Permutations, Combinations and The Binomial Theorem

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22 August, 2013

Outline

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1. Permutations
2. Combinations

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Using the multiplication principle,

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Permutations (contd.)

Example

Compute $P(7, 3)$, $P(n, 0)$, $P(n, 1)$, and $P(n, n)$.

Permutations (contd.)

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Compute $P(7, 3)$, $P(n, 0)$, $P(n, 1)$, and $P(n, n)$.
Solution: 210,

Permutations (contd.)

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Compute $P(7,3)$, $P(n,0)$, $P(n,1)$, and $P(n,n)$.
Solution: 210, 1, n, and $n!$.

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How many 3 letter words can be formed using the letters in the word “compiler”?

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In how many ways can a president and vice-president be chosen from a group of 20 people?

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Solution: 210, 1, n, and $n!$.

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How many 3 letter words can be formed using the letters in the word “compiler”?
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In how many ways can a president and vice-president be chosen from a group of 20 people?
Solution: $P(20,2)$.

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A library has 4 books on programming, 7 on algorithms and 3 on complexity. In how many ways can the books be ordered on a shelf?

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Solution: If there is no restriction, the number of arrangements is $P(14, 14) = 14!$.

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Now consider the case in which the books of a given subject are required to be together. First arrange the three subjects.

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Provided that the books of a subject are required to be together?
Solution: If there is no restriction, the number of arrangements is $P(14, 14) = 14!$.
Now consider the case in which the books of a given subject are required to be together. First arrange the three subjects. This can be done in $P(3, 3) = 3!$ ways.

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A library has 4 books on programming, 7 on algorithms and 3 on complexity. In how many ways can the books be ordered on a shelf?
Provided that the books of a subject are required to be together?
Solution: If there is no restriction, the number of arrangements is $P(14, 14) = 14!$.
Now consider the case in which the books of a given subject are required to be together. First arrange the three subjects. This can be done in $P(3, 3) = 3!$ ways.
Corresponding to each such arrangement, the programming books can be permuted in $P(4, 4) = 4!$ ways, the algorithms books can be permuted in $P(7, 7) = 7!$ ways and the complexity books can be permuted in $P(3, 3) = 3!$ ways.

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Now consider the case in which the books of a given subject are required to be together. First arrange the three subjects. This can be done in $P(3, 3) = 3!$ ways.
Corresponding to each such arrangement, the programming books can be permuted in $P(4, 4) = 4!$ ways, the algorithms books can be permuted in $P(7, 7) = 7!$ ways and the complexity books can be permuted in $P(3, 3) = 3!$ ways. Using the multiplication principle, the total number of arrangements is $3! \cdot 4! \cdot 7! \cdot 3!$.

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Solution: 35,

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Solution: 35, 1, n, 1.

Combinations (examples)

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2. it must contain exactly one freshman.

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3. it can contain at most one freshman. Solution: $C(34, 8)$

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3. it can contain at most one freshman. Solution: $C(34,8) + C(19,1) \cdot C(34,7)$.

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3. it can contain at most one freshman. Solution: $C(34,8) + C(19,1) \cdot C(34,7)$.
4. it contains at least one freshman.

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2. it must contain exactly one freshman. Solution: $C(19, 1) \cdot C(34, 7)$.
3. it can contain at most one freshman. Solution: $C(34, 8) + C(19, 1) \cdot C(34, 7)$.
4. it contains at least one freshman. Solution: $C(53, 8) - C(34, 8)$.

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(ii) $(a+b)^2 = a^2 + 2ab + b^2$.

(iii) $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

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(iv) $(a+b)^4 =???$

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(iii) $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

(iv) $(a+b)^4 =???$

We want a general formula that permits us to write down the terms of $(a+b)^n$ without actual multiplication.

