Topology 2, Math 681, Spring 2017: Notes

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Class of Tuesday, January 10:

- Note that the last semester's abbreviated notes are still available on my web page, at:

  http://www.math.wvu.edu/˜kcies/teach/Fall2016/Fall2016.html

- Next class, *January 12, I will give extended quiz*, to check the background of everybody. I will ask for definitions and, possibly, statements of some fundamental theorems.

Written assignment for Tuesday, January 17: (*Mr Cook and Ms Le excluded.*) Solve the exercises from last semester final test. **Turn, in writing, only Exercises 3 and 4.**

Quick Review (Also definition of topological space and continuous maps)

- X is *Hausdorff* (or a *T_2 space*) provided for every distinct $x, y \in X$ there exists disjoint open sets $U, V \subset X$ such that $x \in U$ and $y \in V$.

- If X is a Hausdorff topological space, then any sequence $\langle x_n \rangle_{n=1}^{\infty}$ of points of X *converges* to at most one point in X.

- The product of two Hausdorff topological spaces is a Hausdorff space. A subspace of a Hausdorff topological space is a Hausdorff space.

- A function $f \colon X \to Y$ is *continuous* provided $f^{-1}(V)$ is open in X for every open subset V of Y.

- If $\mathcal{B}$ a basis for Y, then $f \colon X \to Y$ is continuous if, and only if, $f^{-1}(B)$ is open in X for every $B \in \mathcal{B}$.

- *Product topology* $\mathcal{T}_{prod}$ on $X = \prod_{\alpha \in J} X_\alpha$ is generated by subbasis $\mathcal{S}_{prod} = \{\pi_\beta^{-1}(U_\beta)$ for all $\beta \in J$ and open subsets U_β of $X_\beta\}$

- *If $f_\alpha \colon A \to X_\alpha$ and $f \colon A \to X$ is given by $f(a)(\alpha) = f_\alpha(a)$, then*

  continuity of f implies the continuity of each f_α;

  continuity of all f_α's implies the continuity of $f \colon A \to \langle X, \mathcal{T}_{prod} \rangle$;

- A *metric space* is a pair $\langle X, d\rangle$, where d is a metric on X. $\mathcal{B}_d = \{B(x,\varepsilon) \colon x \in X \ \&\ \varepsilon > 0\}$ is a basis for a topology on X.
- $\mathcal{T}(\mathcal{B}_d)$ is the metric topology on X (for metric d).
- Subspace of a metric space is metric.
- Every metrizable space is Hausdorff.
- Finite and countable product of metric spaces is metrizable.
- A topological space $\langle X, \mathcal{T}\rangle$ is *first countable* provided for every $x \in X$ there exists a countable basis $\mathcal{B}_x$ of X at x;
- *Let X be a first countable space and let $A \subset X$. Then $x \in \mathrm{cl}(A)$ if, and only if, there is a sequence of points of A converging to x.*

- A topological space X is *connected* provided it **does not** exist a pair U, V of open, non-empty disjoint sets with $X = U \cup V$.
- **(Star Lemma)** Let $\{A_\alpha\}_{\alpha\in J}$ be a family of connected subspaces of X. If $\bigcap_{\alpha\in J} A_\alpha \neq \emptyset$, then $\bigcup_{\alpha\in J} A_\alpha$ is connected.
- A closure of a connected space is connected.
- Finite product of connected spaces is connected: sketch proof.
- $\mathbb{R}^\omega$ with the product topology is connected: sketch proof.
- Continuous image of connected space is connected: sketch proof.
- $A \subset \mathbb{R}$ is connected if, and only if, A is convex (an interval).
- Intermediate Value Theorem.
- Definition of *path connectedness*.
- *Topologists sine curve*: it is connected but not path connected.

Class of January 12:

Review, continuation.

- Show that every continuous function $f\colon [0,1] \to [0,1]$ has a fixed point.

- Show that the shapes given by the characters A, I, and T are not homeomorphic to each other. What about the characters K and L, with respect to A, I, and T?

- X is *compact* provided for every open cover $\mathcal{U}$ of X contains a finite subcover $\mathcal{U}_0$ of $\mathcal{U}$ that covers X.

- Closed subspace of compact space is compact.

- Every compact subspace of a Hausdorff space is closed.

- For every compact subspace Y of a Hausdorff space X and every $x \in X \setminus Y$ there are disjoint open U and V such that $x \in U$ and $Y \subset V$.

- Every closed interval $[a,b]$ in $\mathbb{R}$ is compact.

- A subspace X of $\mathbb{R}$ is compact if, and only if, it is closed and bounded.

- Continuous image of a compact space is compact.

- **(Extreme Value Theorem for $\mathbb{R}^n$)** If R is a closed bounded subset of $\mathbb{R}^n$, then for every continuous function $f\colon R \to \mathbb{R}$ there exist $c, d \in R$ such that $f(c) \le f(x) \le f(d)$ for every $x \in R$.

- A collection $\mathcal{C}$ of subsets of X has *finite intersection property, fip,* provided $\bigcap \mathcal{C}_0 \neq \emptyset$ for every finite $\mathcal{C}_0 \subset \mathcal{C}$.

- X is compact if, and only if, $\bigcap \mathcal{C} \neq \emptyset$ for every family $\mathcal{C}$ of closed subsets of X having fip.

Written assignment for Tuesday, January 17: Ex 4 p. 178. (It has a short, easy solution.)

Start material on compact spaces not covered last semester:

Definition 1 A function f from a metric space $\langle X, d\rangle$ into a metric space $\langle Y, \rho\rangle$ is said to be *uniformly continuous* provided for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $x_1, x_2 \in X$

$$d(x_0, x_1) < \delta \Longrightarrow \rho(f(x_0), f(x_1)) < \varepsilon.$$

The rest was stated only:

Theorem 1 (Thm 27.6, Uniform continuity theorem) *Let f be a continuous function from a compact metric space $\langle X, d\rangle$ into a metric space $\langle Y, \rho\rangle$. Then f is uniformly continuous.*

Prove using the Lebesgue number lemma, where a *diameter* of a subset D of a metric space $\langle X, d\rangle$ is defined as $\operatorname{diam}(D) = \sup\{d(x, y) \colon x, y \in D\}$.

