Analysis of Algorithms

K. Subramani
LCSEE,
West Virginia University,
Morgantown, WV
ksmani@csee.wvu.edu

1 Induction

Mathematical induction is a general purpose tool of testing out a hypothesis. The drawback of this technique is that the hypothesis must be supplied!

For example, let us say that we want a closed form representation of the sum

$$S_n = 1 + 2 + \dots n = \sum_{i=1}^{n} i \tag{1}$$

Further assume that some good samaritan gave us the hypothesis that the closed form function is $\frac{n.(n+1)}{2}$. We need to check that the samaritan is not deceiving us.

So our Proposition is :

$$P(n) : S_n = \frac{n.(n+1)}{2}, \forall n = 1, 2, ...$$

Induction proceeds by two steps:

1. Step 1: The base case. Is $P(1)$ true? The Left Hand Side (LHS) is $S_1 = 1$ and the RHS is $\frac{1.(1+1)}{2}$, which is 1. Thus, LHS = RHS and $P(1)$ is true. Thus, the samaritan is correct at least when $n = 1$. This step is called *base confirmation*.

2. Step 2: The Inductive step: Here we assume that the samaritan is correct when $n = k$ i.e $P(k)$ is true and show that the formula must also hold for $n = k + 1$ i.e. $P(k + 1)$. Thus, we want to show that

$$P(k) \Rightarrow P(k+1)$$

In our case, assuming that $P(k)$ is true gives $S_k = \frac{k.(k+1)}{2}$. What is $P(k + 1)$? On the LHS it is

$$S_{k+1} = 1 + 2 + \dots k + (k+1)$$

$$\Rightarrow S_{k+1} = (1 + 2 + \dots k) + (k+1)$$

$$\Rightarrow S_{k+1} = S_k + (k+1)$$

Since $P(k)$ is true by assumption, we can set $S_k = \frac{k.(k+1)}{2}$

$$\Rightarrow S_{k+1} = \frac{k.(k+1)}{2} + (k+1)$$

$$\Rightarrow S_{k+1} = \frac{(k+1).(k+2)}{2}$$

which is exactly what we would obtain by substituting $(k+1)$ in the samaritan's formula (= RHS). Thus we have,

$$P(k) \Rightarrow P(k+1).$$

Since $P(1)$ is true, we can then conclude that $P(2)$ is true and hence $P(3)$ is true and .... $P(n)$ is true for all n. Thus the samaritan was good after all.

Remark: 1.1 *Show that*

$$\sum_{i=1}^{n} i^2 = \frac{n.(n+1).(2.n+1)}{6}$$

The induction technique used above is called the First Principle of induction. In some cases though (especially when proving the correctness of Algorithms!), we need to use another form of induction, called the *Second Principle of Induction.* The second principle is based on the following observation:

If

1. $P(1)$ is true, and
2. $\forall r \; 1 \leq r \leq k, \; P(r)$ is true implies that $P(k+1)$ is true, then

$P(n)$ is true for all n.

Remark: 1.2 *Show that for every $n \geq 2$, n is either prime or a product of primes. Does the first principle of induction work?*