Practice Midterm Solutions

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1. We are given the probability distribution described by:

$$\begin{aligned} \mathbf{Pr}(X) &= \frac{1}{\ell - k + 1}, x = k, k+1, ..., \ell \\ &= 0, \ otherwise \end{aligned} \quad (1)$$

For the above PMF to describe a valid probability distribution, it must satisfy two conditions:

(a) $0 \leq \mathbf{Pr}(X = x) \leq 1$

$\mathbf{Pr}(X = x) = \frac{1}{\ell-k+1}$, for $x = k, k+1, ..., \ell$
Since $\ell > k$ then $\ell - k + 1 > 1$. Thus, $0 < \frac{1}{\ell-k+1} < 1$
Hence, $0 \leq \mathbf{Pr}(X = x) \leq 1$

(b) $\sum_x \mathbf{Pr}(X = x) = 1$

$$\begin{aligned} \sum_x \mathbf{Pr}(X = x) &= \sum_{i=k}^{\ell} \frac{1}{\ell - k + 1} \\ &= \frac{1}{\ell - k + 1} \sum_{x=k}^{\ell} 1 \\ &= \frac{\ell - k + 1}{\ell - k + 1} \\ &= 1. \end{aligned}$$

Thus, we may conclude that the PMF descibed by (1) is a valid probability distribution.

- The mean of the distribution $\mathbf{E}(X)$ is calculated as:

$$\begin{aligned} \mathbf{E}[X] &= \sum_x x\mathbf{Pr}(x) \\ &= \sum_{x=k}^{\ell} \frac{x}{\ell - k + 1} = \frac{1}{\ell - k + 1} \sum_{x=k}^{\ell} x \\ &= \frac{1}{\ell - k + 1} \times \frac{(\ell - k + 1)(\ell + k)}{2} \\ &= \frac{\ell + k}{2}. \end{aligned}$$

- The standard deviation of the distribution $\sigma_{\mathbf{x}}$ is calculated as:

$$\sigma_{\mathbf{x}}^{\mathbf{2}} = \mathbf{E}[X^2] - (\mathbf{E}[x])^2$$

$$
\begin{aligned}
\mathbf{E}[X^2] &= \sum_x x^2 \mathbf{Pr}(x) \\
&= \sum_{x=k}^{\ell} \frac{x^2}{\ell - k + 1} \\
&= \frac{1}{\ell - k + 1} \sum_{x=k}^{\ell} x^2 \\
&= \frac{1}{\ell - k + 1} \left\{ \sum_{x=1}^{\ell} x^2 - \sum_{x=1}^{k-1} x^2 \right\} \\
&= \frac{1}{\ell - k + 1} \left\{ \frac{(\ell)(\ell+1)(2\ell+1)}{6} - \frac{k(k-1)(2k-1)}{6}] \right\} \\
&= \frac{1}{\ell - k + 1} \left\{ \frac{2(\ell^3 - k^3) + 3(\ell^2 + k^2) + (\ell - k)}{6} \right\}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sigma_{\mathbf{x}}^{\mathbf{2}} &= \frac{1}{\ell - k + 1} \left\{ \frac{2(\ell^3 - k^3) + 3(\ell^2 + k^2) + (\ell - k)}{6} \right\} - \left(\frac{\ell + k}{2} \right)^2 \\
&= \frac{4\left(\ell^3 - k^3\right) + 6\left(\ell^2 + k^2\right) + 2\left(\ell - k\right) - 3\left(\ell + k\right)^2\left(\ell - k + 1\right)}{12\left(\ell - k + 1\right)} \\
&= \frac{4\left(\ell^3 - k^3\right) + 6\left(\ell^2 + k^2\right) + 2\left(\ell - k\right) - 3\left(\ell + k\right)^2\left(\ell - k\right) - 3\left(\ell + k\right)^2}{12\left(\ell - k + 1\right)} \\
&= \frac{4\left(\ell^3 - k^3\right) + 3\ell^2 - 6\ell k + 3k^2 + \left(\ell - k\right)\left(2 - 3\left(\ell + k\right)^2\right)}{12\left(\ell - k + 1\right)} \\
&= \frac{4\left(\ell^3 - k^3\right) + 3\left(\ell - k\right)^2 + \left(\ell - k\right)\left(2 - 3\left(\ell + k\right)^2\right)}{12\left(\ell - k + 1\right)} \\
&= \frac{\left(\ell - k\right)\left[4\left(\ell^2 + \ell k + k^2\right) + 3\left(\ell - k\right) + 2 - 3\left(\ell + k\right)^2\right]}{12\left(\ell - k + 1\right)} \\
&= \frac{\left(\ell - k\right)\left[\ell^2 - 2\ell k + k^2 + 3\left(\ell - k\right) + 2\right]}{12\left(\ell - k + 1\right)} \\
&= \frac{\left(\ell - k\right)\left[\left(\ell - k\right)^2 + 3\left(\ell - k\right) + 2\right]}{12\left(\ell - k + 1\right)} \\
&= \frac{\left(\ell - k\right)\left(\ell - k + 2\right)\left(\ell - k + 1\right)}{12\left(\ell - k + 1\right)} \\
&= \frac{\left(\ell - k\right)\left(\ell - k + 2\right)}{12}
\end{aligned}
$$

