$$\begin{aligned} yb + (\lambda - \mu)\beta &= \tilde{y}b + \tilde{\lambda}\beta - \lceil \mu^* \rceil \beta \\ &\le y^*b + (\lambda^* + \lceil \mu^* \rceil - \mu^*)\beta - \lceil \mu^* \rceil \beta \end{aligned}$$

since $(y^*, \lambda^* + \lceil \mu^* \rceil - \mu^*)$ is feasible for the minimum in (5.3), and $(\tilde{y}, \tilde{\lambda})$ is an optimum solution. We conclude that

$$yb + (\lambda - \mu)\beta \le y^*b + (\lambda^* - \mu^*)\beta,$$

proving that (y, λ, μ) is an integral optimum solution for the minimum in (5.2). $\square$

The following statements are straightforward consequences of the definition of TDI-systems: A system $Ax = b,\ x \ge 0$ is TDI if $\min\{yb : yA \ge c\}$ has an integral optimum solution y for each integral vector c for which the minimum is finite. A system $Ax \le b,\ x \ge 0$ is TDI if $\min\{yb : yA \ge c,\ y \ge 0\}$ has an integral optimum solution y for each integral vector c for which the minimum is finite. One may ask whether there are matrices A such that $Ax \le b,\ x \ge 0$ is TDI for each integral vector b. It will turn out that these matrices are exactly the totally unimodular matrices.

5.4 Totally Unimodular Matrices

Definition 5.18. *A matrix A is* **totally unimodular** *if each subdeterminant of A is 0, $+1$, or -1.*

In particular, each entry of a totally unimodular matrix must be 0, $+1$, or -1. The main result of this section is:

Theorem 5.19. (Hoffman and Kruskal [1956]) *An integral matrix A is totally unimodular if and only if the polyhedron $\{x : Ax \le b,\ x \ge 0\}$ is integral for each integral vector b.*

Proof: Let A be an $m \times n$-matrix and $P := \{x : Ax \le b,\ x \ge 0\}$. Observe that the minimal faces of P are vertices.

To prove necessity, suppose that A is totally unimodular. Let b be some integral vector and x a vertex of P. x is the solution of $A'x = b'$ for some subsystem $A'x \le b'$ of $\begin{pmatrix} A \\ -I \end{pmatrix} x \le \begin{pmatrix} b \\ 0 \end{pmatrix}$, with A' being a nonsingular $n \times n$-matrix. Since A is totally unimodular, $|\det A'| = 1$, so by Cramer's rule $x = (A')^{-1}b'$ is integral.

We now prove sufficiency. Suppose that the vertices of P are integral for each integral vector b. Let A' be some nonsingular $k \times k$-submatrix of A. We have to show $|\det A'| = 1$. W.l.o.g., A' contains the elements of the first k rows and columns of A.

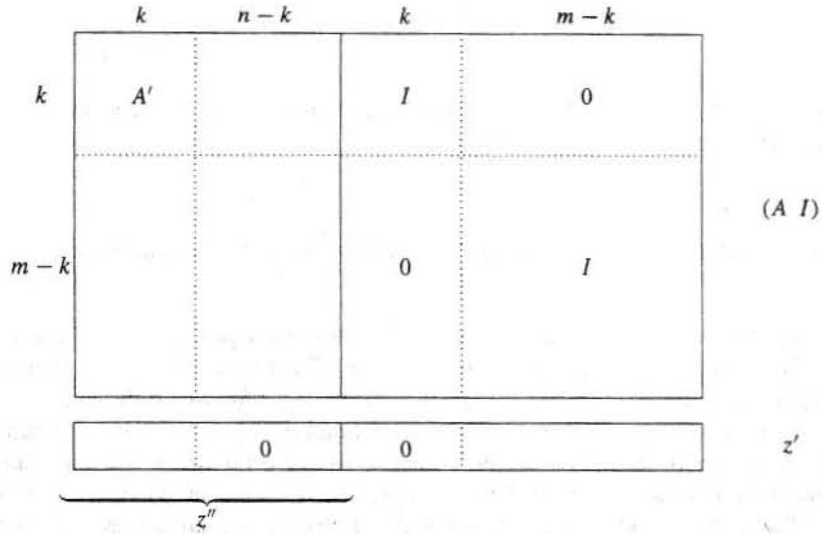

Fig. 5.2.

Consider the integral $m \times m$-matrix B consisting of the first k and the last $m-k$ columns of $(\ A \quad I\ )$ (see Figure 5.2). Obviously, $|\det B| = |\det A'|$.

To prove $|\det B| = 1$, We shall prove that B^{-1} is integral. Since $\det B \det B^{-1} = 1$, this implies that $|\det B| = 1$, and we are done.

Let $i \in \{1, \dots, m\}$; we prove that $B^{-1}e_i$ is integral. Choose an integral vector y such that $z := y + B^{-1}e_i \geq 0$. Then $b := Bz = By + e_i$ is integral. We add zero components to z in order to obtain z' with

$$(\ A \quad I\ )z' = Bz = b.$$

Now z'', consisting of the first n components of z', belongs to P. Furthermore, n linearly independent constraints are satisfied with equality, namely the first k and the last $n-k$ inequalities of $\begin{pmatrix} A \\ -I \end{pmatrix} z'' \leq 0$. Hence z'' is a vertex of P. By our assumption z'' is integral. But then z' must also be integral: its first n components are the components of z'', and the last m components are the slack variables $b - Az''$ (and A and b are integral). So z is also integral, and hence $B^{-1}e_i = z - y$ is integral. □

The above proof is due to Veinott and Dantzig [1968].

Corollary 5.20. *An integral matrix A is totally unimodular if and only if for all integral vectors b and c both optima in the LP duality equation*

$$\max\{cx : Ax \leq b,\ x \geq 0\} = \min\{yb : y \geq 0,\ yA \geq c\}$$

are attained by integral vectors (if they are finite).