5 Theorems of the Alternatives

5.1 Systems of Equations

Let's start with a system of linear equations:

$$Ax = b.$$

Suppose you wish to determine whether this system is feasible or not. One reasonable approach is to use Gaussian elimination. If the system has a solution, you can find a particular one, $\overline{x}$. (You remember how to do this: Use elementary row operations to put the system in row echelon form, select arbitrary values for the independent variables and use back substitution to solve for the dependent variables.) Once you have a feasible $\overline{x}$ (no matter how you found it), it is straightforward to convince someone else that the system is feasible by verifying that $A\overline{x} = b$.

If the system is infeasible, Gaussian elimination will detect this also. For example, consider the system

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 1 \\ 2x_1 - x_2 + 3x_3 &= -1 \\ 8x_1 + 2x_2 + 10x_3 + 4x_4 &= 0 \end{aligned}$$

which in matrix form looks like

$$\left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 0 & -1 \\ 8 & 2 & 10 & 4 & 0 \end{array}\right].$$

Perform elementary row operations to arrive at a system in row echelon form:

$$\left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{array}\right]\left[\begin{array}{rrr} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -8 & 0 & 1 \end{array}\right]\left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 0 & -1 \\ 8 & 2 & 10 & 4 & 0 \end{array}\right] = \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 1 \\ 0 & -3 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & -2 \end{array}\right],$$

which implies

$$\left[\begin{array}{rrr} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -4 & -2 & 1 \end{array}\right]\left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 0 & -1 \\ 8 & 2 & 10 & 4 & 0 \end{array}\right] = \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 1 \\ 0 & -3 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & -2 \end{array}\right].$$

Immediately it is evident that the original system is infeasible, since the resulting equivalent system includes the equation $0x_1 + 0x_2 + 0x_3 + 0x_4 = -2$.

This equation comes from multiplying the matrix form of the original system by the third row of the matrix encoding the row operations: $[-4, -2, 1]$. This vector satisfies

$$\begin{bmatrix} -4 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 0 \\ 8 & 2 & 10 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} -4 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = -2.$$

In matrix form, we have found a vector $\overline{y}$ such that $\overline{y}^T A = O$ and $\overline{y}^T b \neq 0$. Gaussian elimination will always produce such a vector if the original system is infeasible. Once you have such a $\overline{y}$ (regardless of how you found it), it is easy to convince someone else that the system is infeasible.

Of course, if the system is feasible, then such a vector $\overline{y}$ cannot exist, because otherwise there would also be a feasible $\overline{x}$, and we would have

$$0 = O^T \overline{x} = (\overline{y}^T A)\overline{x} = \overline{y}^T (A\overline{x}) = \overline{y}^T b \neq 0,$$

which is impossible. (Be sure you can justify each equation and inequality in the above chain.) We have established our first Theorem of the Alternatives:

Theorem 5.1 *Either the system*

$$(I) \quad Ax = b$$

has a solution, or the system

$$(II) \quad \begin{array}{c} y^T A = O^T \\ y^T b \neq 0 \end{array}$$

has a solution, but not both.

As a consequence of this theorem, the following question has a "good characterization": Is the system (I) feasible? I will not give an exact definition of this concept, but roughly speaking it means that whether the answer is yes or no, there exists a "short" proof. In this case, if the answer is yes, we can prove it by exhibiting any particular solution to (I). And if the answer is no, we can prove it by exhibiting any particular solution to (II).

Geometrically, this theorem states that precisely one of the alternatives occurs:

1. The vector b is in the column space of A.

2. There is a vector y orthogonal to each column of A (and hence to the entire column space of A) but not orthogonal to b.

5.2 Fourier-Motzkin Elimination — A Starting Example

Now let us suppose we are given a system of linear inequalities

$$Ax \le b$$

and we wish to determine whether or not the system is feasible. If it is feasible, we want to find a particular feasible vector $\overline{x}$; if it is not feasible, we want hard evidence!

It turns out that there is a kind of analog to Gaussian elimination that works for systems of linear inequalities: Fourier-Motzkin elimination. We will first illustrate this with an example:

$$(I) \quad \begin{array}{c} x_1 - 2x_2 \le -2 \\ x_1 + x_2 \le 3 \\ x_1 \le 2 \\ -2x_1 + x_2 \le 0 \\ -x_1 \le -1 \\ 8x_2 \le 15 \end{array}$$

Our goal is to derive a second system (II) of linear inequalities with the following properties:

1. It has one fewer variable.
2. It is feasible if and only if the original system (I) is feasible.
3. A feasible solution to (I) can be derived from a feasible solution to (II).

