Homework I - Solutions

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1. The key observation to make is that in INSERTION-SORT, when the i^{th} element is being inserted, the sum of the number of moves and the number of comparisons is $O(i)$. Consequently, the insertion of n elements will take time $O(n^2)$. Two special cases are worth studying:

   (a) The input array is sorted - In this case, each element is compared with all the elements prior to it. Although there is no data movement, the total number of comparisons is $1 + 2 + \ldots n = O(n^2)$;

   (b) The input array is reverse sorted - In this case, each element suffers exactly two comparisons, one against the number that was inserted before it and another with the element that is inserted after it. However the insertion of the i^{th} element causes $i-1$ items to shift. Thus the insertion of all n elements would require $1 + 2 + \ldots n - 1 = O(n^2)$ shifts.

   We conclude that even if binary search were given to us as a tool, we could not speed up the worst-case time of INSERTION-SORT, which is $\Theta(n^2)$.

2. The only way to compare functions is by taking limits, We have,

$$
\begin{aligned}
\lim_{n\to\infty} \frac{f(n)}{g(n)} &= \lim_{n\to\infty} \frac{n^{0.281}}{\log n} \\
&= \lim_{n\to\infty} \frac{\log n^{0.281}}{\log\log n} \\
&= \lim_{n\to\infty} \frac{0.281.\log n}{\log\log n} \\
&= \lim_{n\to\infty} \frac{0.281.\frac{1}{n}}{\frac{1}{\log n}.\frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{0.281.n.\log n}{n} \\
&= \lim_{n\to\infty} 0.281\ \log n \\
&= \infty
\end{aligned}
$$

   Hence $f(n)$ grows at a *faster* rate than $g(n)$; we can therefore write $f(n) = \omega(g(n))$ or $g(n) = o(f(n))$. I have also given credit for the weak $g(n) = O(f(n))$.

3. Let

$$
\begin{aligned}
k &= a^{\log_b n} \\
\Rightarrow \log_b k &= \log_b a^{\log_b n} \\
\Rightarrow \log_b k &= \log_b n . \log_b a
\end{aligned}
$$

$$\begin{aligned}\Rightarrow k &= b^{(\log_b n . \log_b a)} \\ \Rightarrow k &= (b^{\log_b n})^{\log_b a} \\ \Rightarrow k &= n^{\log_b a}\end{aligned}$$

4. (a) Using Stirling's approximation from Page 35, (2.12), we have,

$$\begin{aligned}\sqrt{2.\pi.n}(\frac{n}{e})^n \leq n! \leq \sqrt{2.\pi.n}(\frac{n}{e})^{n+\frac{1}{12.n}} \\ \Rightarrow \log(\sqrt{2.\pi.n}(\frac{n}{e})^n) \leq \log(n!) \leq \log(\sqrt{2.\pi.n}(\frac{n}{e})^{n+\frac{1}{12.n}}) \\ \Rightarrow \frac{1}{2}\log 2.\pi + \frac{1}{2}.\log n + n.\log n - n\log e \leq \log(n!) \leq \frac{1}{2}\log 2.\pi + \frac{1}{2}.\log n + (n + \frac{1}{12.n}).(\log n - \log e) \\ \Rightarrow n.\log n - n.\log e \leq \log(n!) + \frac{1}{2}\log 2.\pi \leq \frac{1}{2}.\log n + (n + \frac{1}{12.n})\log n \\ \Rightarrow 0.5.n\log n \leq \log(n!) + \frac{1}{2}\log 2.\pi \leq 3.n.\log n \\ \Rightarrow c_1.n\log n \leq \log(n!) + \frac{1}{2}\log 2.\pi \leq c_2.n.\log n\end{aligned}$$

The two key substitutions being $\frac{1}{2}.n.\log n$ for $n.\log e$ and $n.\log n$ for the two additional terms of the second inequality. It follows that $log(n!) = \Theta(n.\log n)$.