Hence,

$$
\sigma_{\mathbf{x}} = \sqrt{\frac{\left(\ell - k\right)\left(\ell - k + 2\right)}{12}}
$$

2. When the coin is tossed twice, the sample space is: $S = \{HH, TT, HT, TH\}$. Since the coin turns up HEADS with probability $p \neq \frac{1}{2}$, the goal in this problem is to devise a revised sample space, in which HEADS and TAILS are equiprobable. Note that the events $\{HT\}$ and $\{TH\}$ both occur with probability $p \cdot (1 - p)$. Consequently, we can create a new sample space $S' = \{HT, TH\}$, in which both events are equiprobable. Denote either one as HEADS and the other as TAILS!

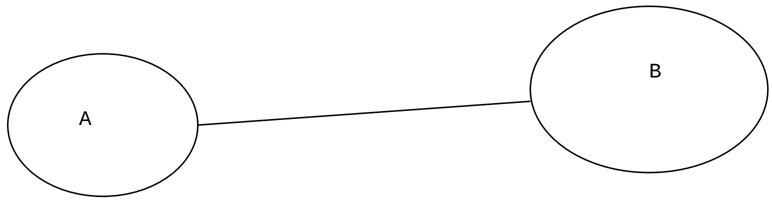

Figure 1: A bad input instance for the vertex-contraction min-cut algorithm

3. Consider the graph in Figure (1).

   A and B are complete graphs on n vertices connected by a single edge. Clearly this edge is the min-cut of this input instance. We need to study the probability that the edge survives a round of vertex contraction, i.e. this cut is not lost as a result of contraction. Any vertex contraction will destroy the min-cut, if it picks one vertex from A and the other vertex from B. The process of picking 2 random vertices from the graph can be done in one of the following 2 ways:

   (a) Pick a vertex, uniformly at random; then pick another vertex uniformly at random. Under this scheme, the probability that a particular vertex-pair is picked is $\frac{1}{2n\cdot(2n-1)}$ (sequential model).

   (b) Pick a vertex pair, uniformly at random; under this scheme, the probability that a particular vertex-pair is picked in $\frac{1}{C(2n,2)}$, where $C(2n,2)$ is the number of combinations of $2n$ objects, taken 2 at a time (parallel model).

   Both schemes differ only by a constant insofar as this analysis is concerned; we will assume that the sampling is done as per the sequential model.

   Observe that the only way in which the min-cut survives a round of $2n-2$ vertex contractions, is if A has been reduced to a single vertex and B has been reduced to a single vertex, i.e. each of the $2n-2$ contraction operations chose both vertices from A or both vertices from B. Let E_A denote the event that A was reduced to a single vertex at some point during the $2n-2$ contractions and let E_b denote the event that B was reduced to a single vertex, at some point during the $2n-2$ contractions. We need to study the probability of the event $E = E_A \cap E_B$.

0.1 A Probabilistic Recurrence

Consider a graph G with 2 components A and B, connected by a singe edge as shown in Figure (1). Let $a = |A|$ and $b = |B|$

Let E_1^A denote the event that both vertices were picked from A in the first contraction. Clearly $\mathbf{Pr}[E_1^A] = \frac{a}{a+b} \times \frac{a-1}{a+b-1}$, likewise, $\mathbf{Pr}[E_1^B] = \frac{b}{a+b} \times \frac{b-1}{a+b-1}$,

Observe that a min-cut preserving contraction in G, results in either A losing a vertex or B losing a vertex. Both these events are mutually exclusive. Let $T(a,b)$ denote the probability with which the min-cut survives a round of $a+b-2$ contractions. It follows that

$$\begin{aligned} T(a,b) &= \mathbf{Pr}[E_1^A]\cdot T(a-1,b) + \mathbf{Pr}[E_1^B]\cdot T(a,b-1) \\ &= \frac{a}{a+b}\cdot\frac{a-1}{a+b-1}\cdot T(a-1,b) + \frac{b}{a+b}\cdot\frac{b-1}{a+b-1}\cdot T(a,b-1) \end{aligned} \tag{2}$$

Equation (2) is true for all $a, b \geq 2$.

