Lecture 5 Alternatives

- theorem of alternatives for linear inequalities
- Farkas' lemma and other variants

Theorem of alternatives for linear inequalities

for given A, b, exactly one of the following two statements is true

- 1. there exists an x that satisfies $Ax \leq b$
- 2. there exists a z that satisfies $z \ge 0$, $A^T z = 0$, $b^T z < 0$

• it is clear that 1 and 2 cannot be both true:

$$Ax \le b, \quad z \ge 0 \qquad \Longrightarrow \qquad z^T (Ax - b) \le 0$$
$$A^T z = 0, \quad b^T z < 0 \qquad \Longrightarrow \qquad z^T (Ax - b) > 0$$

- proof that 1 and 2 cannot be both false is less obvious (see page 5–7)
- z in statement 2 is a **certificate** of infeasibility of $Ax \leq b$

Farkas' lemma

for given A, b, exactly one of the following statements is true:

- 1. there exists an x with with Ax = b, $x \ge 0$
- 2. there exists a y with $A^T y \ge 0$, $b^T y < 0$

proof: apply previous theorem to

$$\begin{bmatrix} A \\ -A \\ -I \end{bmatrix} x \le \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix}$$

• this system is infeasible if and only if there exist u, v, w such that

$$u \ge 0, v \ge 0, w \ge 0, \qquad A^T(u-v) = w, \qquad b^T(u-v) < 0$$

• in simpler notation (defining y = u - v): $A^T y \ge 0$, $b^T y < 0$

Alternatives

Geometric interpretation of Farkas' lemma

assume A is $m \times n$ with columns a_i

first alternative

$$b = \sum_{i=1}^{n} x_i a_i, \qquad x_i \ge 0, \quad i = 1, \dots, n$$

 \boldsymbol{b} is in the cone generated by the columns of \boldsymbol{A}

second alternative

$$y^T a_i \ge 0, \quad i = 1, \dots, m, \qquad y^T b < 0$$

the hyperplane $y^T z = 0$ separates b from a_1, \ldots, a_m

Alternatives





Mixed inequalities and equalities

given A, b, C, d, exactly one of the following statements is true

1. there exists an x that satisfies

$$Ax \le b, \qquad Cx = d$$

2. there exist y, z that satisfy

$$z \ge 0, \qquad A^T z + C^T y = 0, \qquad b^T z + d^T y < 0$$

proof: apply theorem of page 5-2 to

$$\begin{bmatrix} A \\ C \\ -C \end{bmatrix} x \le \begin{bmatrix} b \\ d \\ -d \end{bmatrix}$$

Exercise: strict inequalities

show that exactly one of the following statements is true

1. there exists an x that satisfies

$$Ax < b, \qquad Bx \le c$$

2. there exist y, z that satisfy

$$y \ge 0, \qquad z \ge 0, \qquad A^Ty + B^Tz = 0,$$
 and
$$b^Ty + c^Tz < 0 \qquad \text{or} \qquad b^Ty + c^Tz = 0, \quad y \ne 0$$

hint. statement 1 is equivalent to: there exist u, t such that

$$Au \le tb - 1, \qquad Bu \le tc, \qquad t \ge 1$$

Proof of the theorem of alternatives

- we show that if statement 1 on page 5–2 is false, then 2 is true
- $\bullet\,$ the proof is by induction on the column dimension of A

basic case: if A has zero columns, the alternatives are

1. $b \ge 0$

2. there exists a $z \ge 0$ with $b^T z < 0$ clearly, if 1 is false ($b_i < 0$ for some i), then 2 is true (take $z = e_i$)

induction step

- assume the theorem holds for sets of inequalities with n-1 variables
- consider an inequality $Ax \leq b$ with an $m \times n$ matrix A

• we divide the inequalities $Ax \leq b$ in three groups:

$$I_{+} = \{i \mid A_{in} > 0\}, \qquad I_{0} = \{i \mid A_{in} = 0\}, \qquad I_{-} = \{i \mid A_{in} < 0\}$$

• scale the inequalities with $A_{in} \neq 0$ to get an equivalent system

$$\sum_{k=1}^{n-1} C_{ik} x_k + x_n \le d_i \quad \text{for } i \in I_+$$

$$\sum_{k=1}^{n-1} C_{ik} x_k - x_n \le d_i \quad \text{for } i \in I_-$$

$$\sum_{k=1}^{n-1} A_{ik} x_k \le b_i \quad \text{for } i \in I_0$$

where

$$C_{ik} = \begin{cases} A_{ik}/A_{in} & i \in I_+ \\ -A_{ik}/A_{in} & i \in I_- \end{cases} \qquad d_i = \begin{cases} b_i/A_{in} & i \in I_+ \\ -b_i/A_{in} & i \in I_- \end{cases}$$

• the inequalities indexed by I_+ and I_- hold for some x_n if and only if

$$\max_{i \in I_{-}} \left(\sum_{k=1}^{n-1} C_{ik} x_k - d_i \right) \le \min_{i \in I_{+}} \left(d_i - \sum_{k=1}^{n-1} C_{ik} x_k \right)$$

• therefore $Ax \leq b$ is solvable if and only if there exist (x_1, \ldots, x_{n-1}) s.t.

$$\sum_{k=1}^{n-1} (C_{ik} + C_{jk}) x_k \le d_i + d_j \quad \text{for all } i \in I_-, \ j \in I_+$$
$$\sum_{k=1}^{n-1} A_{ik} x_k \le b_i \quad \text{for all } i \in I_0$$

this is a system of inequalities with n-1 variables

• if this system is infeasible, there exist u_{ij} $(i \in I_-, j \in I_+)$, v_i $(i \in I_0)$,

$$u_{ij} \ge 0$$
 for $i \in I_-$, $j \in I_+$, $v_i \ge 0$ for $i \in I_0$

$$\sum_{i \in I_{-}, j \in I_{+}} (C_{ik} + C_{jk}) u_{ij} + \sum_{i \in I_{0}} v_{i} A_{ik} = 0, \quad k = 1, \dots, n-1$$
$$\sum_{i \in I_{-}, j \in I_{+}} (d_{i} + d_{j}) u_{ij} + \sum_{i \in I_{0}} b_{i} v_{i} < 0$$

• now define

$$z_{i} = \frac{1}{-A_{in}} \sum_{j \in I_{+}} u_{ij} \quad \text{for } i \in I_{-}$$
$$z_{j} = \frac{1}{A_{jn}} \sum_{i \in I_{-}} u_{ij} \quad \text{for } j \in I_{+}$$
$$z_{i} = v_{i} \quad \text{for } i \in I_{0}$$

to get a vector z that satisfies $z\geq 0$, $A^Tz=0,\ b^Tz<0$