}$ be an open cover of a metric space $\langle X, d\rangle$. If X is compact, then there exists a $\delta > 0$, known as a* **Lebesgue number**, *such that for every $D \subset X$ of diameter $< \delta$, there exists an $A \in \mathcal{A}$ with $D \subset A$.*

Written assignment for Monday, February 9: Ex 11 p. 171. (Hint, given in the book, is very good. Notice, that open sets U and V suggested in this hint exist by Ex 5, which you can use without a proof.)

Other suggested exercises (to solve at home, no homework):

- 2 page 177 and 6 page 178;
- also, from section 26: exercises 8 and 10.

Class of February 9:

Section 28: Limit Point Compactness

Definition 6 A space X is *limit point compact* provided every infinite subset of X has a limit point.

Theorem 18 (Thm 28.1) *If X is compact, then X is limit point compact, but not conversely.*

Go over Examples 1 and 2.

Definition 7 A space X is *sequentially compact* provided every sequence in X has a convergent subsequence.

Theorem 19 (Thm 28.2) *For a metrizable space X, the following are equivalent:*

(1) X is compact.

(2) X is limit point compact.

(3) X is sequentially compact.

Remark: "(2) implies (3)" requires only first countability of X.

Go over Exercises 1 and 2 page 181.

Class of February 11:

Recall that:

- X is: *limit point compact* provided every infinite subset of X has a limit point; *sequentially compact* provided every sequence in X has a convergent subsequence.
- **Thm** For metrizable spaces, the three notions, *compactness*, *limit point compactness*, and *sequential compactness*, are equivalent.

Mention Example 3: the space satisfies (1)–(3), but is not first countable, so not metrizable.

Discuss:
Exercise 7 pages 181–182.

Written assignment for Monday, February 16: Ex 3 p. 181.

Class of February 18:

Class of February 16 was cancelled, due to weather conditions.

Section 29: Local Compactness

Definition 8 A space X is *locally compact* provided every $x \in X$ there is an open set $U \ni x$ such that $\mathrm{cl}(U)$ is compact.

Compact implies locally compact.
Go over Examples 1 and 2.
$\mathbb{Q}$ is not locally compact: Exercise 1 page 186.
State and prove Theorem 29.1.
Define *one-point compactification* of a locally compact space.
Go over Example 4.

Time permitting
State and prove Theorem 29.2.
State and prove Corollary 29.3.
State and prove Corollary 29.4.
Go over Exercise 3 page 186.

Time permitting, consider going over (not covered):
Exercises 2 page 177 and 6 page 178;
from section 26 – Exercises 8 and 10.

Next class we will start new chapter:
Countability and Separation Axioms

Class of February 23:

Section 30: The Countability Axioms

The next definition and theorem were covered last semester.

Definition 9 A topological space X is *first countable* (or *satisfies the first countability axiom*) provided for every $x \in X$ there exists a countable basis $\mathcal{B}_x$ of X at x.

Theorem 20 *Let X be a first countable topological space and let $A \subset X$. Then $x \in \mathrm{cl}(A)$ if, and only if, there is a sequence of points of A converging to x. Moreover, the implication "$\Longleftarrow$" does not require the assumption of first countability.*

New material:

Definition 10 A topological space X is *second countable* (or *satisfies the second countability axiom*) provided X has a countable basis.

Go over Examples 1 and 2.
Go over Theorem 30.2.

Definition 11

- A subset A of a space X is *dense (in X)* provided $\mathrm{cl}(A) = X$.
- A topological space X is *separable* provided X has a countable dense subset D, that is, such that $\mathrm{cl}(D) = X$.
- A topological space X is *Lindelöf* provided every open cover of X has a countable subcover.

Go over Theorem 30.3.

Written assignment for Wednesday, February 25: Exercise 14, page 194. Hint: use the ideas form the proof, that the product of two compact spaces is compact.

Go over Examples 3 and 4. (Very important!)

Mid term test will be in class on Monday, March 9.
I can extend the test time for up to two hours.

Class of February 25:

Mid term test will be in class on Monday, March 9. I can extend the test time for up to two hours.

The solutions of all homework assignments will be given to you by Wednesday, March 4. No rewrites will be accepted after that.

Recall that:

- A topological space X is: *second countable* provided X has a countable basis; *separable* provided X has a countable dense subset; *Lindelöf* provided every open cover of X has a countable subcover.
- Second countability implies: first countability, separability, and Lindelöf property. None of these implications can be reversed, as proved by $\mathbb{R}_l$.
- Product of Lindelöf spaces need not be Lindelöf, as proved by $\mathbb{R}_l$.

