Primal LP (n variables, m equations):

$$\begin{array}{lll} & \max \sum_{j=1}^{n} c_j x_j & \\ \text{s.t.} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i & (i = 1 \dots m) \\ & x_j \geq 0 & (j = 1 \dots n) \end{array}$$

Dual LP (m variables, n equations):

$$\begin{array}{lll} & \min \sum_{i=1}^{m} b_i y_i & \\ \text{s.t.} & \sum_{i=1}^{m} a_{ij} y_i \geq c_j & (j = 1 \dots n) \\ & y_i \geq 0 & (i = 1 \dots m) \end{array}$$

Theorem 4.1 (Weak Duality Theorem)
For every primal feasible solution x, and every dual feasible solution y, we have:

$$\sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{m} b_i y_i.$$

Proof:
The proof of this theorem is really easy and follows almost by definition.

$$\sum_{j=1}^{n} c_j x_j \leq \sum_{j=1}^{n} (\sum_{i=1}^{m} a_{ij} y_i) x_j = \sum_{i=1}^{m} (\sum_{j=1}^{n} a_{ij} x_j) y_i \leq \sum_{i=1}^{m} b_i y_i$$

□

It is easy to see that if we obtain x^* and y^*, such that the equation in the Weak Duality theorem is met with equality, then both the solutions, x^*, y^* are optimal solutions for the primal and dual programs. By the Weak Duality theorem, we know that for any solution x, the following is true:

$$\sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{m} b_i y_i^* = \sum_{j=1}^{n} c_j x_j^*$$

Hence x^* is an optimal solution for the primal LP. Similarly, we can show that y^* is an optimal solution to the Dual LP.

Theorem 4.2 (Strong Duality Theorem)
If the primal LP has an optimal solution x^*, then the dual has an optimal solution y^* such that:

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*.$$

Proof:
To prove the theorem, we only need to find a (feasible) solution y^* that satisfies the constraints of the Dual LP, and satisfies the above equation with equality. We solve the primal program by the simplex method, and introduce m slack variables in the process.

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j \quad (i = 1, \dots, m)$$

Assume that when the simplex algorithm terminates, the equation defining z reads as:

$$z = z^* + \sum_{k=1}^{n+m} \overline{c}_k x_k.$$

Since we have reached optimality, we know that each $\overline{c}_k$ is a nonpositive number (in fact, it is 0 for each basic variable). In addition z^* is the value of the objective function at optimality, hence $z^* = \sum_{j=1}^{n} c_j x_j^*$. To produce y^* we pull a rabbit out of a hat ! Define $y_i^* = -\overline{c}_{n+i} \quad (i = 1, \dots, m)$. To show that y^* is an optimal dual feasible solution, we first show that it is feasible for the Dual LP, and then establish the strong duality condition.

¿From the equation for z we have:

$$\sum_{j=1}^{n} c_j x_j = z^* + \sum_{k=1}^{n} \overline{c}_k x_k - \sum_{i=1}^{m} y_i^* (b_i - \sum_{j=1}^{n} a_{ij} x_j).$$

Rewriting it, we get

$$\sum_{j=1}^{n} c_j x_j = (z^* - \sum_{i=1}^{m} b_i y_i^*) + \sum_{j=1}^{n} (\overline{c}_j + \sum_{i=1}^{m} a_{ij} y_i^*) x_j.$$

Since this holds for all values of x_i, we obtain:

$$z^* = \sum_{i=1}^{m} b_i y_i^*$$

(this *establishes the equality*) and

$$c_j = \overline{c}_j + \sum_{i=1}^{m} a_{ij} y_i^* \quad (j = 1, \ldots, n).$$

Since $\overline{c}_k \leq 0$, we have

$$y_i^* \geq 0 \quad (i = 1, \ldots, m).$$

$$\sum_{i=1}^{m} a_{ij} y_i^* \geq c_j \quad (j = 1, \ldots, n)$$

This *establishes the feasibility* of y^*. □

Complementary Slackness Conditions:

Theorem 4.3 *Necessary and Sufficient conditions for x^* and y^* to be optimal solutions to the primal and dual are as follows.*

$$\sum_{i=1}^{m} a_{ij} y_i^* = c_j \text{ or } x_j^* = 0 \text{ (or both) for } j = 1, \ldots, n$$

$$\sum_{j=1}^{n} a_{ij} x_j^* = b_i \text{ or } y_i^* = 0 \text{ (or both) for } i = 1, \ldots, m$$

In other words, if a variable is non-zero then the corresponding equation in the dual is met with equality, and vice versa.

Proof:

We know that

$$c_j x_j^* \leq (\sum_{i=1}^{m} a_{ij} y_i^*) x_j^* \qquad (j = 1, \ldots, n)$$

$$(\sum_{j=1}^{n} a_{ij} x_j^*) y_i^* \leq b_i y_i^* \qquad (i = 1, \ldots, m)$$

We know that at optimality, the equations are met with equality. Thus for any value of j, either $x_j^* = 0$ or $\sum_{i=1}^{m} a_{ij} y_i^* = c_j$. Similarly, for any value of i, either $y_i^* = 0$ or $\sum_{j=1}^{n} a_{ij} x_j^* = b_i$. □