Discrete Probability

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Note

Discrete Probability combines aspects of combinatorics, logic and inference.

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1. Preliminaries
   - Sample Space and Events
   - Defining Probabilities on Events
2. Conditional Probability
3. Independent Events
4. Bayes' Formula

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Hence, $P(E \mid F) = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$. Notice that $P(E) = \frac{1}{4} \neq P(E \mid F)$.

Some more examples

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A family has two children. What is the conditional probability that both are boys given that at least one of them is a boy? Assume that the sample space is $S = \{(b,g),\ (b,b),\ (g,b),\ (g,g)\}$ and that all outcomes are equally likely.

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Bayes' Formula

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Thus, the probability of an event E is the weighted average of the conditional probability of E, given that event F has occurred and the conditional probability of E, given that event F has not occurred, each conditional probability being given as much weight as the probability of the event that it is conditioned on, has of occurring.

One Final Example

Example

Consider two urns.

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Consider two urns. Urn 1 has 2 white balls and 7 black balls. Urn 2 has 5 white balls and 6 black balls. A fair coin is tossed.

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Therefore, $P(H \mid W) = \frac{\frac{1}{9}}{\frac{67}{198}}$

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Therefore, $P(H \mid W) = \frac{\frac{1}{9}}{\frac{67}{198}} = \frac{22}{67}$, i.e., the conditional probability that the ball was drawn from Urn 1, given that it is white, is $\frac{22}{67}$.