New material:

Review Example 3: $\mathbb{R}_\ell$ is Lindelöf. Main steps:

- Note, that it is enough to consider only the covers $\mathcal{V}$ of $\mathbb{R}_\ell$ composed of the standard basic open sets, that is, $\mathcal{V}$ of the form $\{[a_\xi, b_\xi)\}_{\xi \in J}$.
- Note that $\mathcal{U} = \{(a_\xi, b_\xi)\}_{\xi \in J}$ is an open (w.r.t. the standard topology) cover of $C = \bigcup_{\xi \in J}(a_\xi, b_\xi)$.
- Since C (with the standard topology) is second countable, we can find countable $J_0 \subset J$ with $C = \bigcup_{\xi \in J_0}(a_\xi, b_\xi)$.
- Prove that $\mathbb{R} \setminus C$ is countable. Then, find countable $J_1 \subset J$ with $\mathbb{R} \setminus C \subset \bigcup_{\xi \in J_1}[a_\xi, b_\xi)$.
- Notice that $\mathcal{V}_0 = \{[a_\xi, b_\xi)\}_{\xi \in J_0 \cup J_1} \subset \mathcal{V}$ is countable and covers $\mathbb{R} = C \cup (\mathbb{R} \setminus C)$.

Go over Exercises 2, 4, 5, and 1, page 194. Suggested Exercises to examine by the students: 12, 13, and 16, page 194.

Written assignment for Monday, March 2: Exercise 10, page 194:

Show that a countable product of separable spaces is separable.

Class of March 2:

- A topological space X is *second countable* (or *satisfies the second countability axiom*) provided X has a countable basis.
- A topological space X is *separable* provided X has a countable dense subset D, that is, such that $\mathrm{cl}(D) = X$.
- A topological space X is *Lindelöf* provided every open cover of X has a countable subcover.

	subspace	closed subspace	countable product	arbitrary product	continuous image
compact	N, $[0,1)$; $[0,1]$	Y	Y, Tych. Thm	Y, Tych. Thm	Y
connected	N, $\mathbb{R}\setminus(0,1)$	N, $\mathbb{R}\setminus(0,1)$	Y	Y	Y
Hausdorff	Y	Y	Y	Y	N, Ex a
2nd count	Y	Y	Y	N, $\mathbb{R}^{\text{uncount}}$ p. 133	N, Ex c
1st count	Y	Y	Y	N, $\mathbb{R}^{\text{uncount}}$ p. 133	N, Ex b,c
separable	N, $L \subset (\mathbb{R}_\ell)^2$	N, $L \subset (\mathbb{R}_\ell)^2$	Y, 10 p. 194	N, 16 p. 195	Y, 11 p. 194
Lindelöf	N, $\mathbb{R}_\infty$	Y, 9 p. 194	N, $(\mathbb{R}_\ell)^2$	N, $(\mathbb{R}_\ell)^2$	Y, 11 p. 194
metrizable	Y	Y	Y	N, $\mathbb{R}^{\text{uncount}}$ p. 133	N, Ex a

Here the space X_∞, in particular $\mathbb{R}_\infty$, is the one point compactification of a discrete space X, that is, $X_\infty = X \cup \{\infty\}$, where $\infty \notin X$, has the topology $\tau = \mathcal{P}(X) \cup \{X_\infty \setminus F : F \text{ is a finite subset of } X\}$.

Example. For a set X let τ_d be a discrete topology on X and $\mathcal{T}$ an arbitrary topology on X. Then a function $f : \langle X, \tau_d\rangle \to \langle X, \mathcal{T}\rangle$, given by $f(x) = x$, is continuous bijection.

(a) If $\mathcal{T} = \{\emptyset, X\}$ is anti-discrete topology and $X = \mathbb{N}$, then domain of f is metric, while $f[X]$ is not Hausdorff.

(b) If $X = \mathbb{R}^\omega$ and $\mathcal{T}$ is a box topology, then domain of f is first countable (as metric), while $f[X]$ is not first countable.

(c) Let $X = \mathbb{N}$ and $\mathcal{T}$ be such that $\langle X, \mathcal{T}\rangle$ is not second countable. (This will be proved together with Tychonoff's theorem.) Then domain of f is second countable, while $f[X]$ is not (since it is not first countable).

Go over Exercises 9 and 11 page 194, pointing their role in the table. Point to Exercise 16, page 194. Read the details.

