Introduction to Analysis and Topology, Math 381, Spring 2017
Class Notes

Krzysztof Chris Ciesielski

Class of Tuesday, January 10:
Discuss what to expect from the course, including syllabus.

Reading assignment for Thursday, January 12: Read Chapter 1: *Introduction: Intuitive Topology*, pages 1–6.

- Go over section 2.1: *Sets*.
- Go over section 2.2: *Functions* (covered only partially).

Written assignment for Thursday, January 12: Solve

Exercise 1 from page 15: *Show that* $X \setminus \bigcap_{\alpha\in\Lambda} A_\alpha = \bigcup_{\alpha\in\Lambda}(X \setminus A_\alpha)$ *for any set* X *and an idexed family* $\{A_\alpha : \alpha \in \Lambda\}$.

Exercise 3 from page 24: *Prove or disprove that* $f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$ *for any function* $f: X \to Y$ *and* $B \subset Y$.

Class of Thursday, January 12:
Collect homework.
Be ready for a quiz next class.

- Finish going over sec. 2.2: *Functions*, including Ex. 2, 5, 6, 7, and 8.
- Go over section 2.4: *Induction*, including Exercises 1, 2, 3.
- Go over section 2.5: *Cardinal numbers*, including Exercises 1, 2, 3.

Written assignment for Tuesday, January 17:

Part of Ex. 2, page 31: *Prove the summation formula:* $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

We did not cover section 2.3. We may come back to it at a latter time.

Class of Tuesday, January 17:

Collect homework.
Administer Quiz # 1.
Return graded homework.

Definition 1 Let X be an arbitrary set having at least two elements. A *topology* on X is any family $\mathcal{T}$ of subsets of X having the following properties:

(1) $\emptyset, X \in \mathcal{T}$.

(2) The union of any subfamily of $\mathcal{T}$ is in $\mathcal{T}$, that is, $\bigcup \mathcal{U} \in \mathcal{T}$ for every $\mathcal{U} \subset \mathcal{T}$.

(3) The intersection of any *finite* subfamily of $\mathcal{T}$ is in $\mathcal{T}$, that is, $\bigcap \mathcal{U} \in \mathcal{T}$ for every finite $\mathcal{U} \subset \mathcal{T}$.

The pair $\langle X, \mathcal{T} \rangle$ is called a *topological space*. For a fixed topological space $\langle X, \mathcal{T} \rangle$, the sets belonging to the family $\mathcal{T}$ will be refereed to as the *open (or $\mathcal{T}$-open) sets* (with respect to this topology).

In the above definition, we used the following notation:

- Arbitrary unions and intersections of sets: Let $\mathcal{A}$ be a family of sets, say $\mathcal{A} = \{A_t : t \in T\}$. Then $\bigcup \mathcal{A} = \bigcup_{t \in T} A_t$ denotes the same set: $\{x : \exists A \in \mathcal{A} (x \in A)\}$, that is, $\{x : \exists t \in T (x \in A_t)\}$.

- Similarly, $\bigcap \mathcal{A} = \bigcap_{t \in T} A_t$ denotes the same set: $\{x : \forall A \in \mathcal{A} (x \in A)\}$, that is, $\{x : \forall t \in T (x \in A_t)\}$.

Remark 1 In the definition, condition (3) can be replaced with

(3') The intersection of any two sets in $\mathcal{T}$ is in $\mathcal{T}$, that is, if $U, V \in \mathcal{T}$, the also $U \cap V \in \mathcal{T}$.

PROOF. Easy induction. See assigned homework. ■

Example 2 *Here are some examples of topological spaces $\langle X, \mathcal{T} \rangle$, where X is an arbitrary set.*

- $\mathcal{D} = \mathcal{P}(X)$, *where $\mathcal{P}(X)$ is the* **power set of** X, *that is, the family of all subsets of X. This topology is called the* **discrete topology**.

- *Give definition of* finer *and* coarser *topologies. Note that the discrete topology on X is the finest among all topologies on X.*
- $\mathcal{I} = \{\emptyset, X\}$. *This topology is called* **trivial** *or* **indiscrete** **topology**.
- *For a four elements set $X = \{a, b, c, d\}$, there are many different possible topologies. E.g. $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$—check the axioms. However, $\sigma = \{\emptyset, X, \{a, b\}, \{b, c, d\}\}$ is not a topology.*
- **The usual (or standard) topology** $\mathcal{U}$ *on* $\mathbb{R}$*:*

$$\mathcal{U} = \{V \subset \mathbb{R}\colon \text{ for every } x \in V \text{ there is an interval } (a, b) \text{ with } x \in (a, b) \subset V\}.$$

Check the axioms.

- **The left-hand (or lower limit) topology** $\mathcal{L}$ *on* $\mathbb{R}$*:*

$$\mathcal{L} = \{V \subset \mathbb{R}\colon \text{ for every } x \in V \text{ there is an interval } [a, b) \text{ with } x \in [a, b) \subset V\}.$$

Checking the axioms is left as an exercise.

- *Note that* $\mathcal{I} \subsetneq \mathcal{U} \subsetneq \mathcal{L} \subsetneq \mathcal{D}$.
- **The right ray topology** $\mathcal{RR}$ *on* $\mathbb{R}$*:*

$$\mathcal{RR} = \{V \subset \mathbb{R}\colon \text{ for every } x \in V \text{ there is an } a \in \mathbb{R} \text{ with } x \in (a, \infty) \subset V\}.$$

Checking the axioms is assigned homework.

- **The finite complement topology** *on X:*

$$\mathcal{FC} = \{V \subset \mathbb{R}\colon V = \emptyset \text{ or } X \setminus V \text{ is finite}\}.$$

Checking the axioms is left as an exercise.

Written assignment for Thursday, January 19:

Ex. 6, page 29: *Prove, by induction, that if $\tau \subset \mathcal{P}(X)$ is such that*

- $V_1 \cap V_2 \in \tau$ *for every* $V_1, V_2 \in \tau$,

then τ is closed under finite intersection (i.e., satisfies (3) from the definition of a topology).

Ex. 8, page 29: *Prove that $\mathcal{RR}$ is a topology on* $\mathbb{R}$. (I will accept solutions for this exercise until Tuesday, January 24.)

Class of Thursday, January 19:

Return graded homework and quiz.

Recall that a *topology* on X is a family $\mathcal{T}$ of subsets of X such that

(1) $\emptyset, X \in \mathcal{T}$;

(2) $\bigcup \mathcal{U} \in \mathcal{T}$ for every $\mathcal{U} \subset \mathcal{T}$;

(3) $\bigcap \mathcal{U} \in \mathcal{T}$ for every finite $\mathcal{U} \subset \mathcal{T}$.