Outline

Pascal's Triangle

The coefficient table

Consider the following table:

Pascal's Triangle

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Row 0: $C(0, 0)$

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Consider the following table:

Row 0:		$C(0,0)$	
Row 1:	$C(1,0)$		$C(1,1)$

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Row 0:			$C(0,0)$		
Row 1:		$C(1,0)$		$C(1,1)$	
Row 2:	$C(2,0)$		$C(2,1)$		$C(2,2)$

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Row 1:			$C(1,0)$		$C(1,1)$			
Row 2:		$C(2,0)$		$C(2,1)$		$C(2,2)$		
Row 3:	$C(3,0)$		$C(3,1)$		$C(3,2)$		$C(3,3)$	

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Row 3:	$C(3, 0)$		$C(3, 1)$		$C(3, 2)$		$C(3, 3)$	

$\vdots$

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Row 2:				$C(2,0)$		$C(2,1)$		$C(2,2)$					
Row 3:			$C(3,0)$		$C(3,1)$		$C(3,2)$		$C(3,3)$				
$\vdots$													
Row n:	$C(n,0)$			$C(n,1)$	$\ldots$	$\ldots$		$C(n,n-1)$		$C(n,n)$			

Pascal's triangle (contd.)

The Value Table

Writing down the values of the terms gives the following table:

Pascal's triangle (contd.)

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Row 0:	1

Pascal's triangle (contd.)

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Pascal's triangle (contd.)

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Row 2:			1		2		1			
Row 3:		1		3		3		1		
$\vdots$										
Row n:	1		n		$\ldots$		n		1	

Pascal's formula

Theorem

$C(n, k) = C(n-1, k-1) + C(n-1, k), 1 \leq k \leq n-1.$

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Proof.

Observe that,

$$C(n-1, k-1) + C(n-1, k) \quad = \quad \frac{(n-1)!}{(k-1)![(n-1-(k-1)!]} + \frac{(n-1)!}{k!(n-1-k)!}$$

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$C(n, k) = C(n-1, k-1) + C(n-1, k), 1 \leq k \leq n-1.$

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Observe that,

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C(n-1, k-1) + C(n-1, k) &= \frac{(n-1)!}{(k-1)![(n-1-(k-1)!]} + \frac{(n-1)!}{k!(n-1-k)!} \\
&= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-1-k)!} \\
&= \frac{k(n-1)!}{k!(n-k)!} + \frac{(n-k)(n-1)!}{k!(n-k)!} \\
&= \frac{(n-1)!}{k!(n-k)!}[k + (n-k)]
\end{aligned}
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&= \frac{(n-1)!}{k!(n-k)!}[k + (n-k)] \\
&= \frac{n(n-1)!}{k!(n-k)!}
\end{aligned}$$

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&= \frac{n!}{k!(n-k)!}
\end{aligned}
$$

Pascal's formula

Theorem

$C(n, k) = C(n-1, k-1) + C(n-1, k), 1 \leq k \leq n-1.$

Proof.

Observe that,

$$\begin{aligned} C(n-1, k-1) + C(n-1, k) &= \frac{(n-1)!}{(k-1)![(n-1-(k-1)!]} + \frac{(n-1)!}{k!(n-1-k)!} \\ &= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-1-k)!} \\ &= \frac{k(n-1)!}{k!(n-k)!} + \frac{(n-k)(n-1)!}{k!(n-k)!} \\ &= \frac{(n-1)!}{k!(n-k)!}[k+(n-k)] \\ &= \frac{n(n-1)!}{k!(n-k)!} \\ &= \frac{n!}{k!(n-k)!} \\ &= C(n, k) \end{aligned}$$

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Note

The above proof is called a combinatorial proof and is always preferred on account of its elegance.

Outline

The Theorem

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Let $P(n)$ denote the proposition in the above theorem. We prove that $P(n)$ holds for all n by using mathematical induction.

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Proof of Binomial Theorem

Proof.

INDUCTIVE STEP: Assume that $P(k)$ is true, for some $k \geq 0$, i.e., assume that

$$(a+b)^k = \sum_{i=0}^{k} C(k, i)\ a^{k-i} \cdot b^i, \text{ for some k } \geq 0.$$

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&= a \cdot (\sum_{i=0}^{k} C(k,i)\ a^{k-i} \cdot b^i) + b \cdot (\sum_{i=0}^{k} C(k,i)\ a^{k-i} \cdot b^i)
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&= \sum_{i=0}^{k} C(k,i)\ a^{k+1-i} \cdot b^i + \sum_{i=0}^{k} C(k,i)\ a^{k-i} \cdot b^{i+1} \\
&= C(k,0)\ a^{k+1} \cdot b^0 + \sum_{i=1}^{k} C(k,i)\ a^{k+1-i} \cdot b^i + \sum_{i=0}^{k-1} C(k,i)\ a^{k-i} \cdot b^{i+1} + C(k,k)\ a^0 \cdot b^{k+1}
\end{aligned}$$

$\square$

Proof (contd.)