Class of March 9:

Class of March 4 was cancelled, due to weather conditions.

The mid term test is postponed till Wednesday, March 11. You will be able to stay longer, for up to two hours, for the mid term test.

Possibly, review Example 2 page 133.

Is there any need for more review?

Section 31: The Separation Axioms

- (already seen) X is a T_0 *space* provided for every distinct $x, y \in X$ there exists an open set $U \subset X$ such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$ (i.e., such that U contains precisely one of the points x and y).
- (already seen) X is a T_1 *space* provided for every distinct $x, y \in X$ there exists an open set $U \subset X$ such that $x \in U$ and $y \notin U$.
- (already seen) X is *Hausdorff* (or a T_2 *space*) provided for every distinct $x, y \in X$ there exist disjoint open sets $U, V \subset X$ such that $x \in U$ and $y \in V$.
- (new) X is *regular* (or a T_3 *space*) provided it is a T_1 space and for every closed set K in X and $x \in X \setminus K$ there exist disjoint open sets $U, V \subset X$ such that $x \in U$ and $K \subset V$.
- (new) X is *normal* (or a T_4 *space*) provided it is a T_1 space and for every disjoint closed sets K and L in X there exist disjoint open sets $U, V \subset X$ such that $K \subset U$ and $L \subset V$.

Go over Lemma 31.1.

Go over Exercises 1 and 2.

Go over Theorem 31.2: a subspace of regular space is regular; the product of regular spaces is regular. (Same for Hausdorff spaces, as proved last semester.) Note that it is false for the normal spaces. (Point to where the subspace part of the proof for the regular spaces brakes for the normal spaces.)

Class of March 11:

In class, mid term test.

Class of March 16:

Return Mid Term Test. Discuss the solutions of the test's problems.

Recall

- X is *regular* (or a T_3 *space*) provided it is a T_1 space and: for every closed set K in X and $x \in X \setminus K$ there exist disjoint open sets $U, V \subset X$ such that $x \in U$ and $K \subset V$ (equivalently, for every open U and $x \in U$, there an exists open V with $x \in V \subset \text{cl}(V) \subset U$).
- X is *normal* (or a T_4 *space*) provided it is a T_1 space and: for every disjoint closed sets K and L in X there exist disjoint open sets $U, V \subset X$ such that $K \subset U$ and $L \subset V$ (equivalently, for every open U and closed $F \subset U$, there exists an open V with $F \subset V \subset \text{cl}(V) \subset U$).
- Theorem 31.2: a subspace of regular space is regular; the product of regular spaces is regular.

Go over Example 1: $\mathbb{R}_K$ is Hausdorff but not regular.
Go over Exercise 4.
Go over Example 2: $\mathbb{R}_\ell$ is normal.
Go over Theorem 7.8 (to be covered next class).
Use it, in Example 3, to show that $(\mathbb{R}_\ell)^2$ is not normal.

Note that the product of normal spaces need not be normal. Also, $(\mathbb{R}_\ell)^2$ is regular but not normal.

Latter we will prove that $(\mathbb{R}_\ell)^2$ is homeomorphic to a subspace of some normal spaces. So, a subspace of normal space need not be normal.

Written assignment for Monday, March 30: Exercise 5, page 199.

Class of March 18:

Recall, that to finish the proof that $(\mathbb{R}_\ell)^2$ is not normal, we need to show that there is no injective function from $\mathcal{P}(L)$ into $\mathcal{P}(D)$, where L is a line and D is countable.

Look at page 198 and go over Theorem 7.8.

Section 32: Normal spaces

Show that every regular Lindelöf space is normal. This is Ex 4 page 205. Proof the same as for Thm 32.1.

Corollary: the product of two Lindelöf spaces need not be Lindelöf, justified by $(\mathbb{R}_\ell)^2$.

Thm 32.2: Every metrizable space is normal.

Thm 32.3: Every compact Hausdorff space is normal.

Go (briefly) over Example 1.

Go over Exercise 1.

Students, try to solve Exercise 3 (not a homework).

Written assignment for Monday, March 30: Exercise 2, page 205. (You have also assigned Exercise 5, page 199, for the same date.)

Class of March 30:

Section 33: The Urysohn Lemma

Prove:

Urysohn Lemma: *If X is normal and A and B are closed disjoint subsets of X, then there exists continuous $f\colon X \to [0,1]$ such that $f[A] \subseteq \{0\}$ and $f[B] \subseteq \{1\}$.*

Define *completely regular* (or $T_{3.5}$) spaces.