(b) Let $f(n) = n^n$ and $g(n) = \sqrt{2.\pi.n}(\frac{n}{e})^{n+\frac{1}{12.n}}$. Then, we have,

$$\begin{aligned}\lim_{n\to\infty}\frac{f(n)}{g(n)} &= \lim_{n\to\infty}\frac{n^n}{\sqrt{2.\pi.n}(\frac{n}{e})^{n+\frac{1}{12.n}}} \\ &= \infty\end{aligned}$$

Hence, we can conclude that $g(n) = o(f(n))$. (Make sure you can go through all the steps.)

5. Base case:
LHS = 0.
RHS = $\frac{\phi^0 - \hat{\phi}^0}{\sqrt{5}} = 0$.
Hence LHS = RHS and the base case is true.

Inductive Step: We assume that the formula

$$F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}$$

is true for all $i < n$, using strong induction. Then, we have,

$$\begin{aligned}F_n &= F_{n-1} + F_{n-2} \\ &= \frac{\phi^{n-1} - \hat{\phi}^{n-1}}{\sqrt{5}} + \frac{\phi^{n-2} - \hat{\phi}^{n-2}}{\sqrt{5}} \\ &= \frac{\phi^{n-1} + \phi^{n-2}}{\sqrt{5}} - \frac{\hat{\phi}^{n-1} + \hat{\phi}^{n-2}}{\sqrt{5}} \\ &= \frac{\phi^n}{\sqrt{5}}(\frac{1}{\phi} + \frac{1}{\phi^2}) - \frac{\hat{\phi}^n}{\sqrt{5}}(\frac{1}{\hat{\phi}} + \frac{1}{\hat{\phi}}^2)\end{aligned}$$

It is straightforward to show that

$$\frac{1}{\phi} + \frac{1}{\phi^2} = 1$$

and

$$\frac{1}{\hat{\phi}} + \frac{1}{\hat{\phi}^2} = 1$$

We can thus conclude that

$$F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}, \forall i$$

6. (a) Take $f(n) = e^n$. Hence $f(\frac{n}{2}) = e^{\frac{n}{2}}$ Hence

$$\begin{aligned}
\lim_{n\to\infty} \frac{f(n)}{f(\frac{n}{2})} &= \lim_{n\to\infty} \frac{e^n}{e^{\frac{n}{2}}} \\
&= \lim_{n\to\infty} e^{\frac{n}{2}} \\
&= \infty
\end{aligned}$$

Consequently, $f(n) \neq O(f(\frac{n}{2}))$ for this choice of f and the proposition is false.

(b) Let us take some $g(n) = o(f(n))$. Be definition, we have,

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = 0$$

Let us first show that $f(n) + o(f(n)) = O(f(n))$ i.e. $f(n) + g(n) = O(f(n))$. Taking limits, we get

$$\begin{aligned}
\lim_{n\to\infty} \frac{f(n) + g(n)}{f(n)} &= \lim_{n\to\infty} 1 + \frac{g(n)}{f(n)} \\
&= 1
\end{aligned}$$

Similarly, we can show that $f(n) + o(f(n)) = \Omega(f(n))$ Hence, we conclude that $f(n) + o(f(n)) = \Theta(f(n))$.

7. Break up the sum as

$$S = \sum_{k=0}^{\infty} \frac{k}{2^k} - \sum_{k=0}^{\infty} \frac{1}{2^k}$$

Using the formula (3.6) from text (Pg. 45), we know that

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{k}{2^k} &= \frac{\frac{1}{2}}{(1 - \frac{1}{2}^2)} \\
&= 2
\end{aligned}$$

Using the formula for geometric series (3.4) on Pg. 35, we know that

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{1}{2^k} &= \frac{1}{(1 - \frac{1}{2})} \\
&= 2
\end{aligned}$$

That pretty much proves what we want to prove!