We use mathematical induction on the sum of the indices $(a+b)$ to show that $T(a,b) \leq (\frac{1}{2})^{a+b-2}$.

It is not hard to verify the following initial condtions:

$$\begin{aligned} T(1,0) &= 0 \\ T(0,1) &= 0 \\ T(1,1) &= 1 \\ T(2,1) &= \frac{2}{3}\cdot\frac{1}{2}\cdot T(1,1) + \frac{1}{3}\cdot\frac{0}{2}\cdot T(2,0) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{3} \\
T(1,2) &= \frac{1}{3} \\
T(2,2) &= \frac{2}{4} \cdot \frac{1}{3}.T(1,2) + \frac{2}{4} \cdot \frac{1}{3}.T(2,1) \\
&= \frac{1}{9}
\end{aligned}$$

Clearly, the proposition is true for the base cases $a + b = 3, 4, 5, 6$ (Verify it!).
Let us assume that $T(j, k)$ is true, whenever $6 < j + k \leq n - 1$.
Using (2), we have,

$$\begin{aligned}
T(a,b) &= \frac{a}{a+b} \cdot \frac{a-1}{a+b-1} \cdot T(a-1,b) + \frac{b}{a+b} \cdot \frac{b-1}{a+b-1} \cdot T(a,b-1) \\
&\leq \frac{a}{a+b} \cdot \frac{a-1}{a+b-1} \cdot (\frac{1}{2})^{a+b-3} + \frac{b}{a+b} \cdot \frac{b-1}{a+b-1} \cdot (\frac{1}{2})^{a+b-3} \\
&= (\frac{1}{2})^{a+b-3} \cdot [\frac{a \cdot (a-1) + b \cdot (b-1)}{(a+b)(a+b-1)}] \\
&\leq (\frac{1}{2})^{a+b-3} \cdot \frac{1}{4}
\end{aligned}$$

$(That\ is\ why\ I\ needed\ to\ go\ upto\ a + b = 6!)$

$$\begin{aligned}
&= (\frac{1}{2})^{a+b-1} \\
&\leq (\frac{1}{2})^{a+b-2}
\end{aligned}$$

It follows that $T(n, n) \leq (\frac{1}{2})^{2n-2}$. Thus, the probability of preserving the min-cut is exponential in the size of the graph.

4. Refer Lecture Notes II!

5. Consider an arbitrary language $L \in \mathbf{RP}$. We need to show that $L \in \mathbf{BPP}$. Since $L \in \mathbf{RP}$, there exists an algorithm $\mathbf{A}$ to decide L as follows:

   - $x \in L \Rightarrow \mathbf{Pr}(\mathbf{A}(x)\ \ accepts) \geq \frac{1}{2}$
   - $x \notin L \Rightarrow \mathbf{Pr}(\mathbf{A}(x)\ \ accepts) = 0$

   Now consider the following algorithm, which we denote by $\mathbf{A}'$, to decide L. Given an arbitrary string $x \in \sum^*$, run $\mathbf{A}$ twice on it. If either run accepts, declare $x \in L$, otherwise declare $x \notin L$. If $x \in L$, the probability that $\mathbf{A}$ rejects is $< \frac{1}{2}$; consequently the probability that $\mathbf{A}'$ rejects is $< \frac{1}{4}$; it follows that the probability that $\mathbf{A}'$ accepts $\geq \frac{3}{4}$. If $x \notin L$, the probability that $\mathbf{A}$ accepts $= 0$; hence the probability that $\mathbf{A}'$ accepts $= 0 \leq \frac{1}{4}$. It is clear that $\mathbf{A}'$ is a **BPP** algorithm for L! (see [MR95].) The claim follows.

6. Define

$$\begin{aligned}
f(y) &= 1,\ \ if\ \ h(y) \geq t \\
&= 0\ \ otherwise
\end{aligned}$$

   It follows that $\mathbf{E}[f(Y)] = 1.\mathbf{Pr}[h(Y) \geq t] + 0.\mathbf{Pr}[h(Y) < t]$; hence $\mathbf{Pr}[h(Y) \geq t] = \mathbf{E}[f(Y)]$. Since $f(y) \leq \frac{h(y)}{t}, \forall h(y)$ (note it is not true for all y!), we have

$$\begin{aligned}
\mathbf{E}[f(Y)] &\leq \mathbf{E}[\frac{h(Y)}{t}] \\
&= \frac{\mathbf{E}[Y]}{t}
\end{aligned}$$

References

[MR95] Rajeev Motwani and Prabhakar Raghavan. *Randomized Algorithms.* Cambridge University Press, Cambridge, England, June 1995.