Prove Theorem 33.2.

Go over the expanded table:

	subspace	closed subspace	countable product	arbitrary product	continuous image
2nd countable	Y	Y	Y	N, $\mathbb{R}^{\text{uncountable}}$	N, Ex c
1st countable	Y	Y	Y	N, $\mathbb{R}^{\text{uncountable}}$	N, Ex b,c
separable	N, $L \subset (\mathbb{R}_\ell)^2$	N, $L \subset (\mathbb{R}_\ell)^2$	Y, 10 p. 194	N, 16 p. 195	Y, 11 p. 194
Lindelöf	N, $\mathbb{R}_\infty$	Y	N, $(\mathbb{R}_\ell)^2$	N, $(\mathbb{R}_\ell)^2$	Y, 11 p. 194
compact	N, $[0,1); [0,1]$	Y	Y, Tych. Thm	Y, Tych. Thm	Y
connected	N, $\mathbb{R} \setminus (0,1)$	N, $\mathbb{R} \setminus (0,1)$	Y	Y	Y
Hausdorff	Y	Y	Y	Y	N, Ex a
regular	Y	Y	Y	Y	N, Ex a
completely reg	Y	Y	Y	Y	N, Ex a
normal	N, p. 203	Y, 1 p. 205	N, $(\mathbb{R}_\ell)^2$	N, $(\mathbb{R}_\ell)^2$	N, Ex a
metrizable	Y	Y	Y	N, $\mathbb{R}^{\text{uncountable}}$	N, Ex a

Answers

Example. For a set X let τ_d be a discrete topology on X and $\mathcal{T}$ an arbitrary topology on X. Then a function $f\colon \langle X, \tau_d \rangle \to \langle X, \mathcal{T} \rangle$, given by $f(x) = x$, is continuous bijection.

(a) If $\mathcal{T} = \{\emptyset, X\}$ is anti-discrete topology and $X = \mathbb{N}$, then domain of f is metric, while $f[X]$ is not Hausdorff.

(b) If $X = \mathbb{R}^\omega$ and $\mathcal{T}$ is a box topology, then domain of f is first countable (as metric), while $f[X]$ is not first countable.

(c) Let $X = \mathbb{N}$ and $\mathcal{T}$ be such that $\langle X, \mathcal{T} \rangle$ is not second countable. (This will be proved together with Tychonoff's theorem.) Then domain of f is second countable, while $f[X]$ is not (since it is not first countable).

Class of April 1:

Recall

- **Urysohn Lemma:** *If X is normal and A and B are closed disjoint subsets of X, then there exists continuous $f: X \to [0,1]$ such that $f[A] \subseteq \{0\}$ and $f[B] \subseteq \{1\}$.*
- X is *completely regular* (or a $T_{3.5}$ *space*) provided it is a T_1 space and: for every closed set K in X and $x \in X \setminus K$ there exists a continuous $f: X \to [0,1]$ such that $f(x) = 0$ and $f[K] \subseteq \{1\}$.
- Thm 33.2: Subspace of a completely regular space is completely regular. Product of completely regular spaces is completely regular.

New material:

- normality implies complete regularity but not conversely (ex: $(\mathbb{R}_l)^2$);
- complete regularity implies regularity but not conversely (example to be given latter);

Reprove: Product of completely regular spaces is completely regular.

Go over a part of Exercise 4, page 213:

(i) *If $f: X \to [0,1]$ is continuous, then $A = f^{-1}(0)$ is a G_δ-set (that is, A is an intersection of countably many open sets).*

Written assignment for Monday, April 6: A more difficult direction of Exercise 4, page 213:

(a) Prove that if X is normal, then for every closed G_δ set $A \subset X$ there is a continuous $f: X \to [0,1]$ such that $f^{-1}(0) = A$.

Exercise 5, page 213 (a version of Urysohn Lemma): Let X be normal. *There exists a continuous $f: X \to [0,1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$ if, and only if, A and B are disjoint closed G_δ sets.*

PROOF. "$\Longrightarrow$" follows from (i).

"$\Longleftarrow$" By (a) there exists continuous functions $f_A, f_B: X \to [0,1]$ with $f_A^{-1}(0) = A$ and $f_B^{-1}(0) = B$. Then $f(x) = \frac{f_A(x)}{f_A(x)+f_B(x)}$ is as needed. ∎

Go over Exercise 2, page 212 and Exercise 5 page 205.

Suggestion: Look over Exercises 1 and 3, page 212; 7 and 8, page 213.