Proof.

We focus on the quantity

$$(\mathbf{F}) \sum_{i=1}^{k} C(k,\, i)\; a^{k+1-i} \cdot b^{i} + (\mathbf{S}) \sum_{i=0}^{k-1} C(k,\, i)\; a^{k-i} \cdot b^{i+1} \tag{1}$$

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Observe that the first k terms in **F** are $a^k \cdot b^1,\ a^{k-1} \cdot b^2, \ldots, a^1 \cdot b^k$, while the first k terms in **S** are also $a^k \cdot b^1,\ a^{k-1} \cdot b^2, \ldots,\ a^1 \cdot b^k$.

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In other words, the terms in **F** and **S** are identical, except for the coefficents.

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Further, all the terms can be generated using the term formula, $a^{k+1-p} \cdot b^{p},\ 1 \leq p \leq k$.

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Accordingly, the coefficient of $a^{k+1-p} \cdot b^p$ in the sum $(\mathbf{F} + \mathbf{S})$ is $C(k,\, p) + C(k,\, p-1)$, which is $C(k+1,\, p)$, using Pascal's formula.

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Thus, the LHS can be written as:

$$C(k,\, 0)\ a^{k+1} \cdot b^0 + \sum_{i=1}^{k} C(k+1,\, i)\ a^{k+1-i} \cdot b^i + C(k,\, k)\ a^0 \cdot b^{k+1}$$

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Thus, the LHS can be written as:

$$\begin{aligned} & C(k, 0)\ a^{k+1} \cdot b^0 + \sum_{i=1}^{k} C(k+1, i)\ a^{k+1-i} \cdot b^i + C(k, k)\ a^0 \cdot b^{k+1} \\ = \quad & C(k+1, 0)\ a^{k+1} \cdot b^0 + \sum_{i=1}^{k} C(k+1, i)\ a^{k+1-i} \cdot b^i + C(k+1, k+1)\ a^0 \cdot b^{k+1} \end{aligned}$$

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Thus, the LHS can be written as:

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□

Proof (contd.)

Proof.

It follows that

$$\begin{aligned} LHS &= \sum_{i=0}^{k+1} C(k+1,\, i)\ a^{k+1-i} \cdot b^{i} \\ &= RHS \end{aligned}$$

Proof (contd.)

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$$\begin{aligned} LHS &= \sum_{i=0}^{k+1} C(k+1, i)\ a^{k+1-i} \cdot b^{i} \\ &= RHS \end{aligned}$$

We have thus shown that $P(k) \rightarrow P(k+1)$ and hence by applying the first principle of mathematical induction, we can conclude that $P(n)$ is true, for all $n \geq 0$. $\square$

Outline

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Expand $(x-3)^4$.

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$$\begin{aligned}(x-3)^4 &= C(4,0)x^4\cdot(-3)^0 + C(4,1)x^3\cdot(-3)^1 + C(4,2)x^2\cdot(-3)^2 \\ &\quad + C(4,3)x^1\cdot(-3)^3 + C(4,4)x^0\cdot(-3)^4\end{aligned}$$

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One more example

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Show that

$$\sum_{i=0}^{n} C(n, i) = 2^n$$

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Proof using the binomial theorem

As per the binomial theorem,

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One more example

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Show that

$$\sum_{i=0}^{n} C(n, i) = 2^n$$

Proof using the binomial theorem

As per the binomial theorem,

$$\begin{aligned}(1+x)^n &= \sum_{i=0}^{n} C(n, i)1^{n-i} \cdot x^i \\ &= C(n, 0)1^n \cdot x^0 + C(n, 1)1^{n-1} \cdot x^1 + \ldots C(n, n)1^0 \cdot x^n\end{aligned}$$

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\begin{aligned}
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Thus LHS=RHS and the inductive step is proven. Applying the first principle of mathematical induction, we conclude that the conjecture is true.