(Do you see why Gaussian elimination does the same thing for systems of linear equations?) Here is how it works. Let's eliminate the variable x_1. Partition the inequalities in (I) into three groups, (I_-), (I_+), and (I_0), according as the coefficient of x_1 is negative, positive, or zero, respectively.

$$(I_-) \ \begin{array}{c} -2x_1 + x_2 \le 0 \\ -x_1 \le -1 \end{array} \qquad (I_+) \ \begin{array}{c} x_1 - 2x_2 \le -2 \\ x_1 + x_2 \le 3 \\ x_1 \le 2 \end{array} \qquad (I_0) \ 8x_2 \le 15$$

For each pair of inequalities, one from (I_-) and one from (I_+), multiply by positive numbers and add to eliminate x_1. For example, using the first inequality in each group,

$$\begin{array}{c} (\frac{1}{2})(-2x_1 + x_2 \le 0) \\ +(1)(x_1 - 2x_2 \le -2) \\ \hline -\frac{3}{2}x_2 \le -2 \end{array}$$

System (II) results from doing this for all such pairs, and then also including the inequalities in (I_0):

$$
(II) \quad
\begin{array}{c}
-\frac{3}{2}x_2 \le -2 \\
\frac{3}{2}x_2 \le 3 \\
\frac{1}{2}x_2 \le 2 \\
-2x_2 \le -3 \\
x_2 \le 2 \\
0x_2 \le 1 \\
8x_2 \le 15
\end{array}
$$

The derivation of (II) from (I) can also be represented in matrix form. Here is the original system:

$$
\left[\begin{array}{rr|r}
1 & -2 & -2 \\
1 & 1 & 3 \\
1 & 0 & 2 \\
-2 & 1 & 0 \\
-1 & 0 & -1 \\
0 & 8 & 15
\end{array}\right]
$$

Obtain the new system by multiplying on the left by the matrix that constructs the desired nonnegative combinations of the original inequalities:

$$
\left[\begin{array}{cccccc}
1 & 0 & 0 & 1/2 & 0 & 0 \\
0 & 1 & 0 & 1/2 & 0 & 0 \\
0 & 0 & 1 & 1/2 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\left[\begin{array}{rr|r}
1 & -2 & -2 \\
1 & 1 & 3 \\
1 & 0 & 2 \\
-2 & 1 & 0 \\
-1 & 0 & -1 \\
0 & 8 & 15
\end{array}\right]
$$

$$
= \left[\begin{array}{cc|c}
0 & -3/2 & -2 \\
0 & 3/2 & 3 \\
0 & 1/2 & 2 \\
0 & -2 & -3 \\
0 & 1 & 2 \\
0 & 0 & 1 \\
0 & 8 & 15
\end{array}\right].
$$

To see why the new system has the desired properties, let's break down this process a bit. First scale each inequality in the first two groups by positive numbers so that each coefficient of x_1 in (I_-) is -1 and each coefficient of x_1 in (I_+) is $+1$.

$$(I_-)\ \begin{array}{c} -x_1 + \frac{1}{2}x_2 \le 0 \\ -x_1 \le -1 \end{array} \quad (I_+)\ \begin{array}{c} x_1 - 2x_2 \le -2 \\ x_1 + x_2 \le 3 \\ x_1 \le 2 \end{array} \quad (I_0)\ 8x_2 \le 15$$

Isolate the variable x_1 in each of the inequalities in the first two groups.

$$(I_-)\ \begin{array}{c} \frac{1}{2}x_2 \le x_1 \\ 1 \le x_1 \end{array} \quad (I_+)\ \begin{array}{c} x_1 \le 2x_2 - 2 \\ x_1 \le -x_2 + 3 \\ x_1 \le 2 \end{array} \quad (I_0)\ 8x_2 \le 15$$

For each pair of inequalities, one from (I_-) and one from (I_+), create a new inequality by "sandwiching" and then eliminating x_1. Keep the inequalities in (I_0).