Lemma 2 (Lem 27.5, the Lebesgue number lemma) *Let $\mathcal{A}$ be an open cover of a metric space $\langle X, d\rangle$. If X is compact, then there exists a $\delta > 0$, known as a* **Lebesgue number***, such that for every $D \subset X$ of diameter $< \delta$, there exists an $A \in \mathcal{A}$ with $D \subset A$.*

Class of January 17:

Administer Review Quiz #0. Collect homework. Recall that:

- A function f from a metric space $\langle X, d\rangle$ into a metric space $\langle Y, \rho\rangle$ is said to be *uniformly continuous* provided for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $x_1, x_2 \in X$
$$d(x_0, x_1) < \delta \Longrightarrow \rho(f(x_0), f(x_1)) < \varepsilon.$$
- **(Thm 27.6, Uniform continuity theorem)** Let f be a continuous function from a compact metric space $\langle X, d\rangle$ into a metric space $\langle Y, \rho\rangle$. Then f is uniformly continuous.
- A *diameter* of a subset D of a metric space $\langle X, d\rangle$ is defined as the number $\text{diam}(D) = \sup\{d(x, y) \colon x, y \in D\}$.
- **(Lem 27.5, the Lebesgue number lemma)** Let $\mathcal{A}$ be an open cover of a metric space $\langle X, d\rangle$. If X is compact, then there exists a $\delta > 0$, known as a **Lebesgue number**, such that for every $D \subset X$ of diameter $< \delta$, there exists an $A \in \mathcal{A}$ with $D \subset A$.

New material:

- For a metric space $\langle X, d\rangle$, $x \in X$, and non-empty $A \subset X$ we define the *distance from x to A* as $d(x, A) = \inf\{d(x, a) \colon a \in A\}$.
- Show that $X \ni x \mapsto d(x, A) \in \mathbb{R}$ is continuous.
- Prove the Lebesgue number lemma.
- Prove the uniform continuity theorem.

Definition 2 A point x in a topological space X is an *isolated point* provided $\{x\}$ is open in X.

Theorem 3 (Thm 27.7) *Let X be a compact Hausdorff space. If X has no isolated points, then X is uncountable.*

Expect other quiz soon, possibly the next class. It may include the same questions as in Quiz #0. It will also concern connectivity, to be reviewed today.

Go over Exercise 5 page 178. This is Baire category theorem.

Written assignment for Tuesday, January 24: Ex 11 p. 171. You can use, without a proof, the result from Ex 5 p. 171.

Class of January 19:

To be covered this semester

Finish Chapter 3: Sections 25, 27, 28

Countability axioms

- A topological space X is *first countable* (or *satisfies the first countability axiom*) provided for every $x \in X$ there exists a countable basis $\mathcal{B}_x$ of X at x.
- A topological space X is *second countable* (or *satisfies the second countability axiom*) provided X has a countable basis.
- A topological space X is *separable* provided X has a countable dense subset D, that is, such that $\text{cl}(D) = X$.
- A topological space X is *Lindelöf* provided every open cover of X has a countable subcover.

Separation axioms

- (already seen) X is a T_0 *space* provided for every distinct $x, y \in X$ there exists an open set $U \subset X$ such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$ (i.e., such that U contains precisely one of the points x and y).
- (already seen) X is a T_1 *space* provided for every distinct $x, y \in X$ there exists an open set $U \subset X$ such that $x \in U$ and $y \notin U$.
- (already seen) X is *Hausdorff* (or a T_2 *space*) provided for every distinct $x, y \in X$ there exist disjoint open sets $U, V \subset X$ such that $x \in U$ and $y \in V$.
- (new) X is *regular* (or a T_3 *space*) provided it is a T_1 space and for every closed set K in X and $x \in X \setminus K$ there exist disjoint open sets $U, V \subset X$ such that $x \in U$ and $K \subset V$.

- (new) X is *normal* (or a T_4 *space*) provided it is a T_1 space and for every disjoint closed sets K and L in X there exist disjoint open sets $U, V \subset X$ such that $K \subset U$ and $L \subset V$.
- (new) X is *completely regular* (or a $T_{3\frac{1}{2}}$ *space*) provided it is a T_1 space and for every closed set K in X and $x \in X \setminus K$ there exist a continuous function $f\colon X \to [0, 1]$ such that $f(x) = 0$ and $f[K] \subset \{1\}$.

Important related theorems

- *Urysohn Lemma*: Every T_4 space is a $T_{3\frac{1}{2}}$ space.
- *Tietze Extension Theorem*: If X is normal, $K \subset X$ is closed, and $f\colon K \to [0, 1]$ is continuous, then f can be extended to a continuous $F\colon X \to [0, 1]$.
- *Urysohn Metrization Theorem*: If X is regular and second countable, then it is metrizable.

The Tychonoff Theorem

- *Tychonoff Theorem*: Arbitrary product of compact spaces is compact.

Some material from Chapters 6-8

Other suggested exercises (to solve at home, no homework):

- 2 page 177 and 6 page 178;
- also, from section 26: exercises 8 and 10.

Go over Section 25

Define components and path components. Go over thms 25.1 and 25.2.

Go over Examples 1 and 2.

Define locally connected spaces and locally path connected spaces.

Go over Example 3.

Briefly discuss Exercise 10: quasi components.

Go over Theorems 25.3, 25.4, and 25.5.

Class of January 24:

Collect homework.

Section 28: Limit Point Compactness

Definition 3 A space X is *limit point compact* provided every infinite subset of X has a limit point.

Theorem 4 (Thm 28.1) *If X is compact, then X is limit point compact, but not conversely.*

Go over Examples 1 and 2.

Definition 4 A space X is *sequentially compact* provided every sequence in X has a convergent subsequence.

Theorem 5 (Thm 28.2) *For a metrizable space X, the following are equivalent:*

(1) X is compact.

(2) X is limit point compact.

(3) X is sequentially compact.

Remark: "(2) implies (3)" requires only first countability of X.

Discuss Exercise 7 pages 181–182.

Class of January 26:

Administer Quiz #1.
Recall that:

- X is: *limit point compact* provided every infinite subset of X has a limit point; *sequentially compact* provided every sequence in X has a convergent subsequence.
- **Thm** For metrizable spaces, the three notions, *compactness*, *limit point compactness*, and *sequential compactness*, are equivalent.

