Analysis of Algorithms - Homework I (Solutions)

L. Kovalchick
LCSEE,
West Virginia University,
Morgantown, WV
{lynn@csee.wvu.edu}

1 Problems

1. Show using mathematical induction (3 points)

$$\sum_{i=1}^{n} i^3 = [\frac{n \cdot (n+1)}{2}]^2$$

<u>Proof</u>: *Base case $P(1)$:*

$$\begin{aligned}
LHS &= \sum_{i=1}^{1} i^3 \\
&= 1^3 \\
&= 1 \\
RHS &= [\frac{1 \cdot (1+1)}{2}]^2 \\
&= [\frac{1 \cdot (2)}{2}]^2 \\
&= [\frac{2}{2}]^2 \\
&= [1]^2 \\
&= 1
\end{aligned}$$

Thus, $LHS = RHS$ and $P(1)$ is true.

Let us assume that $P(k)$ is true, i.e.

$$\sum_{i=1}^{k} i^3 = [\frac{k \cdot (k+1)}{2}]^2$$

We need to show that $P(k+1)$ is true.

$$\begin{aligned}
LHS &= \sum_{i=1}^{k+1} i^3 \\
&= 1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3
\end{aligned}$$

$$
\begin{aligned}
&= [\frac{k \cdot (k+1)}{2}]^2 + (k+1)^3 \quad (\textit{using the inductive hypothesis}) \\
&= \frac{k^2 \cdot (k+1)^2}{4} + (k+1)^3 \\
&= \frac{k^2 \cdot (k+1)^2 + 4 \cdot (k+1)^3}{4} \\
&= \frac{(k+1)^2 \cdot [k^2 + 4 \cdot (k+1)]}{4} \\
&= \frac{(k+1)^2 \cdot [k^2 + 4k + 4)]}{4} \\
&= \frac{(k+1)^2 \cdot (k+2)^2}{4} \\
&= [\frac{(k+1) \cdot (k+2)}{2}]^2 \\
RHS &= [\frac{(k+1) \cdot ((k+1)+1)}{2}]^2 \\
&= [\frac{(k+1) \cdot (k+2)}{2}]^2
\end{aligned}
$$

LHS=RHS. Thus, we have shown that $P(k) \rightarrow P(k+1)$*; applying the principle of mathematical induction, we conclude that the conjecture is true.* □

2. Show using mathematical induction (2 points)

$$\sum_{i=0}^{n} a^i = \frac{1-a^{n+1}}{1-a}, \quad 0 < a \neq 1$$

Proof: *Base case* $P(0)$*:*

$$
\begin{aligned}
LHS &= \sum_{i=0}^{0} a^i \\
&= a^0 \\
&= 1 \\
RHS &= \frac{1-a^{0+1}}{1-a} \\
&= \frac{1-a^1}{1-a} \\
&= \frac{1-a}{1-a} \\
&= 1
\end{aligned}
$$

Thus, $LHS = RHS$ *and* $P(0)$ *is true.*

Let us assume that $P(k)$ *is true, i.e.*

$$\sum_{i=0}^{k} a^i = \frac{1-a^{k+1}}{1-a}, \quad 0 < a \neq 1$$

We need to show that $P(k+1)$ is true.

$$
\begin{aligned}
LHS &= \sum_{i=0}^{k+1} a^i \\
&= a^0 + a^1 + a^2 + \ldots + a^k + a^{k+1} \\
&= \frac{1-a^{k+1}}{1-a} + a^{k+1} \text{ (using the inductive hypothesis)} \\
&= \frac{1-a^{k+1} + (1-a)a^{k+1}}{1-a} \\
&= \frac{1-a^{k+1} + a^{k+1} - a^{k+2}}{1-a} \\
&= \frac{1-a^{k+2}}{1-a} \\
RHS &= \frac{1-a^{(k+1)+1}}{1-a} \\
&= \frac{1-a^{k+2}}{1-a}
\end{aligned}
$$

Thus, LHS=RHS. We have shown that $P(k) \rightarrow P(k+1)$; applying the principle of mathematical induction, we conclude that the conjecture is true. □

3. Show that $O(\max\{f(n), g(n)\}) = O(f(n) + g(n))$ (3 points)

   Proof: *We must show that if $h(n) \in O(\max\{f(n), g(n)\})$ then $h(n) \in O(f(n) + g(n))$ and vice versa.*
   First, let $h(n) \in O(\max\{f(n), g(n)\})$. This implies that:

$$
\begin{aligned}
h(n) &\leq c \cdot \max\{f(n), g(n)\} \text{ (since } f(n), g(n) \geq 0 \text{ and } c > 0) \\
h(n) &\leq c \cdot (f(n) + g(n))
\end{aligned}
$$

   Then, by definition of "O", $h(n) \in O(f(n) + g(n))$.
   Now, let $h(n) \in O(f(n) + g(n))$. This implies that:

$$
\begin{aligned}
h(n) &\leq c \cdot (f(n) + g(n)) \\
&\leq 2c \cdot \max\{f(n), g(n)\} \text{ (since } f(n), g(n) \geq 0 \text{ and } c > 0) \\
&\leq c' \cdot \max\{f(n), g(n)\}
\end{aligned}
$$

   Then, by definition of "O", $h(n) \in O(\max\{f(n), g(n)\})$.
   We have thus shown that $h(n) \in O(f(n)+g(n)) \Rightarrow h(n) \in O(\max\{f(n), g(n)\})$ and $h(n) \in O(\max\{f(n), g(n)\}) \Rightarrow h(n) \in O(f(n) + g(n))$, which implies that $O(\max\{f(n), g(n)\}) = O(f(n) + g(n))$.
   □

4. Consider the experiment of throwing a pair of dice. Let A be the event that the first die shows up prime and B be the event that the sum of the two dice is 8. Are events A and B independent? (2 points)

   There are 3 possible prime numbers that can come up on the first die (A=2, A=3, or A=5). If the first die is a 2 (A=2), in order for the sum of the two dice to equal 8 (B=8), the second die must be a 6. Similarly, if the first die is a 3 (A=3), the second die must be a 5, and if the first die is a 5 (A=5), the second die must be a 3.

There are 5 possible ways to get a sum of 8 i.e. {(2,6), (3,5),(4,4),(5,3),(6,2)}.

From this we have:

$$\begin{aligned}
A &= \{2, 3, 5\} \\
Pr(A) &= \frac{3}{6} \\
&= \frac{1}{2} \\
B &= \{(2,6), (3,5), (4,4), (5,3), (6,2)\} \\
Pr(B) &= \frac{5}{36} \\
A \cap B &= \{(2,6), (3,5), (5,3)\} \\
Pr(A \cap B) &= \frac{3}{36} \\
&= \frac{1}{12} \\
Pr(A) \cdot Pr(B) &= \frac{1}{2} \cdot \frac{5}{36} \\
&= \frac{5}{72}
\end{aligned}$$

Since, $\frac{1}{12} \neq \frac{5}{72}$, events A and B are not independent.