$$(IIa)\quad \begin{array}{ccc} \left\{ \begin{array}{c} \frac{1}{2}x_2 \\ 1 \end{array} \right\} & \le x_1 \le & \left\{ \begin{array}{c} 2x_2 - 2 \\ -x_2 + 3 \\ 2 \end{array} \right\} \\ 8x_2 & \le & 15 \end{array} \quad \longrightarrow \quad (IIb)\quad \begin{array}{c} \frac{1}{2}x_2 \le x_1 \le 2x_2 - 2 \\ \frac{1}{2}x_2 \le x_1 \le -x_2 + 3 \\ \frac{1}{2}x_2 \le x_1 \le 2 \\ 1 \le x_1 \le 2x_2 - 2 \\ 1 \le x_1 \le -x_2 + 3 \\ 1 \le x_1 \le 2 \\ 8x_2 \le 15 \end{array}$$

$$\longrightarrow \quad (IIc)\quad \begin{array}{c} \frac{1}{2}x_2 \le 2x_2 - 2 \\ \frac{1}{2}x_2 \le -x_2 + 3 \\ \frac{1}{2}x_2 \le 2 \\ 1 \le 2x_2 - 2 \\ 1 \le -x_2 + 3 \\ 1 \le 2 \\ 8x_2 \le 15 \end{array} \quad \longrightarrow \quad (II)\ \begin{array}{c} -\frac{3}{2}x_2 \le -2 \\ \frac{3}{2}x_2 \le 3 \\ \frac{1}{2}x_2 \le 2 \\ -2x_2 \le -3 \\ x_2 \le 2 \\ 0x_2 \le 1 \\ 8x_2 \le 15 \end{array}$$

Observe that the system (II) does not involve the variable x_1. It is also immediate that if (I) is feasible, then (II) is also feasible. For the reverse direction, suppose that (II) is feasible. Set the variables (in this case, x_2) equal to any specific feasible values (in this case we choose a feasible value $\overline{x}_2$). From the way the inequalities in (II) were derived, it is evident that

$$\max \left\{ \begin{array}{c} \frac{1}{2}\overline{x}_2 \\ 1 \end{array} \right\} \le \min \left\{ \begin{array}{c} 2\overline{x}_2 - 2 \\ -\overline{x}_2 + 3 \\ 2 \end{array} \right\}.$$

So there exists a specific value $\overline{x}_1$ of x_1 such that

$$\begin{array}{ccc}
\left\{ \begin{array}{c} \frac{1}{2}\overline{x}_2 \\ 1 \end{array} \right\} & \leq \overline{x}_1 \leq & \left\{ \begin{array}{c} 2\overline{x}_2 - 2 \\ -\overline{x}_2 + 3 \\ 2 \end{array} \right\} \\
8\overline{x}_2 & \leq & 15
\end{array}$$

We will then have a feasible solution to (I).

5.3 Showing our Example is Feasible

From this example, we now see how to eliminate one variable (but at the possible considerable expense of increasing the number of inequalities). If we have a solution to the new system, we can determine a value of the eliminated variable to obtain a solution of the original system. If the new system is infeasible, then so is the original system.

From this we can tackle any system of inequalities: Eliminate all of the variables one by one until a system with no variables remains! Then work backwards to determine feasible values of all of the variables.

In our previous example, we can now eliminate x_2 from system (II):

$$\left[\begin{array}{ccccccc}
2/3 & 2/3 & 0 & 0 & 0 & 0 & 0 \\
2/3 & 0 & 2 & 0 & 0 & 0 & 0 \\
2/3 & 0 & 0 & 0 & 1 & 0 & 0 \\
2/3 & 0 & 0 & 0 & 0 & 0 & 1/8 \\
0 & 2/3 & 0 & 1/2 & 0 & 0 & 0 \\
0 & 0 & 2 & 1/2 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/2 & 1 & 0 & 0 \\
0 & 0 & 0 & 1/2 & 0 & 0 & 1/8 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\left[\begin{array}{cc|c}
0 & -3/2 & -2 \\
0 & 3/2 & 3 \\
0 & 1/2 & 2 \\
0 & -2 & -3 \\
0 & 1 & 2 \\
0 & 0 & 1 \\
0 & 8 & 15
\end{array}\right]$$

$$= \left[\begin{array}{cc|c}
0 & 0 & 2/3 \\
0 & 0 & 8/3 \\
0 & 0 & 2/3 \\
0 & 0 & 13/24 \\
0 & 0 & 1/2 \\
0 & 0 & 5/2 \\
0 & 0 & 1/2 \\
0 & 0 & 3/8 \\
0 & 0 & 1
\end{array}\right].$$

Each final inequality, such as $0x_1 + 0x_2 \leq 2/3$, is feasible, since the left-hand side is zero and the right-hand side is nonnegative. Therefore the original system is feasible. To find one specific feasible solution, rewrite (II) as

$$\{4/3, 3/2\} \leq x_2 \leq \{2, 4, 15/8\}\,.$$

We can choose, for example, $\overline{x}_2 = 3/2$. Substituting into (I) (or (IIa)), we require

$$\{3/4, 1\} \leq x_1 \leq \{1, 3/2, 2\}\,.$$

So we could choose $\overline{x}_1 = 1$, and we have a feasible solution $(1, 3/2)$ to (I).