Mention Example 3: the space satisfies (1)–(3), but is not first countable, so not metrizable.
Go over Exercises 1 and 2 page 181.

Written assignment for Tuesday, January 31: Ex 3 p. 181.

Section 29: Local Compactness

Definition 5 A space X is *locally compact* provided every $x \in X$ there is an open set $U \ni x$ such that $\mathrm{cl}(U)$ is compact.

Compact implies locally compact.
Go over Examples 1 and 2.
$\mathbb{Q}$ is not locally compact: Exercise 1 page 186.
State and prove Theorem 29.1.
Define *one-point compactification* of a locally compact space.
Go over Example 4.

Time permitting
State and prove Theorem 29.2.
State and prove Corollary 29.3. (Stated only.)
State and prove Corollary 29.4. (Stated only.)
Go over Exercise 3 page 186.

Time permitting, consider going over (not covered):
Exercises 2 page 177 and 6 page 178;
from section 26 – Exercises 8 and 10.

Next class we will start new chapter:
Countability and Separation Axioms

Class of January 31:

Return quiz. Collect homework.

Be ready for quiz next class!

Section 30: The Countability Axioms

The next definition and theorem were covered last semester.

Definition 6 A topological space X is *first countable* (or *satisfies the first countability axiom*) provided for every $x \in X$ there exists a countable basis $\mathcal{B}_x$ of X at x.

Theorem 6 *Let X be a first countable topological space and let $A \subset X$. Then $x \in \text{cl}(A)$ if, and only if, there is a sequence of points of A converging to x. Moreover, the implication "$\Longleftarrow$" does not require the assumption of first countability.*

New material:

Definition 7 A topological space X is *second countable* (or *satisfies the second countability axiom*) provided X has a countable basis.

Go over Examples 1 and 2.

Go over Theorem 30.2.

Definition 8

- A subset A of a space X is *dense (in X)* provided $\text{cl}(A) = X$.
- A topological space X is *separable* provided X has a countable dense subset D, that is, such that $\text{cl}(D) = X$.
- A topological space X is *Lindelöf* provided every open cover of X has a countable subcover.

Go over Theorem 30.3.

Written assignment for Tuesday, February 7: Exercise 14, page 194. Hint: use the ideas form the proof, that the product of two compact spaces is compact.

Go over Examples 3 and 4. (Very important!)

Class of February 2:

Collect homework. Administer Quiz #2.

Recall that:

- A topological space X is: *second countable* provided X has a countable basis; *separable* provided X has a countable dense subset; *Lindelöf* provided every open cover of X has a countable subcover.
- Second countability implies: first countability, separability, and Lindelöf property. None of these implies can be reversed, as proved by $\mathbb{R}_l$.
- $\mathbb{R}_\ell$ is Lindelöf; main steps:
  - Note, that it is enough to consider only the covers $\{[a_\xi, b_\xi)\}_{\xi\in J}$.
  - $\mathcal{U} = \{(a_\xi, b_\xi)\}_{\xi\in J}$ is an open cover of $C = \bigcup_{\xi\in J}(a_\xi, b_\xi)$.
  - We can find countable $J_0 \subset J$ with $C = \bigcup_{\xi\in J_0}(a_\xi, b_\xi)$.
  - $\mathbb{R}\backslash C$ is countable. Find countable $J_1 \subset J$ with $\mathbb{R}\backslash C \subset \bigcup_{\xi\in J_1}[a_\xi, b_\xi)$.
  - $\mathcal{V}_0 = \{[a_\xi, b_\xi)\}_{\xi\in J_0\cup J_1} \subset \mathcal{V}$ is countable and covers $\mathbb{R} = C\cup(\mathbb{R}\backslash C)$.
- Product of Lindelöf spaces need not be Lindelöf, as proved by $\mathbb{R}_l$.

New material: (Last column of table not covered.)
Go over Exercises 2, 4, and 5, page 194; 16, page 195.
Suggested Exercises to examine by the students: 12, 13, 16, page 194.

Written assignment for Tuesday, February 7: Exercise 10, page 194:

Show that a countable product of separable spaces is separable.

	subspace	closed subspace	countable product	arbitrary product	continuous image
compact	N, $[0,1)$; $[0,1]$	Y	Y, Tych. Thm	Y, Tych. Thm	Y
connected	N, $\mathbb{R}\setminus(0,1)$	N, $\mathbb{R}\setminus(0,1)$	Y	Y	Y
Hausdorff	Y	Y	Y	Y	N, Ex a
2nd count	Y	Y	Y	N, $\mathbb{R}^{\text{uncount}}$ p. 133	N, Ex c
1st count	Y	Y	Y	N, $\mathbb{R}^{\text{uncount}}$ p. 133	N, Ex b,c
separable	N, $L \subset (\mathbb{R}_\ell)^2$	N, $L \subset (\mathbb{R}_\ell)^2$	Y, 10 p. 194	N, 16 p. 195	Y, 11 p. 194
Lindelöf	N, $\mathbb{R}_\infty$	Y, 9 p. 194	N, $(\mathbb{R}_\ell)^2$	N, $(\mathbb{R}_\ell)^2$	Y, 11 p. 194
metrizable	Y	Y	Y	N, $\mathbb{R}^{\text{uncount}}$ p. 133	N, Ex a

Class of February 7:

Collect homework. Administer Quiz #3.

Go over Exercises 1 and 11 page 194. Discuss last column of the table.

	subspace	closed subspace	countable product	arbitrary product	continuous image
compact	N, $[0,1); [0,1]$	Y	Y, Tych. Thm	Y, Tych. Thm	Y
connected	N, $\mathbb{R} \setminus (0,1)$	N, $\mathbb{R} \setminus (0,1)$	Y	Y	Y
Hausdorff	Y	Y	Y	Y	N, Ex a
2nd count	Y	Y	Y	N, $\mathbb{R}^{\text{uncount}}$ p. 133	N, Ex c
1st count	Y	Y	Y	N, $\mathbb{R}^{\text{uncount}}$ p. 133	N, Ex b,c
separable	N, $L \subset (\mathbb{R}_\ell)^2$	N, $L \subset (\mathbb{R}_\ell)^2$	Y, 10 p. 194	N, 16 p. 195	Y, 11 p. 194
Lindelöf	N, $\mathbb{R}_\infty$	Y, 9 p. 194	N, $(\mathbb{R}_\ell)^2$	N, $(\mathbb{R}_\ell)^2$	Y, 11 p. 194
metrizable	Y	Y	Y	N, $\mathbb{R}^{\text{uncount}}$ p. 133	N, Ex a

Here the space X_∞, in particular $\mathbb{R}_\infty$, is the one point compactification of a discrete space X, that is, $X_\infty = X \cup \{\infty\}$, where $\infty \notin X$, has the topology $\tau = \mathcal{P}(X) \cup \{X_\infty \setminus F \colon F$ is a finite subset of $X\}$.

