By Farkas' Lemma (Corollary = xC'.

3.24). By Lemma 5.4 there ξK_F . Let S_F be the system of

 $\max{\{a_i x : x \in P\}}.$

 S_F (for all minimal faces F). $P = \{x : Ax \le b\}$. It remains

$\geq 0, yA = c$

P]}. F is a face of P, so let stem $a_1x \le \beta_1, \ldots, a_tx \le \beta_t$. egers $\lambda_1, \ldots, \lambda_t$. We add zero i vector $\overline{\lambda} \ge 0$ with $\overline{\lambda}A = c$ ". So $\overline{\lambda}$ attains the minimum

ral. Conversely, if b can be al.

ere is a unique minimal TDI-, we prove that each "face"

 β be a TDI-system, where a TDI.

tor such that

 $+ (\lambda - \mu)a = c\}$ (5.2)

 $c' := c + [\mu^*]a$ and observe

 $\lambda \ge 0, \ yA + \lambda a = c' \}$ (5.3)

mum and $y := y^*, \lambda :=$

3) has an integral optimum $\mu := \lceil \mu^* \rceil$ and claim that num in (5.2).

(5.2). Furthermore,

5.4 Totally Unimodular Matrices 101

$$yb + (\lambda - \mu)\beta = \tilde{y}b + \tilde{\lambda}\beta - \lceil \mu^* \rceil\beta$$

$$\leq y^*b + (\lambda^* + \lceil \mu^* \rceil - \mu^*)\beta - \lceil \mu^* \rceil\beta$$

since $(y^*, \lambda^* + \lceil \mu^* \rceil - \mu^*)$ is feasible for the minimum in (5.3), and $(\tilde{y}, \tilde{\lambda})$ is an optimum solution. We conclude that

$$yb + (\lambda - \mu)\beta \leq y^*b + (\lambda^* - \mu^*)\beta$$

proving that (y, λ, μ) is an integral optimum solution for the minimum in (5.2).

The following statements are straightforward consequences of the definition of TDI-systems: A system Ax = b, $x \ge 0$ is TDI if $\min\{yb : yA \ge c\}$ has an integral optimum solution y for each integral vector c for which the minimum is finite. A system $Ax \le b$, $x \ge 0$ is TDI if $\min\{yb : yA \ge c, y \ge 0\}$ has an integral optimum solution y for each integral vector c for which the minimum is finite. One may ask whether there are matrices A such that $Ax \le b$, $x \ge 0$ is TDI for each integral vector b. It will turn out that these matrices are exactly the totally unimodular matrices.

5.4 Totally Unimodular Matrices

Definition 5.18. A matrix A is totally unimodular if each subdeterminant of A is 0, +1, or -1.

In particular, each entry of a totally unimodular matrix must be 0, +1, or -1. The main result of this section is:

Theorem 5.19. (Hoffman and Kruskal [1956]) An integral matrix A is totally unimodular if and only if the polyhedron $\{x : Ax \le b, x \ge 0\}$ is integral for each integral vector b.

Proof: Let A be an $m \times n$ -matrix and $P := \{x : Ax \le b, x \ge 0\}$. Observe that the minimal faces of P are vertices.

To prove necessity, suppose that A is totally unimodular. Let b be some integral vector and x a vertex of P. x is the solution of A'x = b' for some subsystem

 $A'x \leq b'$ of $\begin{pmatrix} A \\ -I \end{pmatrix} x \leq \begin{pmatrix} b \\ 0 \end{pmatrix}$, with A' being a nonsingular $n \times n$ -matrix.

Since A is totally unimodular, $|\det A'| = 1$, so by Cramer's rule $x = (A')^{-1}b'$ is integral.

We now prove sufficiency. Suppose that the vertices of P are integral for each integral vector b. Let A' be some nonsingular $k \times k$ -submatrix of A. We have to show $|\det A'| = 1$. W.l.o.g., A' contains the elements of the first k rows and columns of A.





Consider the integral $m \times m$ -matrix B consisting of the first k and the last m - k columns of $(A \ I)$ (see Figure 5.2). Obviously, $|\det B| = |\det A'|$.

To prove $|\det B| = 1$, We shall prove that B^{-1} is integral. Since det $B \det B^{-1} = 1$, this implies that $|\det B| = 1$, and we are done.

Let $i \in \{1, ..., m\}$; we prove that $B^{-1}e_i$ is integral. Choose an integral vector y such that $z := y + B^{-1}e_i \ge 0$. Then $b := Bz = By + e_i$ is integral. We add zero components to z in order to obtain z' with

$$(A \quad I)z' = Bz = b.$$

Now z'', consisting of the first *n* components of z', belongs to *P*. Furthermore, *n* linearly independent constraints are satisfied with equality, namely the first *k*

and the last n - k inequalities of $\begin{pmatrix} A \\ -I \end{pmatrix} z'' \le 0$. Hence z'' is a vertex of P. By our assumption z'' is integral. But then z' must also be integral: its first n

components are the components of z'', and the last *m* components are the slack variables b - Az'' (and A and b are integral). So z is also integral, and hence $B^{-1}e_i = z - y$ is integral.

The above proof is due to Veinott and Dantzig [1968].

Corollary 5.20. An integral matrix A is totally unimodular if and only if for all integral vectors b and c both optima in the LP duality equation

 $\max \{ cx : Ax \le b, x \ge 0 \} = \min \{ yb : y \ge 0, yA \ge c \}$

are attained by integral vectors (if they are finite).

Proof: This follows from that the transpose of a tota

Let us reformulate thes

Corollary 5.21. An integration system $Ax \le b$, $x \ge 0$ is T

Proof: If A (and thus A^{T} Theorem min $\{yb : yA \ge$ vector b and each integral words, the system $Ax \le b$,

To show the converse, s b. Then by Corollary 5.14 each integral vector b. By

This is not the only wa a certain system is TDI. Th this will be used several tin

Lemma 5.22. Let $Ax \leq b$ $b \in \mathbb{R}^m$. Suppose that for ecoptimum solution, it has one components of y^* form a top

Proof: Let $c \in \mathbb{Z}^n$, and let 0} such that the rows of A totally unimodular matrix A

 $\min\{yb: yA \ge$

where b' consists of the con the inequality " \leq " of (5.4), the LP on the left-hand side follows from the fact that y the LP on the right-hand side

Since A' is totally unim optimum solution (by the He with zeros we obtain an inte completing the proof.

A very useful criterion f

Theorem 5.23. (Ghouila-H unimodular if and only if for R_2 such that