5.4 An Example of an Infeasible System

Now let's look at the system:

$$(I)\qquad \begin{array}{c} x_1 - 2x_2 \leq -2 \\ x_1 + x_2 \leq 3 \\ x_1 \leq 2 \\ -2x_1 + x_2 \leq 0 \\ -x_1 \leq -1 \\ 8x_2 \leq 11 \end{array}$$

Multiplying by the appropriate nonnegative matrices to successively eliminate x_1 and x_2, we compute:

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{rr|r} 1 & -2 & -2 \\ 1 & 1 & 3 \\ 1 & 0 & 2 \\ -2 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 8 & 11 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 0 & -3/2 & -2 \\ 0 & 3/2 & 3 \\ 0 & 1/2 & 2 \\ 0 & -2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 8 & 11 \end{array}\right] \qquad (II)$$

and

$$\begin{bmatrix} 2/3 & 2/3 & 0 & 0 & 0 & 0 & 0 \\ 2/3 & 0 & 2 & 0 & 0 & 0 & 0 \\ 2/3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2/3 & 0 & 0 & 0 & 0 & 0 & 1/8 \\ 0 & 2/3 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 1/8 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \left[\begin{array}{cc|c} 0 & -3/2 & -2 \\ 0 & 3/2 & 3 \\ 0 & 1/2 & 2 \\ 0 & -2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 8 & 11 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 0 & 0 & 2/3 \\ 0 & 0 & 8/3 \\ 0 & 0 & 2/3 \\ 0 & 0 & 1/24 \\ 0 & 0 & 1/2 \\ 0 & 0 & 5/2 \\ 0 & 0 & 1/2 \\ 0 & 0 & -1/8 \\ 0 & 0 & 1 \end{array}\right] \quad (III)$$

Since one inequality is $0x_1+0x_2 \leq -1/8$, the final system (III) is clearly infeasible. Therefore the original system (I) is also infeasible. We can go directly from (I) to (III) by collecting together the two nonnegative multiplier matrices:

$$\begin{bmatrix} 2/3 & 2/3 & 0 & 0 & 0 & 0 & 0 \\ 2/3 & 0 & 2 & 0 & 0 & 0 & 0 \\ 2/3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2/3 & 0 & 0 & 0 & 0 & 0 & 1/8 \\ 0 & 2/3 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 1/8 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$
= \begin{bmatrix}
2/3 & 2/3 & 0 & 2/3 & 0 & 0 \\
2/3 & 0 & 2 & 4/3 & 0 & 0 \\
2/3 & 1 & 0 & 1/3 & 1 & 0 \\
2/3 & 0 & 0 & 1/3 & 0 & 1/8 \\
1/2 & 2/3 & 0 & 1/3 & 1/2 & 0 \\
1/2 & 0 & 2 & 1 & 1/2 & 0 \\
1/2 & 1 & 0 & 0 & 3/2 & 0 \\
1/2 & 0 & 0 & 0 & 1/2 & 1/8 \\
0 & 0 & 1 & 0 & 1 & 0
\end{bmatrix} = M.
$$

You can check that $M(I) = (III)$. Since M is a product of nonnegative matrices, it will itself be nonnegative. Since the infeasibility is discovered in the eighth inequality of (III), this comes from the eighth row of M, namely, $[1/2, 0, 0, 0, 1/2, 1/8]$. You can now demonstrate directly to anyone that (I) is infeasible using these nonnegative multipliers:

$$
\begin{array}{c}
(\frac{1}{2})(x_1 - 2x_2 \le -2) \\
+(\frac{1}{2})(-x_1 \le -1) \\
+(\frac{1}{8})(8x_2 \le 11) \\
\hline
0x_1 + 0x_2 \le -\frac{1}{8}
\end{array}
$$

In particular, we have found a nonnegative vector y such that $y^T A = O^T$ but $y^T b < 0$.

5.5 Fourier-Motzkin Elimination in General

Often I find that it is easier to understand a general procedure, proof, or theorem from a few good examples. Let's see if this is the case for you.

We begin with a system of linear inequalities

$$
(I) \quad \sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i = 1, \ldots, m.
$$

Let's write this in matrix form as

$$
Ax \le b
$$

or

$$
A^i x \le b_i, \quad i = 1, \ldots, m
$$

where A^i represents the ith row of A.

Suppose we wish to eliminate the variable x_k. Define

$$
\begin{aligned}
I_- &= \{i : a_{ik} < 0\} \\
I_+ &= \{i : a_{ik} > 0\} \\
I_0 &= \{i : a_{ik} = 0\}
\end{aligned}
$$