Example. For a set X let τ_d be a discrete topology on X and $\mathcal{T}$ an arbitrary topology on X. Then a function $f \colon \langle X, \tau_d \rangle \to \langle X, \mathcal{T} \rangle$, given by $f(x) = x$, is continuous bijection.

(a) If $\mathcal{T} = \{\emptyset, X\}$ is anti-discrete topology and $X = \mathbb{N}$, then domain of f is metric, while $f[X]$ is not Hausdorff.

(b) If $X = \mathbb{R}^\omega$ and $\mathcal{T}$ is a box topology, then domain of f is first countable (as metric), while $f[X]$ is not first countable.

(c) Let $X = \mathbb{N}$ and $\mathcal{T}$ be such that $\langle X, \mathcal{T} \rangle$ is not second countable. (This will be proved together with Tychonoff's theorem.) Then domain of f is second countable, while $f[X]$ is not (since it is not first countable).

Start new section:

Section 31: The Separation Axioms

- (already seen) X is a T_0 *space* provided for every distinct $x, y \in X$ there exists an open set $U \subset X$ such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$ (i.e., such that U contains precisely one of the points x and y).
- (already seen) X is a T_1 *space* provided for every distinct $x, y \in X$ there exists an open set $U \subset X$ such that $x \in U$ and $y \notin U$.
- (already seen) X is *Hausdorff* (or a T_2 *space*) provided for every distinct $x, y \in X$ there exist disjoint open sets $U, V \subset X$ such that $x \in U$ and $y \in V$.
- (new) X is *regular* (or a T_3 *space*) provided it is a T_1 space and for every closed set K in X and $x \in X \setminus K$ there exist disjoint open sets $U, V \subset X$ such that $x \in U$ and $K \subset V$.
- (new) X is *normal* (or a T_4 *space*) provided it is a T_1 space and for every disjoint closed sets K and L in X there exist disjoint open sets $U, V \subset X$ such that $K \subset U$ and $L \subset V$.

Go over Lemma 31.1.

Go over Exercises 1 and 2.

Go over Theorem 31.2: a subspace of regular space is regular; the product of regular spaces is regular. (Same for Hausdorff spaces, as proved last semester.) Note that it is false for the normal spaces. (Point to where the subspace part of the proof for the regular spaces brakes for the normal spaces.)

Class of February 9:

Recall

- X is *regular* (or a T_3 *space*) provided it is a T_1 space and: for every closed set K in X and $x \in X \setminus K$ there exist disjoint open sets $U, V \subset X$ such that $x \in U$ and $K \subset V$ (equivalently, for every open U and $x \in U$, there an exists open V with $x \in V \subset \text{cl}(V) \subset U$).

- X is *normal* (or a T_4 *space*) provided it is a T_1 space and: for every disjoint closed sets K and L in X there exist disjoint open sets $U, V \subset X$ such that $K \subset U$ and $L \subset V$ (equivalently, for every open U and closed $F \subset U$, there exists an open V with $F \subset V \subset \text{cl}(V) \subset U$).

- Theorem 31.2: a subspace of regular space is regular; the product of regular spaces is regular.

Go over Example 1: $\mathbb{R}_K$ is Hausdorff but not regular.
Go over Exercise 4.
Go over Example 2: $\mathbb{R}_\ell$ is normal.
Go over Theorem 7.8 (set theoretical).
Use it, in Example 3, to show that $(\mathbb{R}_\ell)^2$ is not normal.

Note that the product of normal spaces need not be normal. Also, $(\mathbb{R}_\ell)^2$ is regular but not normal.

Latter we will prove that $(\mathbb{R}_\ell)^2$ is homeomorphic to a subspace of some normal spaces. So, a subspace of normal space need not be normal.

Written assignment for Tuesday, February 14: Exercise 5, page 199.

Class of February 14:

Collect homework.

Recall

- $\mathbb{R}_K$ is Hausdorff but not regular.
- $\mathbb{R}_\ell$ is normal.
- $(\mathbb{R}_\ell)^2$ is not normal (but regular). So, product of normal spaces need not be normal.

Section 32: Normal spaces

Show that every regular Lindelöf space is normal. This is Ex 4 page 205. Proof the same as for Thm 32.1.

Corollary: the product of two Lindelöf spaces need not be Lindelöf, justified by $(\mathbb{R}_\ell)^2$.

Thm 32.2: Every metrizable space is normal.

Thm 32.3: Every compact Hausdorff space is normal.

Go (briefly) over Example 1.

Go over Exercises 1, 2, 3, 5.

Class of February 16:

Be ready for a quiz next class. *I may also ask in it to state some results.*

Section 33: The Urysohn Lemma

Prove:

Urysohn Lemma: *If X is normal and A and B are closed disjoint subsets of X, then there exists continuous $f\colon X \to [0,1]$ such that $f[A] \subseteq \{0\}$ and $f[B] \subseteq \{1\}$.*

Define *completely regular* (or $T_{3.5}$) spaces.

Prove Theorem 33.2.

Go over a part of Exercise 4, page 213:

(i) *If $f\colon X \to [0,1]$ is continuous, then $A = f^{-1}(0)$ is a G_δ-set (that is, A is an intersection of countably many open sets).*

Written assignment for Tuesday, February 21: A more difficult direction of Exercise 4, page 213:

(a) Prove that if X is normal, then for every closed G_δ set $A \subset X$ there is a continuous $f\colon X \to [0,1]$ such that $f^{-1}(0) = A$.