Examples of topological spaces $\langle X, \mathcal{T} \rangle$:

- **Discrete topology**: $\mathcal{D} = \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the **power set of** X.
- **Indiscrete topology** $\mathcal{I} = \{\emptyset, X\}$.
- **Usual topology** on $\mathbb{R}$: $\mathcal{U} = \{V \subset \mathbb{R} \colon \forall x \in V \exists a < b \; x \in (a, b) \subset V\}$.
- **Left-hand topology** on $\mathbb{R}$: $\mathcal{L} = \{V \subset \mathbb{R} \colon \forall x \in V \exists a < b \; x \in [a, b) \subset V\}$.
- **Right ray topology** on $\mathbb{R}$: $\mathcal{RR} = \{V \subset \mathbb{R} \colon \forall x \in V \exists a \in \mathbb{R} \; x \in (a, \infty) \subset V\}$.
- **Finite complement topology** $\mathcal{FC} = \{\emptyset\} \cup \{X \setminus F \colon F \text{ is finite}\}$.

Proved, again, that $\mathcal{U}$ is a topology.

New material:

Prove that $\mathcal{FC}$ is a topology.

Section 3.3: Basics on open and closed sets

Definition 2 A set $A \subset X$ is *closed* (or $\mathcal{T}$*-closed*) in the topological space $\langle X, \mathcal{T} \rangle$ if its complement $X \setminus A$ is open.

Theorem 3 *For any the topological space* $\langle X, \mathcal{T} \rangle$

(1) $\emptyset$ *and* X *are closed.*

(2) The intersection of any family of $\mathcal{T}$*-closed sets is a* $\mathcal{T}$*-closed set.*

(3) The union of any finite family of $\mathcal{T}$*-closed sets is a* $\mathcal{T}$*-closed set.*

Prove Theorem 3.

Class of Tuesday, January 24:

Collect homework.
Administer Quiz # 2.
Return graded homework.

Recall that

- $A \subset X$ is *closed* in $\langle X, \mathcal{T} \rangle$ if, and only of, $X \setminus A \in \tau$.
- $\emptyset$ and X are closed.
- Arbitrary intersection of closed sets is closed.
- Finite union of closed sets is closed.

Define *neighborhood* of a point in a topological space $\langle X, \mathcal{T} \rangle$.
State and prove Theorem 3.3.2.
Define the *closure* $\mathrm{cl}(A)$ of $A \subset X$ and go over Theorem 3.3.3.
Show that, in $\mathcal{U}$, $\mathrm{cl}((a, b)) = [a, b]$.
Define the *interior* $\mathrm{int}(A)$ of $A \subset X$ and go over Theorem 3.3.4.
Show that, in $\mathcal{U}$, $\mathrm{int}([a, b]) = (a, b)$.
Go over Theorem 3.3.5.
Solve Exercises 10 page 65 for topology $\mathcal{U}$.
Define a *dense* set in a topological space $\langle X, \mathcal{T} \rangle$, Definition 3.3.4.
Note that $\mathbb{Q}$ is dense in the usual topology on $\mathbb{R}$.
Solve Exercises 2 and 4 page 64.

Written assignment for Tuesday, January 31: Solve Ex. 5 page 64.

Class of Thursday, January 26:

Return graded homework and quiz.
Next class **expect Quiz # 3**.
Recall that

- The *closure* $\mathrm{cl}(A)$ of $A \subset X$ is the smallest closed set containing A.
- The *interior* $\mathrm{int}(A)$ of $A \subset X$ is the largest open set contained in A.
- $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$, $\mathrm{cl}(A \cap B) \subset \mathrm{cl}(A) \cap \mathrm{cl}(B)$; the last inclusion can be strict.
- $A \subset X$ is *dense* when $\mathrm{cl}(A) = X$. $\mathbb{Q}$ is dense in $\langle \mathbb{R}, \mathcal{U} \rangle$.

New material:

Solve Exercise 8 page 64.

Section 3.4: The subspace topology

Definition 3 Let $\langle X, \mathcal{T} \rangle$ be a topological space and Y be any subset of X. Then the family

$$\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\} = \{V \subset Y : V = Y \cap U \text{ for some } U \in \mathcal{T}\}$$

forms a topology on Y called the *subspace topology.*

Prove Theorem 3.4.1 that $\mathcal{T}_Y$ is indeed a topology.
Prove Theorem 3.4.2:

Theorem 4 *Let Y be a subspace of X. Then, $A \subset Y$ is closed in Y if, and only if, $A = Y \cap F$ for some closed subset F of X.*

Solve Exercises 1 and 3 page 66.

Written assignment for Tuesday, January 31: Solve Ex. 4 page 66.

Class of Tuesday, January 31:

Administer Quiz # 3.
Extend due date for the last homework, Ex. 4 page 66, to the next class.
Collect the other homework, Ex. 5, page 64.
Recall that

- If $\mathcal{T}$ is a topology on X and $Y \subset X$, then $\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\} = \{V \subset Y : V = Y \cap U \text{ for some } U \in \mathcal{T}\}$ is the *subspace topology on* Y.
- If $\mathcal{T}$ is a topology on X and $Y \subset X$, then $A \subset Y$ *is closed in* Y *if, and only if,* $A = Y \cap F$ *for some closed subset* F *of* X.

New material:
Solve Exercises 5 and 6 page 66.

Start Section 3.5 *Continuous functions*:

- Recall calculus, ε-δ, definition, 3.5.1, of continuity of $f : \mathbb{R} \to \mathbb{R}$.
- Define topological *continuity* of $f : \langle X, \tau \rangle \to \langle Y, \nu \rangle$, Def 3.5.2.

Definition of continuity of $f\colon\mathbb{R}\to\mathbb{R}$ coincides with the topological continuity for $f\colon\langle\mathbb{R},\mathcal{U}\rangle\to\langle\mathbb{R},\mathcal{U}\rangle$. Not in the textbook.

The proof is more difficult than the rest of the material. I like you to follow this in class and again read at home. I expect you to be able to follow the argument. I understand, that you may have hard time to repeat the argument in its entirety.

Theorem 5 *(Motivational) Let $f\colon\mathbb{R}\to\mathbb{R}$. The following two definitions of continuity of f are equivalent:*

(a) (Topological definition) $f^{-1}(U)\in\mathcal{U}$ for every $U\in\mathcal{U}$.

(b) (ε-δ definition) For every $x\in\mathbb{R}$ and every $\varepsilon>0$ there is a $\delta>0$ such that for every $r\in\mathbb{R}$, if $|x-r|<\delta$, then $|f(x)-f(r)|<\varepsilon$.

PROOF. (a)$\Longrightarrow$(b): Fix an $x\in\mathbb{R}$ and an $\varepsilon>0$. Using (a), we need to find a δ satisfying (b).

Let $U=(f(x)-\varepsilon,f(x)+\varepsilon)$. Notice that $U\in\mathcal{U}$. (This requires checking, that U satisfies the definition of sets in $\mathcal{U}$.) So, by (a), $f^{-1}(U)\in\mathcal{U}$. Note also, that $x\in f^{-1}(U)$, as $f(x)\in(f(x)-\varepsilon,f(x)+\varepsilon)=U$. Therefore, we have $x\in f^{-1}(U)\in\mathcal{U}$ and, by the definition of $\mathcal{U}$, there is a $\delta>0$ such that $(x-\delta,x+\delta)\subset f^{-1}(U)$. We show, that this δ satisfies (b).