Class of April 6:

Prove **Urysohn metrization theorem**:
Every regular second countable space X is metrizable.

1. Notice that every regular second countable space is normal (as it is regular Lindelöf), so we can use Urysohn Lemma.

2. Prove that there exists a countable family $\mathcal{F}$ of continuous functions $f: X \to [0,1]$ such that:

   (∗) For every open $U \subset X$ and $x \in U$ there is an $f \in \mathcal{F}$ such that $f(x) > 0$ and $f[X \setminus U] \subset \{0\}$.

   A family $\mathcal{F}$ of continuous functions $f: X \to \mathbb{R}$ satisfying (∗) is said to *separate points from closed sets* in X.

3. Prove that (Thm 34.2): *For any T_1 space X family $\{f_\alpha\}_{\alpha \in J}$ separating points from closed sets in X the mapping $F: X \to \mathbb{R}^J$, $F(x)(\alpha) = f_\alpha(x)$, is an imbedding.*

4. Notice that, by 1 and 2, our regular second countable space X can be imbedded into $\mathbb{R}^\omega$. Since $\mathbb{R}^\omega$ is metrizable, so is X.

Notice that (second version of the proof of thm 34.1):
Every regular second countable space X can be imbedded into $\mathbb{R}^\omega$ considered with the uniform topology.

PROOF. By 2, there exists a family $\{f_n\}_{n=1}^\infty$ separating points from closed sets in X, with $f_n: X \to [0,1]$. Replacing f_n with f_n/n, if necessary, we can assume that $f_n: X \to [0, 1/n]$. Therefore, by 3, there is an imbedding F of X into $T = \prod_{n=1}^\infty [0, 1/n]$.

Then, the theorem follows from the fact that
The uniform topology on T coincides with the product topology.

State and prove Theorem 34.3.

Class of April 8:

The enclosed two pages represent the rest of material that we will cover this semester. During the remaining time, we will review the material (possibly, in order not matching the suggested dates).

Recall

- **Urysohn Lemma:** *If X is normal and A and B are closed disjoint subsets of X, then there exists continuous $f : X \to [0, 1]$ such that $f[A] \subseteq \{0\}$ and $f[B] \subseteq \{1\}$.*
- **Urysohn metrization theorem**: *Every regular second countable space X is metrizable.*
- *X is completely regular iff X is homeomorphic to a subspace of $[0, 1]^J$ for some J.*

New material:

State and prove **Tietze Extension Theorem**: I*f X is normal, $K \subset X$ is closed, and $f : K \to [0, 1]$ is continuous, then f can be extended to a continuous $F : X \to [0, 1]$.*

Prove that Tietze Extension Theorem is true, when the interval $[0, 1]$ is replaced with $\mathbb{R}$.

Note that, for example, a circle does not have universal extension property (using Brower Fixed Point Theorem).

Go over Exercise 5(a), page 223.
Time permitting, go over exercises 1, 2, 3, 4, and 5 page 218.

Class of April 13: Tychonoff Theorem

Go over incomplete proof, from page 231, of the Tychonoff Theorem: *Arbitrary product of compact spaces is compact.*

Definition of a *filter* on a set X: *a non-empty family* $\mathcal{F} \subset \mathcal{P}(X)$ *(i.e., of subsets of* X*) such that:*

- *if* $A, B \in \mathcal{F}$*, then* $A \cap B \in \mathcal{F}$*;*
- *if* $A \in \mathcal{F}$ *and* $A \subset C \subset X$*, then* $C \in \mathcal{F}$*.*

Filter is *proper* when $\emptyset \notin \mathcal{F}$, that is, when $\mathcal{F} \neq \mathcal{P}(X)$. Note that the filter is proper if, and only if, it has the finite intersection property, fip. We will consider only proper filters.

Give an intuitive argument that any family having the finite intersection property can be extended to a maximal family having the finite intersection property. (Formal prove will be deduced from Zorn's Lemma.)

State Lemma 37.2. and the result that any any family having the finite intersection property can be extended to a maximal family having the finite intersection property.

Use the above results to prove Tychonoff Theorem.

Prove Lemma 37.2. Notice also that every maximal family having the finite intersection property is a proper filter.

State Zorn Lemma and Hausdorff Maximal Principle, section 11.

Prove, using Hausdorff Maximal Principle, that every family having the finite intersection property can be extended to a maximal family having the finite intersection property.

Final test will be on Wednesday, April 29, in this room, in usual time of 7pm