Exercise 5, page 213 (a version of Urysohn Lemma): Let X be normal. *There exists a continuous $f\colon X \to [0,1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$ if, and only if, A and B are disjoint closed G_δ sets.*

PROOF. "$\Longrightarrow$" follows from (i).

"$\Longleftarrow$" By (a) there exists continuous functions $f_A, f_B\colon X \to [0,1]$ with $f_A^{-1}(0) = A$ and $f_B^{-1}(0) = B$. Then $f(x) = \frac{f_A(x)}{f_A(x)+f_B(x)}$ is as needed. ∎

Suggestion: Look over Exercises 1 and 3, page 212; 7 and 8, page 213.

Class of February 21:

Collect homework. Administer Quiz # 4. Recall

- **Urysohn Lemma:** *If X is normal and $A, B \subset X$ are closed disjoint, then there is continuous $f\colon X \to [0,1]$ s.t. $f[A] \subseteq \{0\}$ and $f[B] \subseteq \{1\}$.*
- X is *completely regular* (or a $T_{3.5}$ *space*) when it is a T_1 space and for every closed $K \subset X$ and $x \in X \setminus K$ there is continuous $f\colon X \to [0,1]$ s.t. $f[K] \subseteq \{0\}$ and $f(x) = 1$.
- Thm 33.2: Subspace of a completely regular space is completely regular. Product of completely regular spaces is completely regular.

Go over the expanded table:

	subspace	closed subspace	countable product	arbitrary product	continuous image
2nd countable	Y	Y	Y	N, $\mathbb{R}^{\text{uncountable}}$	N, Ex c
1st countable	Y	Y	Y	N, $\mathbb{R}^{\text{uncountable}}$	N, Ex b,c
separable	N, $L \subset (\mathbb{R}_\ell)^2$	N, $L \subset (\mathbb{R}_\ell)^2$	Y, 10 p. 194	N, 16 p. 195	Y, 11 p. 194
Lindelöf	N, $\mathbb{R}_\infty$	Y	N, $(\mathbb{R}_\ell)^2$	N, $(\mathbb{R}_\ell)^2$	Y, 11 p. 194
compact	N, $[0,1); [0,1]$	Y	Y, Tych. Thm	Y, Tych. Thm	Y
connected	N, $\mathbb{R} \setminus (0,1)$	N, $\mathbb{R} \setminus (0,1)$	Y	Y	Y
Hausdorff	Y	Y	Y	Y	N, Ex a
regular	Y	Y	Y	Y	N, Ex a
completely reg	Y	Y	Y	Y	N, Ex a
normal	N, p. 203	Y, 1 p. 205	N, $(\mathbb{R}_\ell)^2$	N, $(\mathbb{R}_\ell)^2$	N, Ex a
metrizable	Y	Y	Y	N, $\mathbb{R}^{\text{uncountable}}$	N, Ex a

Answers

Example. For a set X let τ_d be a discrete topology on X and $\mathcal{T}$ an arbitrary topology on X. Then a function $f\colon \langle X, \tau_d\rangle \to \langle X, \mathcal{T}\rangle$, given by $f(x) = x$, is continuous bijection.

(a) If $\mathcal{T} = \{\emptyset, X\}$ is anti-discrete topology and $X = \mathbb{N}$, then domain of f is metric, while $f[X]$ is not Hausdorff.

(b) If $X = \mathbb{R}^\omega$ and $\mathcal{T}$ is a box topology, then domain of f is first countable (as metric), while $f[X]$ is not first countable.

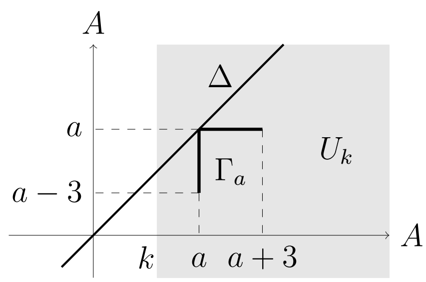
(c) Let $X = \mathbb{N}$ and $\mathcal{T}$ be such that $\langle X, \mathcal{T}\rangle$ is not second countable. (This will be proved together with Tychonoff's theorem.) Then domain of f is second countable, while $f[X]$ is not (since it is not first countable).

Example of a regular not completely regular space
Based on section 2.1 of 2016 Ciesielski and Wojciechowski paper.

Let $A \subseteq \mathbb{R}$ be such that the intersection $A_k = A \cap [k, k+1)$ is uncountable for every integer $k \in \mathbb{Z}$. Let $\Delta = \{\langle a, a\rangle : a \in A\}$ be the diagonal of $X = A^2$ and define the following sets, displayed in the figure:

$$U_k = \{\langle a, b\rangle \in X : a > k\} \quad \text{for } k \in \mathbb{Z},$$

$$\Gamma_a = \{\langle a+\varepsilon, a\rangle \in X : \varepsilon \in [0,3]\} \cup \{\langle a, a-\varepsilon\rangle \in X : \varepsilon \in [0,3]\} \quad \text{for } a \in A.$$



Consider a topology $\mathcal{T}$ on $X = A^2$ generated by a basis consisting of all singletons $\{x\}$ with $x \in X \setminus \Delta$ and all sets $\Gamma_a \setminus F$, where $a \in A$ and F is finite. Clearly X is Hausdorff and zero-dimensional, so, completely regular. Let $\mathcal{E} = \{U_k : k \in \mathbb{Z}\}$.

Definition 9 Let $X_{\mathcal{E}} = X \cup \{-\infty, \infty\}$ be endowed with a topology generated by a basis $\mathcal{T} \cup \mathcal{B}^{\mathcal{E}}_{\infty} \cup \mathcal{B}^{\mathcal{E}}_{-\infty}$, where

$$\mathcal{B}^{\mathcal{E}}_{\infty} = \{\{\infty\} \cup U_k : k \in \mathbb{Z}\}, \quad \mathcal{B}^{\mathcal{E}}_{-\infty} = \left\{\{-\infty\} \cup \widetilde{U_k} : k \in \mathbb{Z}\right\},$$

and $\widetilde{U_k}$ is defined as $X \setminus \mathrm{cl}_X(U_k)$.

