Combinatorial Optimization - Homework I

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1 Problems

1. Consider the feasibility version of the linear programming problem, in which you are given a matrix $\mathbf{A}_{m\times n}$, a vector $\mathbf{b}_{m\times 1}$ and asked, if there exists $\mathbf{x} \in \Re^n$, such that

$$\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}. \tag{1}$$

   In the integer programming counterpart, we are required to find an integral $\mathbf{x}$, which satisfies System (1). Show that an oracle for the integer programming problem can be used to solve the linear programming problem in polynomial time.

2. Is the Fourier-Motzkin elimination procedure polynomially convergent? If yes, provide a proof; if no, provide a counterexample.

   Solution:

   □

3. In class, I showed that the same combinatorial optimization problem can have more than one integer programming formulation. Let F and G denote two formulations for a problem P. Formulation F is said to be stronger than formulation G, if the linear programming relaxation of F is a strict subset of the linear programming relaxation of G.

   Consider the following two integer programming formulations of a problem:

$$\begin{aligned} \mathbf{F} : 2 \cdot x_1 + 2 \cdot x_2 + x_3 + x_4 &\leq 2 \\ x_1 &\leq 1 \\ x_2 &\leq 1 \\ x_3 &\leq 1 \\ x_4 &\leq 1 \\ x_i &\geq 0,\ i = 1, 2, 3, 4 \end{aligned}$$

$$\begin{aligned} \mathbf{G} : x_1 + x_2 + x_3 &\leq 1 \\ x_1 + x_2 + x_4 &\leq 1 \\ x_i &\geq 0,\ i = 1, 2, 3, 4 \end{aligned}$$

   Which one is stronger?

   Solution: □

4. Let $\mathbf{A}$ denote an $m \times n$ matrix and $\mathbf{b}$ denote an $m \times 1$ vector. Using the Strong Duality Theorem, prove that that either

$$\exists \mathbf{x} \in \Re^n_+ \;\; \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$$

or (mutually exclusively)

$$\exists \mathbf{y} \in \Re^m_+ \;\; \begin{aligned} \mathbf{y} \cdot \mathbf{A} \cdot &\leq \mathbf{0} \\ \mathbf{y} \cdot \mathbf{b} &< 0. \end{aligned}$$

Solution:

□

5. Let $x_1, x_2, \ldots x_k$ be points in $\Re^n$. Argue that the following statements are equivalent:

   (i) $x_1, x_2, \ldots x_k$ are affinely independent.

   (ii) $(x_2 - x_1), (x_3 - x_1), \ldots (x_k - x_1)$ are linearly independent.

   (iii) $(x_1, 1), (x_2, 1), \ldots (x_k, 1)$ are linearly independent.

Solution:

□