Indeed, let $r\in\mathbb{R}$ be such that $|x-r|<\delta$. Then, $r\in(x-\delta,x+\delta)\subset f^{-1}(U)$. Therefore, $f(r)\in U=(f(x)-\varepsilon,f(x)+\varepsilon)$ and so, $|f(x)-f(r)|<\varepsilon$, as required.

(b)$\Longrightarrow$(a): Fix a $U\in\mathcal{U}$. We need to show that $f^{-1}(U)$ is in $\mathcal{U}$. For this, take an $x\in f^{-1}(U)$. We need to find a $\delta>0$ for which $(x-\delta,x+\delta)\subset f^{-1}(U)$.

We have $f(x)\in U$, as $x\in f^{-1}(U)$. Since $U\in\mathcal{U}$, there exists an $\varepsilon>0$ for which $(f(x)-\varepsilon,f(x)+\varepsilon)\subset U$. Using (b) for this x and ε, we can find a $\delta>0$ such that $|f(x)-f(r)|<\varepsilon$ provided $|x-r|<\delta$. We will show that for this choice of δ we indeed have $(x-\delta,x+\delta)\subset f^{-1}(U)$.

To see this, take an $r\in(x-\delta,x+\delta)$. We need to show that $r\in f^{-1}(U)$. Since $r\in(x-\delta,x+\delta)$, we have $|x-r|<\delta$. So, by the choice of δ, $|f(x)-f(r)|<\varepsilon$. In particular, $f(r)\in(f(x)-\varepsilon,f(x)+\varepsilon)\subset U$. Thus, $r\in f^{-1}(U)$, as required. ∎

Recitation given, on Wednesday, February 1: 3-4pm, Armstrong 313

Class of Thursday, February 2:

Recall that

- A function $f\colon \langle X, \tau\rangle \to \langle Y, \nu\rangle$ is *continuous* provided $f^{-1}(V) \in \tau$ for every $V \in \nu$.
- For $f\colon \langle \mathbb{R}, \mathcal{U}\rangle \to \langle \mathbb{R}, \mathcal{U}\rangle$, this definition is equivalent to the calculus ε-δ definition of continuity.

Go over Exercises 1 and 2, page 71.
Go over 4 examples just before Theorem 3.5.1.
State and prove Theorem 3.5.1.
Go over Example 3.9.
State and prove Theorem 3.5.2 (on composition). Easy, but very important.
State and prove Theorem 3.5.3 (on restriction).
Go over Exercises 3 and 8.

Written assignment for Tuesday, February 7: Solve Ex. 7 page 71.

Class of Tuesday, February 7:

Collect homework assignment.
Be ready for a quiz next class.

Test #1 will be administered on Tuesday, February 14. It will be on the material from Chapters 2 and 3. We will do review during (a part of) the next class.

New material:
Define homeomorphism, Definition 3.5.3.
Define open mapping, Definition 3.5.4.
Go over example directly after Definition 3.5.4.
Go over Exercises 4 and 6.
Start next chapter. (Will not be included in Test #1.)

Sec 4.1. Basis for a topology

Definition 4 *Basis — Two closely related by different notions*

FROM A BASIS TO TOPOLOGY — **Basis for a topology, Def 4.2.1:** A collection $\mathcal{B}$ of a subsets of a set X such that

(B1) For every $x \in X$ there is a $B \in \mathcal{B}$ with $x \in B$ (i.e., $\bigcup \mathcal{B} = X$).

(B2) For every $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ there is a $B \in \mathcal{B}$ with $x \in B \subset B_1 \cap B_2$.

[FROM A TOPOLOGY TO ITS BASIS — **Basis for a given topology $\mathcal{T}$:**

(Only implicitly in the text, on page 75) Let $\langle X, \mathcal{T} \rangle$ be a fixed topological space. A basis for $\mathcal{T}$ is any collection $\mathcal{B} \subset \mathcal{T}$ such that for every $U \in \mathcal{T}$ and every $x \in U$ there exists a $B \in \mathcal{B}$ with $x \in B \subset U$.

Go over examples $\mathcal{B}_1$ and $\mathcal{B}_2$ on page 74.

The first of these notion is used to create new topologies. The second is used to easier deal with a given, fixed topology $\mathcal{T}$. This second notion is used considerably more often than the first one.

Fact 1 *(Thm 4.2.1) If $\mathcal{B}$ satisfies (B1) and (B2), then the family*

$$\mathcal{T}(\mathcal{B}) = \{U \subset X : \forall x \in U \exists B \in \mathcal{B}(x \in B \subset U)\} = \left\{\bigcup \mathcal{U} : \mathcal{U} \subset \mathcal{B}\right\}$$

is a topology on X. The family $\mathcal{B}$ is a basis to the topology $\mathcal{T}(\mathcal{B})$.

Go over example $\mathcal{B}_3$ on page 75.

Class of Thursday, February 9:

Administer Quiz #4.

Review for Test #1. The test will be given next class. You can start working on it half an hour earlier, that is, at 3:30 pm.

- Make sure that you know how to solve your past homework.
- Show that $\bigcup_{i\in J} g(U_j) = g\left(\bigcup_{i\in J} U_j\right)$ for any $g\colon X \to Y$ and indexed family $\{U_j\colon j \in J\}$ of subsets of X.
- Show that $\text{int}(A) \cap \text{int}(B) = \text{int}(A \cap B)$.
- For $x \in \mathbb{R}$ let $\tau_x = \{U \subset \mathbb{R}\colon x \in U \text{ or } U = \emptyset\}$. Show that τ_x is a topology on $\mathbb{R}$. Describe the family of all τ_x-closed sets. Find the interior and closure of $L = (-\infty, 0)$ and $R = (0, \infty)$ in τ_7.
- Describe subspace topologies of τ_{-1} for subspaces $L = (-\infty, 0)$ and $R = (0, \infty)$.
- Find the closure and the interior of $A = (-1, 0)$ in the topology $\mathcal{U}_L$.

Class of Tuesday, February 14:

Administer Test #1. You can start working on the test half an hour earlier, that is, at 3:30 pm.

Class of Thursday, February 16:

Return Test #1, together with its printed solutions.
Discuss the results and possibility to make up some of the low scores.
Discuss some of the typical errors for:
Ex 1 (set vs statement; missing variable x);
Ex 2 (no clearly stated inductive assumption);
Ex 3 (the definition of topology τ_1 not used, or used incorrectly);
Ex 5 (preimages incorrectly calculated).

Recall: *Basis — Two closely related by different notions*

FROM A BASIS TO TOPOLOGY — **Basis for a topology, Def 4.2.1:** A collection $\mathcal{B}$ of a subsets of a set X such that

(B1) For every $x \in X$ there is a $B \in \mathcal{B}$ with $x \in B$ (i.e., $\bigcup \mathcal{B} = X$).

(B2) For every $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ there is a $B \in \mathcal{B}$ with $x \in B \subset B_1 \cap B_2$.