Theorem 7 *$X_{\mathcal{E}}$ is a regular space such that $f(-\infty) = f(\infty)$ for every continuous $f : X_{\mathcal{E}} \to [0,1]$. In particular, $X_{\mathcal{E}}$ is not completely regular.*

Proof. (Sketch) Indeed, every continuous $f : X \to [0,1]$ is constant on $\Delta \setminus S$ for some countable $S \subset \Delta$. So, if $f : X_{\mathcal{E}} \to [0,1]$ is continuous, then the restriction of f to X gives $f[\Delta \setminus S] = \{z\}$ for some $z \in Z$ and this ensures that $f(-\infty) = z = f(\infty)$. ■

Go over Exercise 2, page 212.

Class of February 23:

Recall

- **Urysohn Lemma:** *If X is normal and A and B are closed disjoint subsets of X, then there exists continuous $f\colon X \to [0,1]$ such that $f[A] \subseteq \{0\}$ and $f[B] \subseteq \{1\}$.*

New material:

Theorem 8 (Urysohn metrization theorem) *Every regular second countable space X is metrizable.*

Proof.

1. Notice that every regular second countable space is normal (as it is regular Lindelöf), so we can use Urysohn Lemma.

2. Prove that there exists a countable family $\mathcal{F}$ of continuous functions $f\colon X \to [0,1]$ such that:

   (∗) For every open $U \subset X$ and $x \in U$ there is an $f \in \mathcal{F}$ such that $f(x) > 0$ and $f[X \setminus U] \subset \{0\}$.

   A family $\mathcal{F}$ of continuous functions $f\colon X \to \mathbb{R}$ satisfying (∗) is said to *separate points from closed sets* in X.

3. Prove that (Thm 34.2): *For any T_1 space X family $\{f_\alpha\}_{\alpha \in J}$ separating points from closed sets in X the mapping $F\colon X \to \mathbb{R}^J$, $F(x)(\alpha) = f_\alpha(x)$, is an imbedding.*

4. Notice that, by 1 and 2, our regular second countable space X can be imbedded into $\mathbb{R}^\omega$. Since $\mathbb{R}^\omega$ is metrizable, so is X.

Notice that (second version of the proof of thm 34.1):
Every regular second countable space X can be imbedded into $\mathbb{R}^\omega$ considered with the uniform topology.

Proof. By 2, there exists a family $\{f_n\}_{n=1}^\infty$ separating points from closed sets in X, with $f_n\colon X \to [0,1]$. Replacing f_n with f_n/n, if necessary, we can assume that $f_n\colon X \to [0,1/n]$. Therefore, by 3, there is an imbedding F of X into $T = \prod_{n=1}^\infty [0,1/n]$.

Then, the theorem follows from the fact that
The uniform topology on T coincides with the product topology.

State and prove Theorem 34.3.

Go over Exercises 1, 2, 3, and 4, page 218.

Class of February 28:

Collect homework.

Mid term test will be given on Thursday, March 16. It will be in class, no notes allowed. I will give you additional time to write it, up to 2 hours. One of the exercises will be in a format of a quiz.

Recall

- **Urysohn Lemma:** *If X is normal and A and B are closed disjoint subsets of X, then there exists continuous $f\colon X \to [0,1]$ such that $f[A] \subseteq \{0\}$ and $f[B] \subseteq \{1\}$.*
- **Urysohn metrization theorem**: *Every regular second countable space X is metrizable.*
- *X is completely regular iff X is homeomorphic to a subspace of $[0,1]^J$ for some J.*

New material:

State and prove **Tietze Extension Theorem**: I*f X is normal, $K \subset X$ is closed, and $f\colon K \to [0,1]$ is continuous, then f can be extended to a continuous $F\colon X \to [0,1]$.*

Notice, that if $X = \mathbb{R}$, then Tietze Extension Theorem is obvious: the linear interpolation of f is continuous.

Prove that Tietze Extension Theorem is true, when the interval $[0,1]$ is replaced with $\mathbb{R}$.

Go over Exercises 1 and 5(a), page 223.

Note that, for example, a circle does not have universal extension property (using Brower Fixed Point Theorem).

Class of March 2:

Return corrected homework and give its printed solution to the students.

Solve the following problems from old Topology Entrance Exams, TEE:

Ex. 1. (TEE Fall 2012) Consider the following subsets, $\vdash$ and $\models$, of $\mathbb{R}^2$, where $\mathbb{R}^2$ is endowed with the standard topology:

$$\vdash = (\{0\}\times[-2,2])\cup([0,2]\times\{0\}) \quad \& \quad \models = (\{0\}\times[-2,2])\cup([0,2]\times\{-1,1\}).$$

Prove, or disprove the following:

(i) There exists a continuous function from $\vdash$ onto $\models$.

(ii) There exists a continuous function from $\models$ onto $\vdash$.

Your argument must be precise, but no great details are necessary.

Ex. 2. (TEE Fall 2012) For the topologies τ and σ on $\mathbb{R}$ let symbol $C(\tau,\sigma)$ stand for the family of all continuous functions from $\langle\mathbb{R},\tau\rangle$ into $\langle\mathbb{R},\sigma\rangle$.

Let $\mathcal{T}_s$ be the standard topology on $\mathbb{R}$ and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on $\mathbb{R}$ such that $C(\mathcal{T}_1,\mathcal{T}_2)=C(\mathcal{T}_s,\mathcal{T}_s)$. Show that:

(i) $\mathcal{T}_2\subseteq\mathcal{T}_1$, that is, $\mathcal{T}_1$ is finer than $\mathcal{T}_2$.

(ii) $\mathcal{T}_2\neq\{\emptyset,\mathbb{R}\}$, that is, $\mathcal{T}_2$ is not trivial.

(iii) $\langle\mathbb{R},\mathcal{T}_1\rangle$ is connected.

(Notice that $C(\mathcal{T}_1,\mathcal{T}_2)=C(\mathcal{T}_s,\mathcal{T}_s)$ does not imply that either of the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ must be equal to the standard topology $\mathcal{T}_s$.)

Ex. 3. (TEE Fall 2015) Let $\mathbb{Q}$ be the set of rational numbers considered with the standard topology. Prove that a topological space X is disconnected if, and only if, there exists a non-constant continuous function from X into $\mathbb{Q}$.

Ex. 4. (TEE Fall 2015 plus extra) Let X be a compact Hausdorff space and Y be Hausdorff. Assume that $f,g\colon X\to Y$ are the continuous functions.

