Advanced Analysis of Algorithms

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1. Large integer multiplication.

   (a) n-bit integers represented in binary.

   (b) $I+J$ and $I-J$ in $O(n)$ time.

   (c) High-school multiplication algorithm $I \cdot J$ takes $O(n^2)$ time.

   (d) Break I into $\langle I_h, I_l \rangle$ and J into $\langle J_h, J_l \rangle$.

   (e) Reconstructing I and J from the breakup.

   $$\begin{aligned} I &= I_h \cdot 2^{\frac{n}{2}} + I_l \\ J &= J_h \cdot 2^{\frac{n}{2}} + J_l \end{aligned}$$

   (f) Computing the product:

   $$\begin{aligned} I \cdot J &= (I_h \cdot 2^{\frac{n}{2}} + I_l) \cdot (J_h \cdot 2^{\frac{n}{2}} + J_l) \\ &= I_h \cdot J_h \cdot 2^{\frac{n}{2}} + (I_h \cdot J_l + I_l \cdot J_h) \cdot 2^{\frac{n}{2}} \\ &\quad + I_l \cdot J_l \end{aligned}$$

   (g) $T(n) = 4 \cdot T(\frac{n}{2}) + c \cdot n$.

   (h) Consider the product:

   $$\begin{aligned} (I_h - I_l) \cdot (J_l - J_h) &= I_h \cdot J_l - I_l \cdot J_l \\ &\quad - I_h \cdot J_h + I_l \cdot J_h \end{aligned}$$

   (i) Compute $I \cdot J$ as follows:

   $$\begin{aligned} I \cdot J &= I_h \cdot J_h \cdot 2^{\frac{n}{2}} + [((I_h - I_l) \cdot (J_l - J_h) \\ &\quad + I_h \cdot J_h + I_l \cdot J_l] \cdot 2^{\frac{n}{2}} + I_l \cdot J_l \end{aligned}$$

   (j) $T(n) = 3 \cdot T(\frac{n}{2}) + c \cdot n \Rightarrow T(n) \in O(n^{1.585})$.

2. Strassen's matrix multiplication algorithm.

   (a) $Z = X \cdot Y$; the naive $O(n^3)$ algorithm.

   (b)

   $$\begin{bmatrix} I & J \\ K & L \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} E & F \\ G & H \end{bmatrix}$$

   (c)

   $$\begin{aligned} I &= A \cdot E + B \cdot G \\ J &= A \cdot F + B \cdot H \\ K &= C \cdot E + L \cdot G \\ L &= C \cdot F + D \cdot H \end{aligned}$$

   (d) $T(n) = 8 \cdot T(\frac{n}{2}) + b \cdot n^2$.

   (e) Strassen's approach:

   $$\begin{aligned} S_1 &= A \cdot (F - H) \\ S_2 &= (A + B) \cdot H \\ S_3 &= (C + D) \cdot E \\ S_4 &= D \cdot (G - E) \\ S_5 &= (A + D) \cdot (E + H) \\ S_6 &= (B - D) \cdot (G + H) \\ S_7 &= = (A - C) \cdot (E + F) \end{aligned}$$

   (f)

   $$\begin{aligned} I &= S_4 + S_5 + S_6 - S_2 \\ J &= S_1 + S_2 \\ K &= S_3 + S_4 \\ L &= S_1 - S_7 - S_3 + S_5 \end{aligned}$$

   (g) $T(n) = 7 \cdot T(\frac{n}{2}) + k \cdot n^2$.

   (h) $T(n) = O(n^{\log_2 7})$.

3. When not to use Divide and Conquer (book).