Analysis of Algorithms - Homework II (Solutions)

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1 Problems

1. Pilot Jim is faced with the task of blowing up a bridge. In each sortie, he carries exactly 5 bombs; each bomb will hit the bridge independently with probability 0.5. The bridge will be blown up, if even one bomb hits it. What is the probability that the bridge is blown up in one sortie? (3 points)

   Solution: If success is defined as a bomb hitting the bridge, the outcome of each bomb drop is a Bernoulli random variable. Since we are interested in the event that at least one bomb hits the bridge, our goal is essentially to *count* the number of bombs that actually hit the bridge, i.e., we are interested in the Binomial random variable X defined by:

$$\begin{aligned} \mathbf{Pr}\{X = k\} &= \binom{5}{k} \cdot (0.5)^k \cdot (1 - 0.5)^{(5-k)} \\ & k = \{0, 1, 2, 3, 4, 5\} \end{aligned} \tag{1}$$

   Now observe that the favourable events are $k = 1, 2, 3, 4, 5$ as per the problem specification. So one technique of solving the problem is to substitute $k = 1, 2, 3, 4, 5$ in Equation (1) and sum up the probabilities. A better idea is to realize that the event in which the bridge is blown up is the complement of the event that the bridge is not hit. Now the probability that the bridge is not hit is easily obtained by substituting $k = 0$ in Equation (1) to get $q = (0.5)^5$. It follows that the probability of hitting the bridge is $1 - q \approx 0.96875$.

   □

2. What is the expected number of sorties that Jim should carry out to ensure that the bridge is destroyed? (2 points)

   Solution: Observe that each sortie is a Bernoulli random variable with probability of success $p \approx 0.96875$, where "success" is now defined as the event that the bridge is blown up (as opposed to a bomb hitting the bridge, as was the case with Problem 1). Thus, we are interested in the expected number of trials before the first success, i.e., our goal is to calculate the expected value of the Geometric random variable X, defined by probability of success $p \approx 0.96875$. We know that $E[X] = \frac{1}{p} \approx 1.0322$. □

3. WVU has 11 games in its football season. The season is deemed winning, if it wins at least 6 games. Let us say that WVU wins each game independently, with probability 0.7. What is the probability that WVU wins the season within 10 games? (3 points)

   Solution: There are two interpretations to the above statement of the problem.

   (a) We can look upon the football season as winning, at the precise moment that 6 games are won, i.e., we are interested in the Negative Binomial random variable X, that counts the number of games (upto a maximum of 10) that need to be played, in order to win 6 games. Accordingly, X is described as:

$$\begin{aligned}\mathbf{Pr}\{X = i\} &= \binom{i-1}{6-1} \cdot (0.7)^6 \cdot (1-0.7)^{(i-6)} \qquad (2)\\ & i = \{6, 7, 8, 9, 10\}\end{aligned}$$

Accordingly, the required probability is calculated as:

$$\begin{aligned}\sum_{i=6}^{10} \mathbf{Pr}\{X = i\} &= \binom{i-1}{5} \cdot (0.7)^6 \cdot (0.3)^{(i-6)} \\ &= \binom{5}{5} \cdot (0.7)^6 + \binom{6}{5} \cdot (0.7)^6 \cdot (0.3) + \binom{7}{5} \cdot (0.7)^6 \cdot (0.3)^2 \\ &\quad + \binom{8}{5} \cdot (0.7)^6 \cdot (0.3)^3 + \binom{9}{5} \cdot (0.7)^6 \cdot (0.3)^4 \\ &= (0.7)^6 + 6 \cdot (0.7)^6 \cdot 0.3 + 21 \cdot (0.7)^6 \cdot (0.3)^2 \\ &\quad + 56 \cdot (0.7)^6 \cdot (0.3)^3 + 126 \cdot (0.7)^6 \cdot (0.3)^4 \\ &\approx 0.849732\end{aligned}$$

(b) We can also interpret the problem in the following way: Given that each game has a success probability of 0.7, what is the probability that WVU wins 6 or more games in the first 10 games? This would mean that we are interested in the *Binomial* random variable X defined by:

$$\begin{aligned}\mathbf{Pr}\{X = k\} &= \binom{10}{k} \cdot (0.7)^k \cdot (1-0.7)^{(10-k)} \qquad (3)\\ & k = \{6, 7, 8, 9, 10\} \qquad (4)\end{aligned}$$

Substituting $k = \{6, 7, 8, 9, 10\}$ and summing up the probabilities, we get the probability of the required event. □

In passing, we note that the first interpretation is the more common one in Sports circles.

4. What is the expected number of games that WVU has to play to ensure a winning season? (2 points)

Solution: Observe that when 6 games are won, the season is deemed winning. Accordingly, we are interested in the expected number of games that need to be played till the 6^{th} game is won, i.e., we are interested in the expected value of the Negative Binomial random variable defined by:

$$\mathbf{Pr}\{X = i\} = \binom{i-1}{5} \cdot (0.7)^6 \cdot (1-0.7)^{(i-6)}$$

As shown in class, this value is $\frac{k}{p} = \frac{6}{0.7} \approx 8.57$. □