Random Variables

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Outline

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3. Expectation of a function of a random variable

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$$P\{X = 1\} \quad =$$

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$$\begin{aligned} P\{X = 1\} &= 0 \\ P\{X = 2\} &= \frac{1}{36} \end{aligned}$$

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Definition

A random variable that can take on only a countable number of possible values is said to be *discrete*. For a discrete random variable X, the probability mass function (pmf) $p(a)$ is defined as:

$$p(a) = P\{X = a\}.$$

The Bernoulli Random Variable

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$$\begin{aligned} p(1) &= P\{X = 1\} = p \\ p(0) &= P\{X = 0\} = 1 - p \end{aligned}$$

where $0 \leq p \leq 1$ is the probability that the experiment results in a success.

The Binomial Random Variable

Motivation

Consider an experiment which consists of n independent Bernoulli trials, with the probability of success in each trial being p. If X is the random variable that counts the number of successes in the n trials, then X is said to be a Binomial Random Variable.

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$$p(i) = P\{X = i\} = C(n, i) \cdot p^i \cdot (1 - p)^{n-i}, \ i = 0, 1, 2, \ldots n$$

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Example (contd.)

Solution

Let the event of heads turning up denote a “success.”

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$$p(2) \quad = \quad C(4,2) \cdot (\frac{1}{2})^2$$

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$$
\begin{aligned}
p(2) &= C(4,2) \cdot (\frac{1}{2})^2 \cdot (1 - \frac{1}{2})^2 \\
&= \frac{3}{8}
\end{aligned}
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$$p(i) = P\{X = i\} = (1 - p)^{i-1} \cdot p, \ i = 1, 2, \ldots$$

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$$\begin{aligned} E[X] &= 1 \cdot p + 0 \cdot (1 - p) \\ &= p \end{aligned}$$

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&= \sum_{i=1}^{n} \frac{n!}{(i-1)!(n-i)!} \cdot p^i \cdot (1-p)^{n-i} \\
&= n \cdot p \sum_{i=1}^{n} \frac{(n-1)!}{(i-1)!(n-i)!} \cdot p^{i-1} \cdot (1-p)^{n-i}
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Expectation of a Binomial Random Variable (contd.)

Example

Substituting $k = i - 1$, we get,

$$E[X] = n \cdot p \sum_{k=0}^{n-1} \frac{(n-1)!}{k! \cdot (n-k-1)!} \cdot p^k \cdot (1-p)^{n-k-1}$$

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&= n \cdot p \cdot [p + (1-p)]^{n-1}, \text{ Binomial theorem}
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Solution:

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&= \sum_{i=1}^{\infty} i \cdot q^{i-1} \cdot p, \text{ where } q = 1 - p \\
&= p \cdot \sum_{i=1}^{\infty} i \cdot q^{i-1} \\
&= p \cdot \sum_{i=1}^{\infty} \frac{d}{dq}[q^i]
\end{aligned}
$$

Expectation of a Geometric Random Variable

Example

Let X denote a Geometric Random Variable with parameters n and p. What is $E[X]$?
Solution:

$$\begin{aligned}
E[X] &= \sum_{i=1}^{\infty} i \cdot p(i), \text{ by definition} \\
&= \sum_{i=1}^{\infty} i \cdot (1-p)^{i-1} \cdot p \\
&= \sum_{i=1}^{\infty} i \cdot q^{i-1} \cdot p, \text{ where } q = 1-p \\
&= p \cdot \sum_{i=1}^{\infty} i \cdot q^{i-1} \\
&= p \cdot \sum_{i=1}^{\infty} \frac{d}{dq}[q^i]
\end{aligned}$$

□

Expectation of a Geometric Random Variable (contd.)

Example

Solution:

$$E[X] = p \cdot \frac{d}{dq}[\sum_{i=1}^{\infty} q^i]$$

Expectation of a Geometric Random Variable (contd.)

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Example

Consider yet another variation to the initial game: The die is tossed ten times. For each toss that turns up an even number, A gets 5 dollars. For tosses turning up an odd number, A loses 4 dollars. How much money can A expect to make from this game?

Expectation of a function of a random variable

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Let X be a random variable, with the following pmf:

$$p(0) = 0.3, \ p(1) = 0.5, \ p(2) = 0.2$$

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Accordingly,

$$E[Y] = E[X^2] = 0 \cdot 0.3 + 1 \cdot 0.5 + 4 \cdot 0.2 = 1.3$$

Expectation of functions - The Direct Approach

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If X is a random variable with pmf $p()$, and $g()$ is any real-valued function, then,

$$E[g(X)] = \sum_{x:\ p(x)>0} g(x) \cdot p(x)$$

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Linearity of Expectation

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Note that linearity of expectation holds even when the random variables are **not** *independent.*

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Note

Note that linearity of expectation holds even when the random variables are **not** *independent.*

Example

What is the expected value of the sum of the upturned faces, when two fair dice are tossed?

Another Application

Example

Compute the expected value of the Binomial random variable.

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Solution

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Accordingly, the Binomial random variable (say X) can be expressed as:

$$X = X_1 + X_2 + \ldots X_n$$

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