[FROM A TOPOLOGY TO ITS BASIS — **Basis for a given topology $\mathcal{T}$:**

(Only implicitly in the text, on page 75) Let $\langle X, \mathcal{T} \rangle$ be a fixed topological space. A basis for $\mathcal{T}$ is any collection $\mathcal{B} \subset \mathcal{T}$ such that for every $U \in \mathcal{T}$ and every $x \in U$ there exists a $B \in \mathcal{B}$ with $x \in B \subset U$.

- If $\mathcal{B}$ satisfies (B1)&(B2), then the family is a topology generated by $\mathcal{B}$: $\mathcal{T}(\mathcal{B}) = \{U \subset X : \forall x \in U \exists B \in \mathcal{B} (x \in B \subset U)\} = \{\bigcup \mathcal{U} : \mathcal{U} \subset \mathcal{B}\}$.

New material:

Go over examples on pages 75–78 on open balls $B(p, \varepsilon)$, diamonds $D(p, \varepsilon)$, and squares $S(p, \varepsilon)$ on the plane $\mathbb{R}^2$.

Go over Theorem 4.2.2.

Solve Exercises 1 and 2, page 78. Possibly also Ex 3.

Class of Tuesday, February 21:
Be ready for a quiz next Tuesday.

Sec 4.3. Limit points
Go over the definition 4.3.1 of a *limit point*.
Go over examples below the definition.
State and prove Theorem 4.3.1: A is closed iff $A' \subset A$.
State and prove Theorem 4.3.2.
State and prove Theorem 4.3.3: $\mathrm{cl}(A) = A \cup A'$.
Solve Exercises 1 and 3, page 81.

Written assignment for Tuesday, February 28: Solve Ex. 2 page 81.

Class of Thursday, February 23:

Be ready for a quiz next class.
Recall that

- A is closed iff $A' \subset A$.
- $\mathrm{cl}(A) = A \cup A'$.

New material

Sec 4.5. More on continuity

Go over Theorem 4.5.1. (Very important!)

Sec 4.4. Interior, boundary, and closure

Recall that

- The *closure* $\mathrm{cl}(A)$ of $A \subset X$ is the smallest closed set containing A.
- The *interior* $\mathrm{int}(A)$ of $A \subset X$ is the largest open set contained in A.
- $\mathrm{int}(A) \subset A \subset \mathrm{cl}(A)$ for any $A \subset X$.

State and prove Theorem 4.4.1.
Define the *boundary* $\mathrm{bd}(A)$ of $A \subset X$.
Go over the examples on page 82.
State and prove Theorem 4.4.2: A is closed iff $\mathrm{bd}(A) \subset A$. (Stated only.)
State and prove Theorem 4.4.3: $\mathrm{cl}(A) = \mathrm{int}(A) \cup \mathrm{bd}(A)$. (Stated only.)
Solve Exercise 1, page 83–84.

Class of Tuesday, February 28:

Collect homework.
Administer Quiz #5.
Solve the exercises from the "Rescue" Test #1. (Solved all by the last one.)
Prove the theorems (left it with no proofs)
(1) *A is closed iff* $\mathrm{bd}(A) \subset A$; and
(2) $\mathrm{cl}(A) = \mathrm{int}(A) \cup \mathrm{bd}(A)$.

Class of Thursday, March 2

Return homework and quiz.

Be ready for a quiz next class, right after the break.

Solve homework. Solve the last exercise from the "Rescue" Test #1.

Sections 5.1 and 5.2: *Product spaces.*

Recall (just read from this page) that

- If $\mathcal{T}$ is a topology on X and $Y \subset X$, then $\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\} = \{V \subset Y : V = Y \cap U \text{ for some } U \in \mathcal{T}\}$ is the *subspace topology on* Y.
- If $\mathcal{T}$ is a topology on X and $Y \subset X$, then $A \subset Y$ *is closed in* Y *if, and only if,* $A = Y \cap F$ *for some closed subset* F *of* X.
- If $f: \langle X, \mathcal{T}\rangle \to Y$ is continuous and $B \subset X$, then $f \restriction B: \langle B, \mathcal{T}_B\rangle \to Y$ is continuous.

New material

Definition 5.2.1. *For topological spaces* $\langle X, \mu\rangle$ *and* $\langle Y, \nu\rangle$ *let* $\mathcal{B}_{\mu,\nu}$ *be the family of all open rectangles (denoted in the text as* $\mu \times \nu$*), that is,*

$$\mathcal{B}_{\mu,\nu} = \{U \times V : U \in \mu \ \&\ V \in \nu\}.$$

Note that $\mathcal{B}_{\mu,\nu}$ *satisfies conditions (B1) and (B2) for a topology on* $X \times Y$*. So, the family* $\mathcal{T}(\mathcal{B}_{\mu,\nu})$ *is a topology on* $X \times Y$*.*

The topology $\mathcal{T}(\mathcal{B}_{\mu,\nu})$ *is called the* product topology *on* $X \times Y$*.*

Examples of the product of two discrete spaces, two indiscrete spaces, and $\mathbb{R}_{\mathcal{U}} \times \mathbb{R}_{\mathcal{U}}$.

Go over Theorem 5.2.1: projections are continuous.

Go over Thm 5.2.2: $F = \langle f_1, f_2\rangle : Z \to X \times Y$ is continuous iff f_1 and f_2 are.

Go over Theorem 5.2.3, on inclusion mappings.

Go over Exercises 1 and 2, pages 95 and 96.

After the break, we will skip the rest of Chapter 5 and start going over Chapter 6, on *Connected spaces.*

Classes of Tuesday, March 14 and Thursday, March 16

Administer Quiz #6.

Start going over Chapter 6, on *Connected spaces*.

Definition 5 [Def 6.2.1] Let X be a topological space. A *separation* of X is any pair $\langle U, V\rangle$ of open, non-empty, disjoint sets for which $X = U \cup V$. A topological space X is *disconnected* provided there exists a separation of X.

A topological space X is *connected* provided it is not disconnected, that is, when it admits no separation.

Example 6.1: $\langle \mathbb{R}, \mathcal{L}\rangle$ is disconnected.

Example 6.2: $\langle X, \mathcal{D}\rangle$ is disconnected, provided $card(X) \geq 2$.

Example 6.3: Any X with indiscrete topology is connected.

Go over Example 6.4.

Example 6.5: $\langle \mathbb{R}, \mathcal{FC}\rangle$ is connected.

Go over Thm 6.2.1: connectedness is a topological property.

Go over **very important** Thm 6.2.2: *Continuous image of connected space is connected.*

Go over Example 6.6.

Go over Thm 6.2.3: *If A is a connected subset of X (i.e., A is connected with respect to $\mathcal{T}_A$), then* $\mathrm{cl}(A)$ *is connected (with respect to $\mathcal{T}_{\mathrm{cl}(A)}$).*

State Thm 6.2.4: Let A be a connected subspace of X. If $A \subset B \subset \mathrm{cl}(A)$, then B is connected.

Solve Ex. 2, page 112.

Solve Ex. 4, page 113.

Solve Ex. 6, page 113.

Written assignment for Thursday, March 16: Solve Ex. 1, page 112.

