A Probabilistic Identity

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Theorem: 0.1 *For arbitrary events E_1 and E_2 in a sample space S,*

$$\mathbf{Pr}[E_1] \leq \mathbf{Pr}[E_1|E_2^c] + \mathbf{Pr}[E_2]$$

Proof:

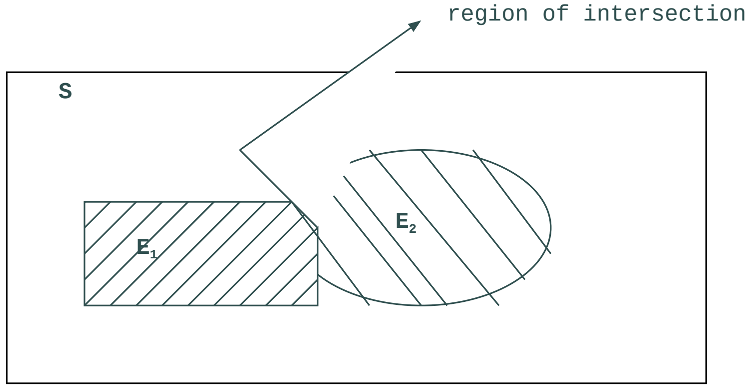

Figure 1: A probabilistic identity

Observe from Figure (1), that the set E_1 can be partioned into the disjoint sets $E_1 - E_2$ and $E_1 \cap E_2$. From elementary set theory, we know that $E_1 - E_2 = E_1 \cap E_2^c$ (Work it out!) Since the sets, $E_1 \cap E_2^c$ and $E_1 \cap E_2$ are disjoint (mutually exclusive), we have,

$$\begin{aligned}
\mathbf{Pr}[E_1] &= \mathbf{Pr}[E_1 \cap E_2^c] + \mathbf{Pr}[E_1 \cap E_2] \\
&\leq \mathbf{Pr}[E_1 \cap E_2^c] + \mathbf{Pr}[E_2] \\
&= \mathbf{Pr}[E_1|E_2^c].Pr[E_2^c] + \mathbf{Pr}[E_2] \\
&\leq \mathbf{Pr}[E_1|E_2^c] + \mathbf{Pr}[E_2]
\end{aligned}$$

□