Advanced Analysis of Algorithms - Homework I

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1 Instructions

1. The homework is due on September 12, in class.
2. Each question is worth 4 points.
3. Attempt as many problems as you can. You will be given partial credit, as per the policy discussed in class.
4. The work must be entirely your own. You are expressly **prohibited** from consulting with colleagues or the internet (with the exception of the material on the course website, your class notes and [NN09]).

2 Problems

1. **Combinatorics:**
   Prove the following combinatorial identities:

   (a)
   $$\binom{2 \cdot n}{2} = 2 \cdot \binom{n}{2} + n^2,\ n \geq 0.$$

   (b)
   $$\sum_{k=0}^{n} k \cdot \binom{n}{k} = n \cdot 2^{n-1},\ n \geq 0.$$

2. **Summation:**
   (a) Show that $\sum_{i=1}^{n} \frac{1}{i^2}$ is bounded above by a constant.
   (b) Give an asymptotically tight upper bound on the sum $\sum_{k=1}^{n} k \cdot \log k$.

3. **Induction:**
   (a) Show that $\sum_{k=1}^{n} k \cdot (k!) = (n+1)! - 1$.
   (b) Assume you are given an array $\mathbf{A}[1 \cdot \cdot n]$ of n sorted integers. Design an algorithm to search $\mathbf{A}$ for the presence of an element key. Prove the correctness of your algorithm.

4. **Asymptotics:**
   (a) Show that $\log n! \in \Theta(n \cdot \log n)$.
   (b) Let $f(n)$ and $g(n)$ be non-negative functions. Show that $\max(f(n), g(n)) = \Theta(f(n) + g(n))$.

5. **Discrete Probability and Random Variables:**

   (a) A laboratory blood test is 95% effective in detecting a disease, when it is present. However, the test also yields a false positive for 1% of the healthy persons tested, i.e., if a healthy person is tested, with probability 0.01, the test result will imply that he has the disease. If 0.5% of the population actually has the disease, what is the probability that a person has the disease, given that his test result is positive?

   (b) Suppose that an airplane engine will fail in flight with probability $1 - p$, independently from engine to engine. A flight will be successful if at least 50% of its engines remain operative in flight. For what values of p is a four-engine airplane preferable to a two-engine airplane?

References

[NN09] Richard Neapolitan and Kumarss Naimipour. *Foundations of Algorithms Using C++ Pseudocode*. Jones and Bartlett, 2009.