Written bonus assignment for Tuesday, March 21: Solve Ex. 5, page 113.

Section 6.3: *Connectedness in* $\mathbb{R}$.

Proposition 6 [Prop. 6.3.1: completeness property of $\mathbb{R}$] *If $A \subset \mathbb{R}$ is non-empty bounded below, that it has the greatest lower bound* $\inf A \in \mathbb{R}$. *Equivalently, if $A \subset \mathbb{R}$ is non-empty bounded above, that it has the least lower bound* $\sup A \in \mathbb{R}$.

Go over **very important** Thm 6.3.2: $\langle \mathbb{R}, \mathcal{U}\rangle$ *is connected.*

Theorem 7 (Version of Corollary 6.3.3) *A subset A of $\mathbb{R}$ (considered with the usual topology) is connected if, and only if, A is an interval (possible degenerated).*

Theorem 8 (Theorem 6.3.4, the Intermediate Value Theorem) *If $f\colon [a,b] \to \mathbb{R}$ is continuous and c is between $f(a)$ and $f(b)$, there there is an $x \in [a,b]$ with $f(x) = c$.*

Solve Ex. 4, page 116.

Class of Tuesday, March 21

Collect the bonus homework, Ex. 5, page 113.
Be ready for a quiz next class.
Quick review:

- Let X be a topological space. A *separation* of X is any pair $\langle U, V\rangle$ of open, non-empty, disjoint sets for which $X = U \cup V$. A topological space X is *disconnected* provided there exists a separation of X.

  A topological space X is *connected* provided it is not disconnected, that is, when it admits no separation.

- *If A is a connected subset of X (i.e., A is connected with respect to $\mathcal{T}_A$), then* $\mathrm{cl}(A)$ *is connected (with respect to $\mathcal{T}_{\mathrm{cl}(A)}$).*

- *Continuous image of connected space is connected.*

- A subset A of $\mathbb{R}$ (considered with the usual topology) is connected if, and only if, A is an interval (possible degenerated).

- **(Thm 6.3.4, the Intermediate Value Theorem)** If $f\colon [a, b] \to \mathbb{R}$ is continuous and c is between $f(a)$ and $f(b)$, there there is an $x \in [a, b]$ with $f(x) = c$.

Review quickly the proofs of the last two bullets above.
Solve Ex. 2, page 116.
Solve Ex. 3, page 116.
Solve Ex. 5, page 116.

Start going over Chapter 7, on *Compact spaces*: Read Section 7.1.

Definition 6 (Def. 7.2.1.) A *cover* of X is any family $\mathcal{V}$ of subsets of X (i.e., $\mathcal{V} \subset \mathcal{P}(X)$) such that $X = \bigcup \mathcal{V}$.

If X is a topological space, $\mathcal{V}$ is an *open cover* provided $\mathcal{V}$ covers X and every $V \in \mathcal{V}$ is open in X.

Definition 7 (Def. 7.2.2.) A topological space $\langle X, \tau \rangle$ is *compact* provided for every open cover $\mathcal{V}$ of X there exists a finite subfamily $\mathcal{V}_0$ of $\mathcal{V}$ that covers X (i.e., $\mathcal{V}_0 \subset \mathcal{V}$ is finite and $X = \bigcup \mathcal{V}_0$). Such a family $\mathcal{V}_0$ will be referred to as a (finite) *subcover* of $\mathcal{V}$.

Compact spaces have some properties of finite spaces.
Example 7.3: A finite space $\langle X, \tau \rangle$ is compact (for any topology τ on X).
Goal: to prove

Theorem 9 (Extreme Value Theorem) *If* $f \colon [a, b] \to \mathbb{R}$ *is continuous (in* $\mathcal{U}$*-*$\mathcal{U}$ *sense) then there exist* $c, d \in [a, b]$ *such that* $f(c) \leq f(x) \leq f(d)$ *for all* $x \in [a, b]$.

Class of Thursday, March 23 Continue going over *Compact spaces.*
Administer Quiz #7.
Return graded bonus homework and its typeset solution.

Test #2 will be administered on Tuesday, April 4. It will concentrate on the material from Chapters 4-7 that we covered in class. However, we will also rely on the material covered in Test #1. We will do a review during (a part of) the next class.

Recall that

- $\mathcal{V}$ is an *open cover* provided of $\langle X\tau\rangle$ provided $\mathcal{V}$ covers X (i.e., $X = \bigcup\mathcal{V}$) and $\mathcal{V} \subset \tau$.
- $\langle X, \tau\rangle$ is *compact* provided every open cover $\mathcal{V}$ of X has a finite *subcover* (i.e., $V_0 \subset \mathcal{V}$ is finite and $X = \bigcup\mathcal{V}_0$).
- Any finite space $\langle X, \tau\rangle$ is compact (for any topology τ on X).

New material
Example 7.1: $\langle\mathbb{R}, \mathcal{U}\rangle$ is not compact.
Example 7.2: $\langle\mathbb{R}, \mathcal{I}\rangle$ is compact.
Example 7.4: If X is infinite, then $\langle X, \mathcal{D}\rangle$ is not compact.
Example 7.5: If $\langle(0, 1), \mathcal{U}_{(0,1)}\rangle$ is not compact.

Theorem 10 (Thm 7.2.1) *Continuous image of a compact space is compact.*

Theorem 11 (Thm 7.2.3) *Closed subspace of compact space is compact.*
Example 7.6: Compact subspace need not be closed.
Solve Ex. 2, page 131.
Solve Ex. 3, page 131.
Solve Ex. 4, page 131.

Written assignment for Tuesday, March 28: Solve Ex. 5, page 131.

Section 7.3: *Hausdorff spaces and compactness.*

Definition 8 A topological space $\langle X, \mathcal{T}\rangle$ is *Hausdorff* (or a T_2 *space*) provided for every distinct $x, y \in X$ there exist disjoint open sets $U, V \subset X$ such that $x \in U$ and $y \in V$.

Example 7.7: $\langle X, \mathcal{D}\rangle$ is Hausdorff.
Example 7.8: $\langle X, \mathcal{I}\rangle$ is not Hausdorff when X has more than one element.
Example 7.9: $\langle\mathbb{R}, \mathcal{U}\rangle$ is Hausdorff.
Example 7.10: $\langle\mathbb{R}, \mathcal{FC}\rangle$ is not Hausdorff.

Written assignment for Tuesday, March 28: Solve Ex. 4, page 134.

Class of Tuesday, March 28

Collect homework. Return graded Quiz #7.

Test #2 will be administered on Tuesday, April 4. It will concentrate on the material from Chapters 4-7 that we covered in class. However, we will also rely on the material covered in Test #1. We will do a review during (a part of) the next class.

Recall that

- $\mathcal{V}$ is an *open cover* of $\langle X, \tau\rangle$ provided $\mathcal{V}$ covers X.
- $\langle X, \tau\rangle$ is *compact* provided every open cover $\mathcal{V}$ of X has a finite *subcover* (i.e., $V_0 \subset \mathcal{V}$ is finite and $X = \bigcup \mathcal{V}_0$).
- *Continuous image of a compact space is compact.*
- *Closed subspace of compact space is compact.*
- $\langle X, \mathcal{T}\rangle$ is *Hausdorff* provided for every distinct $x, y \in X$ there are disjoint open $U, V \subset X$ such that $x \in U$ and $y \in V$.