(i) Show that the graph $G(f)=\{\langle x,f(x)\rangle\colon x\in X\}$ of f is compact (as a subspace of $X\times Y$).

(ii) Assume that $X \times Y$ is a metric space. Use part (i) to show that

($\bullet$) $\mathrm{dist}(G(f), G(g)) > 0$ if, and only if, $f(x) \neq g(x)$ for all $x \in X$.

Recall, that the distance $\mathrm{dist}(A, B)$ between the non-empty subsets A and B of a metric space $\langle Z, \rho \rangle$ is defined via formula $\mathrm{dist}(A, B) = \inf\{\rho(a, b) \colon a \in A \ \&\ b \in B\}$.

(iii) Show, by giving an example, that the characterization ($\bullet$) is false if X is not compact.

Ex. 5. (TEE Spring 2015) Let X be a compact topological space and let $f \colon X \to \mathbb{R}$ be an arbitrary, **not necessary continuous**, function. Assume that f is locally bounded, that is, that for every $x \in X$ there exists an open $U \ni x$ such that $f[U]$ is bounded in $\mathbb{R}$. Show that $f[X]$ is bounded in $\mathbb{R}$.

Ex. 6. (TEE Fall 2014) Let X be a regular space. Show that for every disjoint closed countable subsets A and B of X there exist disjoint open subsets U and V of X such that $A \subset U$ and $B \subset V$. Do not assume that X is normal!

Ex. 7. (TEE Fall 2013) A topological space is a T_0-space provided for every distinct $x, y \in X$ there exists an open set U in X which contains precisely one of the points x and y. Show that X is a T_0-space if, and only if, $\mathrm{cl}(\{x\}) \neq \mathrm{cl}(\{y\})$ for all distinct $x, y \in X$.

Ex. 8. (TEE Fall 2013) Let $\langle X, d \rangle$ be a metric space and let $A \subset X$ be such that it has no limit points in X, that is, such that $A' = \emptyset$. Show that there exists a family $\{U_a\}_{a \in A}$ of pairwise disjoint open sets such that $a \in U_a$ for every $a \in A$.

Ex. 9. (TEE Spring 2013) Let X and Y be topological spaces and let $\pi \colon X \times Y \to X$ be a projection, that is, $\pi(x, y) = x$. Show that if Y is compact, then for every $S \subset X \times Y$ we have $\pi[\mathrm{cl}(S)] = \mathrm{cl}(\pi[S])$.

Class of March 14

Solve the following exercises and e-mail the solutions to me for March 16.

Ex. 10. Let X be connected and let $f, g\colon X \to [-1,1]$ be continuous functions. Assume that $f[X] = [-1,1]$. Show, that there exists an $x \in X$ such that $f(x) = g(x)$.

Ex. 11. Prove or disprove the following assertion.

> For a closed subset X of $\mathbb{R}^3$, X is compact if, and only if, every continuous function $f\colon X \to \mathbb{R}$ is bounded.

Recall, that a function $f\colon X \to \mathbb{R}$ is bounded, when there exist numbers $m, M \in \mathbb{R}$ such that $m \leq f(x) \leq M$ for all $x \in X$.

Ex. 12. Let X be a compact topological space and $\langle Y, d\rangle$ be a metric space. Show that for every pair of continuous functions $f\colon X \to Y$ and $g\colon X \to Y$, the extended real number $B = \sup\{d(f(x_1), g(x_2))\colon x_1, x_2 \in X\} \in [0,\infty]$ is, in fact, a real number.

Ex. 13. Let X be a topological space.

(a) Show that for every closed $F \subset X$ we have $\operatorname{int}(\operatorname{cl}(\operatorname{int}(F))) = \operatorname{int}(F)$.

(b) Give an example of an open $U \subset \mathbb{R}$, for which $\operatorname{int}(\operatorname{cl}(\operatorname{int}(U))) \neq \operatorname{int}(U)$. (This will show that part (a) need not hold for an open set F.)

Ex. 14. Let X and Y be the topological spaces and let $A \subset X$. Let $f\colon \operatorname{cl}(A) \to Y$ and $g\colon \operatorname{cl}(A) \to Y$ be continuous functions. Show that if $f \restriction A = g \restriction A$ and Y is Hausdorff, then $f = g$.

Ex. 15. Let $f\colon X \to Y$ and $g\colon X \to Z$ be continuous. Show that if A is a connected subset of X, then the set $T = \{\langle f(a), g(a)\rangle\colon a \in A\}$ is a connected subset of $Y \times Z$.

Class of March 21

Mid term test will be next class: Thursday, March 23, 5:30pm to 7:30pm.

Review for the mid term test.

Solve the following exercises

Ex. 16. Let X be a topological space.

(a) Show that for every $A \subset X$, $\mathrm{int}(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))) = \mathrm{int}(\mathrm{cl}(A))$ (i.e., that the operation $\mathrm{int}(\mathrm{cl}(\cdot))$ is idempotent). You can use in your argument, without a proof, the fact that the operations int and cl are idempotent (i.e., that $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$ and $\mathrm{int}(\mathrm{int}(A)) = \mathrm{int}(A)$.)

(b) Give an example of a subset A of $\mathbb{R}$, considered with the standard topology, for which $\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A))) \neq \mathrm{cl}(A)$.

Ex. 17. A family $\mathcal{A}$ of subsets of a topological space X is *locally discrete* when every $x \in X$ has a neighborhood that intersects at most one $A \in \mathcal{A}$. Show that if X is compact, then every locally discrete family in X is finite.

Ex. 18. Assume that $f\colon X \to Y$ is a continuous function that is an open map, that is, such that $f[U]$ is open in Y for every open U in X. Show that, for such map, we have $\mathrm{cl}\left(f^{-1}(B)\right) = f^{-1}(\mathrm{cl}(B))$ for every $B \subset Y$.

Ex. 19. Prove or give a counterexample: If $\{X_\alpha\}_{\alpha\in J}$ is a family of connected metric spaces, then $Y = \Pi_{\alpha\in J} X_\alpha$ with the uniform topology is connected.

Ex. 20. Let $\mathbb{I}$ be the set of irrational numbers and let $X \subset \mathbb{R}^2$ be such that $X \cup \mathbb{I}^2 = \mathbb{R}^2$. Show, that X is connected.

