Summation - Tricks of the trade

K. Subramani
Department of Computer Science and Electrical Engineering,
West Virginia University,
Morgantown, WV
ksmani@csee.wvu.edu

1 Factoids

Lemma: 1.1 *Linearity of sums*

$$\sum_{k=1}^{n}(c.a_k + b_k) = c.\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k \tag{1}$$

Lemma: 1.2 *Given the arithmetic progression, of the form*

$$a_1, a_1 + d, a_1 + 2.d, \ldots,$$

the n^{th} term in the sequence is given by

$$T_n = a_1 + (n-1).d \tag{2}$$

and the sum of the first n terms is given by:

$$S_n = \frac{n}{2}.(a_1 + T_n) \tag{3}$$

Lemma: 1.3 *Given a geometric series of the form*

$$1, x, x^2, \ldots,$$

the n^{th} term in the sequence is given by

$$T_n = x^{n-1} \tag{4}$$

and the sum of the first n terms is given by:

$$S_n = 1.\frac{x^n - 1}{x - 1}. \tag{5}$$

If $x < 1$, we can rewrite S_n as:

$$S_n = \frac{1}{1 - x}. \tag{6}$$

Remark: 1.1 *These formulae are slightly different from the ones in the text. Ensure that you understand the differences.*

Remark: 1.2 *Prove all the above formulae using mathematical induction.*

2 Techniques

The following techniques are used to derive upper and lower bounds on series:

1. Telescoping - If the summand can be expressed as the difference between adjacent terms of the corresponding sequence, then telescoping can be used. Clearly, in sequence

$$a_0, a_1, \ldots, a_n$$

$$\sum_{k=1}^{n} a_k - a_{k-1} = a_n - a_0$$

Similarly,

$$\sum_{k=0}^{n-1} a_k - a_{k+1} = a_0 - a_n$$

Let us say that we want

$$\sum_{k=1}^{n-1} \frac{1}{k.(k+1)}$$

Observe that $\frac{1}{k.(k+1)}$ can be rewritten as:

$$\frac{1}{k} - \frac{1}{k+1}$$

Applying the telescoping formula,

$$\sum_{k=1}^{n-1} \frac{1}{k.(k+1)} = 1 - \frac{1}{n}$$

Remark: 2.1 *Make sure that you understand every step!*

2. Mathematical Induction - The hammer! It can be used almost everywhere. For deriving bounds, it is necessary to guess a value and then validate the guess. (Unfortunately, there will not be a samaritan lurking in the exam/quiz to give you the answer. You will have to make an intelligent guess and then check it.) You do not have to guess the exact function e.g. $\frac{n.(n+1)}{2}$; it suffices to guess the form i.e. $O(n^2)$. Consider the geometric series $\sum_{k=0}^{n} 3^k$. You can apply the formula, which gives $\frac{1}{2}.(3^{n+1} - 1)$ i.e. $\frac{3}{2}.3^n - \frac{1}{2}$ i.e. $O(3^n)$. Or you could guess the form as $O(3^n)$. This means that we have to prove

$$\sum_{k=0}^{n} 3^k \leq c.3^n$$

for some $c > 0$.

Base confirmation: Putting $n = 0$, gives $1 \leq c$ or $c \geq 1$. Inductive step: Assuming that the proposition holds for $n = p$, we get

$$\sum_{k=0}^{p} 3^k \leq c.3^p$$

for some $c > 0$. Now,

$$\sum_{k=0}^{p+1} 3^k = \sum_{k=0}^{p} 3^p + 3^{p+1}$$

$$\Rightarrow \sum_{k=0}^{p+1} 3^k \leq c.3^p + 3^{p+1}$$

by the inductive hypothesis

$$\Rightarrow \sum_{k=0}^{p+1} 3^k \leq (\frac{1}{3} + \frac{1}{c}).c.3^{p+1}$$

, which holds as long as $(\frac{1}{3} + \frac{1}{c}) \leq 1$ i.e. $c \geq \frac{3}{2}$. Thus we have proved our case i.e.

$$\sum_{k=0}^{n} 3^k \leq c.3^n$$

for some $c \geq \frac{3}{2}$ or

$$\sum_{k=0}^{n} 3^k = O(3^n)$$

3. Term bounding - Yet another idea is to bound the individual terms of the series, e.g.

$$\sum_{k=1}^{n} k \leq \sum_{k=1}^{n} n = n^2$$

For a geometric series, $\sum_{k=0}^{n} a_k$, if $\frac{a_{k+1}}{a_k} \leq r < 1$, where r is a constant, then $a_k \leq a_0.r^k$. Hence,

$$\sum_{k=0}^{n} a_k \leq \sum_{k=0}^{\infty} a_0.r^k$$
$$= a_0. \sum_{k=0}^{\infty} .r^k$$
$$= a_0. \frac{1}{1-r}$$

Remark: 2.2 *Use the term bounding idea to derive an upper bound for* $\sum_{k=1}^{\infty} \frac{k}{3^k}$. *(Hint: Given in textbook)*

4. Sum splitting - This technique is particularly useful for deriving *lower bounds* e.g.

$$\sum_{k=1}^{n} k = \sum_{k=1}^{\frac{n}{2}} k + \sum_{k=\frac{n}{2}+1}^{n} k$$
$$\Rightarrow \geq 0 + \sum_{k=\frac{n}{2}+1}^{n} \frac{n}{2}$$
$$\Rightarrow \geq (\frac{n}{2})^2$$

5. Integration - This is a versatile technique used in deriving upper and lower bounds when the function (summand) is not amenable to the tricks above. For a monotonically increasing function, we have

$$\int_{m-1}^{n} f(x).dx \leq \sum_{k=m}^{n} f(k) \leq \int_{m}^{n+1} f(x).dx$$

For a monotonically decreasing function, we have

$$\int_{m-1}^{n} f(x).dx \geq \sum_{k=m}^{n} f(k) \geq \int_{m}^{n+1} f(x).dx$$

Take $f(k) = \frac{1}{k}$, a monotonically decreasing function. Hence,

$$\sum_{k=1}^{n} \frac{1}{k} \geq \int_{1}^{n+1} \frac{dx}{x} = \ln(n+1)$$

Likewise,

$$\sum_{k=2}^{n} \frac{1}{k} \leq \int_{1}^{n} \frac{dx}{x} = \ln n$$

and thus

$$\ln(n+1) \leq \sum_{k=1}^{n} \frac{1}{x} \leq 1 + \ln n$$