New material

Solve Ex. 1, page 134.

Solve Ex. 2, page 134.

Solve Ex. 3, page 134.

Theorem 12 (Thm 7.3.1.) *Compact subspace of a Hausdorff space is closed.*

Theorem 13 (Thm 7.3.3.) *If $f\colon X \to Y$ is a continuous bijection, X is compact, and Y is Hausdorff, then f is a homeomorphism.*

Section 7.4: *Compactness in the Real Line*

Very important theorem:

Theorem 14 (Thm 7.4.1.) *Any closed interval $[a, b]$ is compact as subspace of $\langle \mathbb{R}, \mathcal{U}\rangle$.*

Corollary 15 (Thm 7.4.2.) *A subset A of $\langle \mathbb{R}, \mathcal{U}\rangle$ is compact if, and only if, it is closed and bounded (with respect to the standard distance on $\mathbb{R}$).*

Solve Ex. 5, page 137, proving:

Theorem 16 (Extreme Value Theorem) *If* $f\colon [a,b] \to \mathbb{R}$ *is continuous (in* $\mathcal{U}$*-*$\mathcal{U}$ *sense) then there exist* $c, d \in [a,b]$ *such that* $f(c) \le f(x) \le f(d)$ *for all* $x \in [a,b]$.

Solve Ex. 1, page 137.
Solve Ex. 2, page 137.

Class of Thursday, March 30

Go over:

Theorem 17 (Thm 7.4.4.) *If $\langle X, \mathcal{T} \rangle$ is compact, then $A' \neq \emptyset$ for every infinite $A \subset X$.*

Corollary 18 (Thm 7.4.3: Bolzano-Weierstrass Theorem) *Every infinite bounded subset of $\langle \mathbb{R}, \mathcal{U} \rangle$ has a limit point.*

Review for Test #2. The test will be given next class. You can start working on it half an hour earlier, that is, at 3:30 pm.

- Show that $\mathcal{B} = \{(q-1/n, q+1/n) \colon q \in \mathbb{Q} \ \&\ n \in \mathbb{N}\}$ is a basis for $\langle \mathbb{R}, \mathcal{U} \rangle$.
- Is the family $\mathcal{C} = \{[p, q) \colon p, q \in \mathbb{Q}\}$ a basis for $\langle \mathbb{R}, \mathcal{L} \rangle$?
- Let $f \colon \langle X, \tau \rangle \to \langle Y, \nu \rangle$. Show, directly from the definitions, that f is continuous (i.e., $f^{-1}(V) \in \tau$ for every $V \in \nu$) if, and only if, $f^{-1}(F)$ if τ-closed for every ν-closed $F \subset Y$.
- Let $A = \{0\} \cup (1, 2) \cup \{3\}$. Find $\mathrm{bd}(A)$ and A' in $\langle \mathbb{R}, \mathcal{RR} \rangle$.
- Consider X^2 with the product topology $\mathcal{T}$ of $\langle X, \tau \rangle$. Show that the projection map $\pi \colon X \times X \to X$, given as $\pi(x, y) = x$, is an open map.
- A space $\langle X, \tau \rangle$ is *path connected* provided for every $x, y \in X$ there is a continuous map $p \colon [0, 1]_{\mathcal{U}} \to \langle X, \tau \rangle$ such that $p(0) = x$ and $p(1) = y$. Show that every path connected space $\langle X, \tau \rangle$ is connected.
- Let $\mathcal{A}$ be a finite family of subsets of $\langle X, \tau \rangle$ such that each $A \in \mathcal{A}$ is compact, when considered with a subspace topology of $\langle X, \tau \rangle$. Show that $\bigcup \mathcal{A}$ is compact, as a subspace of $\langle X, \tau \rangle$. *Did not have a time to solve it.*

Class of Tuesday, April 4

Administer Test #2. The test will be given next class. You can start working on it half an hour earlier, that is, at 3:30 pm.

Class of Thursday, April 6

Return graded Test #2 and its printed solutions.

Discuss results of Test #2 and the meaning of *your current course score.* [The remaining two tests, #3 and final, **are likely to reduced this score!**]

Give alternative solution to the problem #2.

Final Test will be administered on Wednesday, May 3, 8:00am to 10:00am.

Test #3 will be administered on Thursday, May 20, *with no formal review on the preceding class of Tuesday, May 18.* I may try to organize review session on Wednesday, May 19 (either 2-3pm, or after 5pm).

Alternatively, I can try to move Test #3 to Tuesday, May 25, with review on Thursday, May 20. But I will need to check the legality of it, since according to the university rule I should not administer tests during the dead week. (Would you like me to explore this option?)

Chapter 8: Separation Axioms

We already have seen Hausdorff separation axiom. This will concern other similar axioms.

Definition 9 Let $\langle X, \mathcal{T} \rangle$ be a topological space. We say that:

- X is *Hausdorff* (or a T_2 *space*) provided for every distinct $x, y \in X$ there exists disjoint open sets $U, V \subset X$ such that $x \in U$ and $y \in V$.
- X is a T_1 *space* provided for every distinct $x, y \in X$ there exists an open set $U \subset X$ such that $x \in U$ and $y \notin U$.
- X is a T_0 *space* provided for every distinct $x, y \in X$ there exists an open set $U \subset X$ such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$ (i.e., such that U contains precisely one of the points x and y).

Notice that if X is T_2 then it is also T_1, and if X is T_1 then it is also T_0.

Examples:

8.1: A space X with a trivial topology $\mathcal{T} = \{\emptyset, X\}$ is not T_0.

8.2: $\langle \mathbb{R}, \mathcal{RR} \rangle$ is T_0 but not T_1. (Also $X = \{0, 1\}$ with a topology $\mathcal{T} = \{\emptyset, \{0\}, X\}$ is T_0 but not T_1.)

8.3: $X = \mathbb{R}$ with a cofinite topology $\mathcal{T} = \{\emptyset\} \cup \{X \setminus F : F \text{ is finite}\}$ is T_1 but not T_2.

- The following spaces are T_2: any space with the discrete topology, $\langle \mathbb{R}, \mathcal{U} \rangle$, and $\langle \mathbb{R}, \mathcal{L} \rangle$.

Theorem 19 *(Theorem 8.2.3) A space X is T_1 if, and only if, every finite subset of X is closed.*

Corollary 20 *Every finite subset in a T_1 and in Hausdorff space is closed.*

Theorem 21 *(Theorem 8.2.4) Let $i \in \{0, 1, 2\}$. Then a subspace of a T_i topological space is a T_i space. The product of two T_i topological spaces is a T_i space.*

Theorem 22 *(Theorem 8.2.5) Let $f, g : \langle X, \tau \rangle \to \langle Y, \nu \rangle$ be continuous functions. If $\langle Y, \nu \rangle$ is Hausdorff, that $A = \{x \in X : f(x) = g(x)\}$ is τ-closed.*

Corollary 23 *(Corollay 8.2.6) Let $f, g : \langle X, \tau \rangle \to \langle Y, \nu \rangle$ be continuous functions. If $\langle Y, \nu \rangle$ is Hausdorff and $f(x) = g(x)$ on a dense subset of X, then $f = g$.*

Written assignment for Tuesday, April 11: Solve Exercise 4, page 148.