Ex. 21. Let $\mathbb{R}$ be the set of real numbers.

(a) Define the following two topologies on $\mathbb{R}$: standard $\mathcal{T}_s$ and lower (i.e., Sorgenfrey) $\mathcal{T}_\ell$.

(b) Show that if $f\colon \langle \mathbb{R}, \mathcal{T}_s\rangle \to \langle \mathbb{R}, \mathcal{T}_l\rangle$ is continuous if, and only if, f is constant.

Ex. 22. Let $\langle A_i\rangle_{i=1}^{\infty}$ be an arbitrary sequence of subsets of a topological space X. Show that

$$\mathrm{cl}\left(\bigcup_{i=1}^{\infty} A_i\right) = \left(\bigcup_{i=1}^{\infty} \mathrm{cl}(A_i)\right) \cup \bigcap_{k=1}^{\infty} \mathrm{cl}\left(\bigcup_{i=k}^{\infty} A_i\right).$$

Review continue

Definition 10 A space X is *locally (path) connected at* $x \in X$ provided for every open $U \ni x$ there is (path) connected open $V \subset U$ containing x.

$C \subset X$ is a (path) component of X if it is a maximal (path) connected subset of X.

Go over Ex 5 page 162.

The rest not covered

Go over Ex 10 page 163.

Definition 11 A space X is *locally compact* provided every $x \in X$ there is an open set $U \ni x$ such that $\mathrm{cl}(U)$ is compact.

Theorem 9 *Every Hausdorff locally compact space X admits one point compactification.*

Theorem 10 *If X is Hausdorff and locally compact, for every open $U \ni x$ there is an open $V \subset U$ containing x such that $\mathrm{cl}(V) \subset U$ is compact.*

Go over Ex 2 page 186.
Go over Ex 3 page 205.

Class of March 23:

Administer Mid Term Test, 5:30pm to 7:30pm.

Class of March 28:

Discuss the results of the Mid Term Test. Solve the exercises.
Time permitting: start the proof of Tychonoff Theorem.

Tychonoff Theorem

Go over incomplete proof of the Tychonoff Theorem, page 231.

Definition of a *filter* on a set X: *a non-empty family* $\mathcal{F} \subset \mathcal{P}(X)$ *(i.e., of subsets of* X*) such that:*

- *if* $A, B \in \mathcal{F}$*, then* $A \cap B \in \mathcal{F}$*;*
- *if* $A \in \mathcal{F}$ *and* $A \subset C \subset X$*, then* $C \in \mathcal{F}$*.*

Filter is *proper* when $\emptyset \notin \mathcal{F}$, that is, when $\mathcal{F} \neq \mathcal{P}(X)$. Note that the filter is proper if, and only if, it has the finite intersection property, fip. We will consider only proper filters.

Give an intuitive argument that any family having the finite intersection property can be extended to a maximal family having the finite intersection property. (Formal prove will be deduced from Zorn's Lemma.)

State Lemma 37.2 and the result that any family having the finite intersection property can be extended to a maximal family having the finite intersection property.

Use the above results to prove Tychonoff Theorem.

Class of March 30:

Recall that a *filter* on a set X is *a non-empty family* $\mathcal{F} \subset \mathcal{P}(X)$ *(i.e., of subsets of* X*) such that:*

- *if* $A, B \in \mathcal{F}$*, then* $A \cap B \in \mathcal{F}$*;*
- *if* $A \in \mathcal{F}$ *and* $A \subset C \subset X$*, then* $C \in \mathcal{F}$*.*

Filter is *proper* when $\emptyset \notin \mathcal{F}$, that is, when $\mathcal{F} \neq \mathcal{P}(X)$. Note that the filter is proper if, and only if, it has the finite intersection property, fip. We will consider only proper filters.

New material:

Prove Lemma 37.2. Notice also that every maximal family having the finite intersection property is a proper filter.

State Zorn Lemma and Hausdorff Maximal Principle, section 11.

Prove, using Hausdorff Maximal Principle, that every family having the finite intersection property can be extended to a maximal family having the finite intersection property.

Time permitting, review the proof that these two results imply Tychonoff Theorem.

Next class we will jump to

Section 43: Complete Metric Spaces

Class of April 4:

Students presentations, depending on the number of volunteers. *Used for this entire class time.*

Class of April 6:

Students presentations, depending on the number of volunteers. *Used for most of the class time.*

Defined of *Cauchy sequences* and *complete metric spaces.*

Sketched the construction of the Peano space filling curve, section 44.

Class of April 11:

Finish students presentations.

Section 43: Complete Metric Spaces

Recall definitions of *Cauchy sequences* and *complete metric spaces*.

New material:
Lemma 43.1.
Theorem 43.2.
Lemma 43.3.
Theorem 43.4.
Go over Examples 1 and 2.
Definition of *uniform metric*.
Theorem 43.5.
Define $\mathcal{C}(X, Y)$ and $\mathcal{B}(X, Y)$.
Theorem 43.6.

Definition of *sup metric* on the class $\mathcal{B}(X, Y)$ of all bounded functions from X into Y.

Go over Theorem 43.7 on the completion of a metric space. See also Exercise 9 for a different proof.

Class of April 13:

Review before topology entrance examination (to be held on Wednesday, April 19).

Class of April 18:

Continue reviewing for the topology entrance examination (to be held on Wednesday, April 19).

Solve the following:

Ex. 23. Let $\mathcal{T}$ be the standard topology on $\mathbb{R}$. Does there exist a finer topology $\mathcal{T}' \supset \mathcal{T}$ on $\mathbb{R}$ such that $\langle \mathbb{R}, \tau' \rangle$ is homeomorphic to S^1?

Solution: **No.** Indeed, if $h\colon \langle \mathbb{R}, \tau' \rangle \to S_1$ be a homeomorphism, then So would be its restriction $h \restriction \mathbb{R} \setminus \{0\}$. But then $X = h[\mathbb{R} \setminus \{0\}] = S_1 \setminus \{h(0)\}$ is connected. At the same time its homeomorphic preimage $\mathbb{R} \setminus \{0\}$ cannot be connected, having separation in a finer topology (standard, restricted to $\mathbb{R} \setminus \{0\}$).