Class of Tuesday, April 11

Collect homework. **Be ready for a quiz next class!**
Recall that

- X is *Hausdorff* (or a T_2 *space*) provided for every distinct $x, y \in X$ there exists disjoint open sets $U, V \subset X$ such that $x \in U$ and $y \in V$.
- X is a T_1 *space* provided for every distinct $x, y \in X$ there exists an open set $U \subset X$ such that $x \in U$ and $y \notin U$.
- X is a T_0 *space* provided for every distinct $x, y \in X$ there exists an open set $U \subset X$ such that either $x \in U$ and $y \notin U$.
- Thm: *A space X is T_1 if, and only if, every finite subset of X is closed.*
- Thm: *Let $f, g\colon \langle X, \tau\rangle \to \langle Y, \nu\rangle$ be continuous functions. If $\langle Y, \nu\rangle$ is Hausdorff, that $A = \{x \in X \colon f(x) = g(x)\}$ is τ-closed.*

New material

Section 8.3: Regular and normal spaces

- X is *regular* (or a T_3 *space*) provided it is a T_1 space and for every closed set K in X and $x \in X \setminus K$ there exist disjoint open sets $U, V \subset X$ such that $x \in U$ and $K \subset V$.
- X is *normal* (or a T_4 *space*) provided it is a T_1 space and for every disjoint closed sets K and L in X there exist disjoint open sets $U, V \subset X$ such that $K \subset U$ and $L \subset V$.

Notice, that in our text the regular space (unlike T_3 space) need not be T_1. ***We will identify regular and T_3 spaces.*** *Similarly for normal spaces.*

Notice that if X is T_3 then it is also T_2, and if X is T_4 then it is also T_3.
Examples:

8.11: Any space with the discrete topology is T_4.

8.12: $\langle \mathbb{R}, \mathcal{U}\rangle$ is T_4.

8.13: The following **K-topology** on $\mathbb{R}$ is T_2 but not T_3.

Let $K = \{1/n : n = 1, 2, 3, \dots\}$. Then $\mathcal{T}_K$ is generated by the basis $\mathcal{B}_K = \{(a, b): a, b \in \mathbb{R}, a < b\} \cup \{(a, b) \setminus K: a, b \in \mathbb{R}, a < b\}$, that is, $\mathcal{T}_K = \mathcal{T}(\mathcal{B}_K)$.

8.15: $\mathbb{R}_{\mathcal{L}} \times \mathbb{R}_{\mathcal{L}}$ is not normal, in spite the fact that $\mathbb{R}_{\mathcal{L}}$ is normal. (Both these facts will be left without a proof.)

By the above example, the product of two normal spaces need not be normal. This stands in contrast with the theorem.

Theorem 24 *(Theorem 8.3.3) The product of two regular spaces is regular.*

Proof of this theorem sketched only.

Theorem 25 *(Theorem 8.3.1) The subspace of a regular space is regular.*

Proof of this theorem sketched only.

Note that subspace of a normal space need not be normal. (No example will be provided.) A close subspace of a normal space is normal (as proved in Theorem 8.3.2). We will not go over its proof.

Section 8.4

Theorem 26 *(Theorems 8.4.1 and 8.4.2) Any compact Hausdorff space is normal, so regular.*

Proved only that compact Hausdorff spaces are regular. Normality is proved, from regularity, by almost identical argument.

Class of Thursday, April 13

Administer Quiz # 8.
Return graded homework.
Recall that

- X is T_3 *space* provided it is a T_1 space and for every closed $K \subset X$ and $x \in X \setminus K$ there exist disjoint open $U, V \subset X$ with $x \in U$ and $K \subset V$.
- X is T_4 *space* provided it is a T_1 space and for every disjoint closed sets $K, L \subset X$ there are disjoint open $U, V \subset X$ with $K \subset U$ and $L \subset V$.
- $\langle \mathbb{R}, \mathcal{T}_K \rangle$ is T_2 but not T_3, where $K = \{1/n : n = 1, 2, 3, \ldots\}$ and $\mathcal{T}_K$ is generated by $\{(a, b) : a, b \in \mathbb{R}, a < b\} \cup \{(a, b) \setminus K : a, b \in \mathbb{R}, a < b\}$.
- $\mathbb{R}_{\mathcal{L}} \times \mathbb{R}_{\mathcal{L}}$ is T_3 but not T_4, in spite the fact that $\mathbb{R}_{\mathcal{L}}$ is T_4.
- The product of two T_3 spaces is T_3.
- The subspace of a T_3 space is T_3.
- Neither of these last two properties holds for T_4 spaces.
- Any compact Hausdorff space is T_4, so T_3.

New material
Review the proof of this last theorem.
Same shows that a compact subspace of a Hausdorff space is closed.

Move to the Chapter 9, *Metric Spaces*.
This new material might be in the final test, but will not be on Test #3.
Next class we will do only review for Test #3.

Test #3 will be administered on Thursday, May 20. As usually, it will start half an hour earlier.

Define a *metric* (*distance*) on X as a function $d\colon X \times X \to [0, \infty)$.

A *metric space* is a pair $\langle X, d \rangle$, where d is a metric on X.

Ex. 9.1: $d\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $d(x, y) = |x - y|$ is a (standard) metric on $\mathbb{R}$.

Ex. 9.3: Define discrete metric on X.

In a metric space $\langle X, d \rangle$, define an *open ball* (centered at $x \in X$ with radius $\varepsilon > 0$) as $B_d(x, \varepsilon) = \{y \in X\colon d(x, y) < \varepsilon\}$.

Prove that a family $\mathcal{B}_d = \{B(x, \varepsilon)\colon x \in X \ \& \ \varepsilon > 0\}$ is a basis for a topology on X.

Define a metric topology for a metric space $\langle X, d \rangle$ as $\mathcal{T}(\mathcal{B}_d)$, that is, as a topology generated by the family of all open balls in $\langle X, d \rangle$.

Example: The metric topology associated with the discrete metric on X is the discrete topology.

The following was not covered yet.

Theorem 27 *(Theorem 9.3.2) Any metric space is T_4.*

Definition 10 A topological space $\langle X, \tau \rangle$ is *metrizable* provided there exists a metric d on X such that $\tau = \mathcal{T}(\mathcal{B}_d)$.

An important problem: Which topological spaces are metrizable?

Partial solution:

Theorem 28 *(Theorem 9.3.8, Urysohn's Metrization Theorem) Assume that X has a basis with countably many sets. Then X is metrizable if, and only if, X is T_3.*

Class of Tuesday, April 18

Return graded Quiz #8.

Test #3 will be administered next class. It will start half an hour earlier.

Review for Test #3 by going over the following, including proofs.

- X is a T_0 *space* provided for every distinct $x, y \in X$ there exists an open set $U \subset X$ such that either $x \in U$ and $y \notin U$.
- X is a T_1 *space* provided for every distinct $x, y \in X$ there exists an open set $U \subset X$ such that $x \in U$ and $y \notin U$.
- X is *Hausdorff* (or a T_2 *space*) provided for every distinct $x, y \in X$ there exists disjoint open sets $U, V \subset X$ such that $x \in U$ and $y \in V$.
- X is T_3 *space* provided it is a T_1 space and for every closed $K \subset X$ and $x \in X \setminus K$ there exist disjoint open $U, V \subset X$ with $x \in U$ and $K \subset V$.
- X is T_4 *space* provided it is a T_1 space and for every disjoint closed sets $K, L \subset X$ there are disjoint open $U, V \subset X$ with $K \subset U$ and $L \subset V$.
- T_4 space $\Longrightarrow$ T_3 space $\Longrightarrow$ T_2 space $\Longrightarrow$ T_1 space $\Longrightarrow$ T_0 space
- $\langle \mathbb{R}, \mathcal{RR} \rangle$ as well as $\langle X, \mathcal{T} \rangle = \langle \{0,1\}, \{\emptyset, \{0\}, X\} \rangle$ are T_0 but not T_1.
- $\mathbb{R}$ with a cofinite topology $\mathcal{T} = \{\emptyset\} \cup \{X \setminus F \colon F \text{ is finite}\}$ is T_1 not T_2.
- $\langle \mathbb{R}, \mathcal{T}_K \rangle$ is T_2 but not T_3, where $K = \{1/n \colon n = 1, 2, 3, \ldots\}$ and $\mathcal{T}_K$ is generated by $\{(a, b) \colon a, b \in \mathbb{R}, a < b\} \cup \{(a, b) \setminus K \colon a, b \in \mathbb{R}, a < b\}$.
- $\mathbb{R}_{\mathcal{L}} \times \mathbb{R}_{\mathcal{L}}$ is T_3 but not T_4, in spite the fact that $\mathbb{R}_{\mathcal{L}}$ is T_4.
- For $i \in \{0, 1, 2, 3\}$, the product of two T_i spaces is T_i. False for T_4, as $\mathbb{R}_{\mathcal{L}}$ is T_4 but $\mathbb{R}_{\mathcal{L}} \times \mathbb{R}_{\mathcal{L}}$ is not.
- For $i \in \{0, 1, 2, 3\}$, the subspace of a T_i space is T_i. False for T_4.
- Thm: *A space X is T_1 if, and only if, every finite subset of X is closed.*
- Thm: *Let $f, g \colon \langle X, \tau \rangle \to \langle Y, \nu \rangle$ be continuous functions. If $\langle Y, \nu \rangle$ is Hausdorff, that $A = \{x \in X \colon f(x) = g(x)\}$ is τ-closed.*
- Any compact Hausdorff space is T_4, so T_3.

Additional review exercises

Ex. 1. For a topological space X be a topological we say that a sequence $\langle x_n \rangle_{n=1}^{\infty}$ of points of X *converges* to an $x \in X$ provided for every open set U containing x there exists an N such that $x_n \in U$ for every $n \geq N$. If this is the case, we say also, that x is a *limit* of a sequence $\langle x_n \rangle_{n=1}^{\infty}$.

Prove that if X is a Hausdorff topological space, then any sequence $\langle x_n \rangle_{n=1}^{\infty}$ of points of X *converges* to at most one point in X.

Ex. 2. Show that X is a Hausdorff space if, and only if, the **diagonal** $\Delta = \{\langle x, x \rangle : x \in X\}$ is closed in $X^2 = X \times X$.

SOLUTION: It is enough to prove that

- X is a Hausdorff if, and only if, $\Delta^c = X^2 \setminus \Delta$ is open in X^2.

"$\Longrightarrow$" Let $z = \langle x, y \rangle \in \Delta^c$. It is enough to show that there exists an open $W \subset X^2$ such that $z \in W \subset \Delta^c$.

Indeed, $x \neq y$, since $\langle x, y \rangle \in \Delta^c$. So, by Hausdorff property, there exists disjoint open sets $U \ni x$ and $V \ni y$. Let $W = U \times V$. Then, W is open and $z = \langle x, y \rangle \in W$. Moreover, if $\langle a, b \rangle \in W = U \times V$, then $a \neq b$, as $U \cap V = \emptyset$. In particular, $\langle a, b \rangle \in \Delta^c$. Therefore, $z \in W \subset \Delta^c$, as required.

"$\Longleftarrow$" Choose distinct $x, y \in X$. Then, $\langle x, y \rangle \in \Delta^c$. Since Δ^c is open, there exists a basic open set $U \times V$ (i.e., U and V open in X) such that $\langle x, y \rangle \in U \times V \subset \Delta^c$. Clearly $x \in U$ and $y \in V$. It is enough to prove that $U \cap V = \emptyset$.

Indeed, if $U \cap V \neq \emptyset$, then there exists an $a \in U \cap V$. However, this is impossible, since then $\langle a, a \rangle \in (U \times V) \cap \Delta$, contradicting the fact that $U \times V \subset \Delta^c$. ∎

Class of Thursday, April 20:

Administer Test #3. You can start working on the test half an hour earlier, that is, at 3:30 pm.

Class of Tuesday, April 25:

Return Test #3, together with its printed solutions.

Start reviewing for the final test, beginning with the following:

- Define a *metric* (*distance*) on X as a function $d\colon X \times X \to [0,\infty)$.
- A *metric space* is a pair $\langle X, d\rangle$, where d is a metric on X.
- Examples: (a) $d\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $d(x,y) = |x-y|$ is a metric on $\mathbb{R}$.

  (b) A discrete metric on any set X.
- *Open ball* in $\langle X, d\rangle$: $B_d(x,\varepsilon) = \{y \in X\colon d(x,y) < \varepsilon\}$; $x \in X$ and $\varepsilon > 0$.
- $\mathcal{B}_d = \{B(x,\varepsilon)\colon x \in X \ \& \ \varepsilon > 0\}$ is a basis for the metric topology $\mathcal{T}_d = \mathcal{T}(\mathcal{B}_d)$ on X.

New material, to be covered

Theorem 29 *(Theorem 9.3.2) Any metric space is T_4.*

Prove it.

Definition 11 A topological space $\langle X, \tau\rangle$ is *metrizable* provided there exists a metric d on X such that $\tau = \mathcal{T}(\mathcal{B}_d)$.

An important problem: Which topological spaces are metrizable?

Partial solution (statement only, no proof):

Theorem 30 *(Theorem 9.3.8, Urysohn's Metrization Theorem) Assume that X has a basis with countably many sets. Then X is metrizable if, and only if, X is T_3.*

Class of Thursday, April 27:

Last class. Continue reviewing for a final test.

Recall, that Final test is on Wednesday, May 3, 8am-10am, in the room at which we always meet.