Analysis of Algorithms - Quiz I (Solutions)

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1 Problems

1. Prove using induction:

$$\sum_{i=1}^{n} i^2 = \frac{n \cdot (n+1) \cdot (2n+1)}{6}$$

Proof: *Base case $P(1)$:*

$$\begin{aligned}
LHS &= \sum_{i=1}^{1} i^2 \\
&= 1^2 \\
&= 1 \\
RHS &= \frac{1 \cdot (1+1) \cdot (2(1)+1)}{6} \\
&= \frac{1 \cdot (2)(2+1)}{6} \\
&= \frac{2 \cdot 3}{6} \\
&= \frac{6}{6} \\
&= 1
\end{aligned}$$

Thus, $LHS = RHS$ and $P(1)$ is true.
Let us assume that $P(k)$ is true, i.e.,

$$\sum_{i=1}^{k} i^2 = \frac{k \cdot (k+1) \cdot (2k+1)}{6}$$

We need to show that $P(k+1)$ is true.

$$\begin{aligned}
LHS &= \sum_{i=1}^{k+1} i^2 \\
&= 1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{k\cdot(k+1)\cdot(2k+1)}{6} + (k+1)^2 \quad (using\ \ the\ \ inductive\ \ hypothesis) \\
&= \frac{k\cdot(k+1)\cdot(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)\cdot[k\cdot(2k+1) + 6(k+1)]}{6} \\
&= \frac{(k+1)\cdot[2k^2 + k + 6k + 6]}{6} \\
&= \frac{(k+1)\cdot[2k^2 + 7k + 6]}{6} \\
&= \frac{(k+1)\cdot(k+2)\cdot(2k+3)}{6} \\
RHS &= \frac{(k+1)\cdot((k+1)+1)\cdot(2(k+1)+1)}{6} \\
&= \frac{(k+1)\cdot(k+2)\cdot(2k+2+1)}{6} \\
&= \frac{(k+1)\cdot(k+2)\cdot(2k+3)}{6}
\end{aligned}
$$

LHS=RHS. Thus, we have shown that $P(k) \rightarrow P(k+1)$; applying the principle of mathematical induction, we conclude that the conjecture is true. □

2. Consider the recurrence relation:

$$
\begin{aligned}
T(n) &= 1,\ \ if\ \ n = 1 \\
&= T(n-1) + 2^n,\ \ otherwise
\end{aligned}
$$

Show that $T(n) = 2^{n+1} - 3$.

Proof: *Base Case:*

$$
T(1) = 1
$$

Using expansion:

$$
\begin{aligned}
T(n) &= T(n-1) + 2^n \\
&= T(n-2) + 2^{n-1} + 2^n \\
&= T(n-3) + 2^{n-2} + 2^{n-1} + 2^n \\
&\vdots \\
&= T(n-(n-1)) + 2^{n-(n-2)} + 2^{n-(n-3)} + \ldots + 2^{n-1} + 2^n \\
&= T(1) + 2^2 + 2^3 + \ldots + 2^{n-1} + 2^n \\
&= T(1) + \sum_{i=2}^{n} 2^i \\
&= 1 + \sum_{i=1}^{n} 2^i - 2^1
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=0}^{n} 2^i - 2 \\
&= \frac{2^{n+1}-1}{2-1} - 2 \\
&= 2^{n+1} - 3
\end{aligned}
$$

□

3. Show that $\sum_{i=1}^{n} \log i = O(n \log n)$

   Proof: *We can obtain an upper bound on this series by bounding each term of the series, by the largest term ($\log n$). From this, we have:*

$$
\begin{aligned}
\sum_{i=1}^{n} \log i &= \log 1 + \log 2 + \ldots + \log n \\
&\leq \log n + \log n + \ldots + \log n \\
&\leq n \cdot \log n
\end{aligned}
$$

   Then by definition of 'O', $\sum_{i=1}^{n} \log i = O(n \log n)$. □

4. Show that $\sum_{i=1}^{n} \log i = \Omega(n \log n)$

   Proof: *Assume without loss of generality that n is even. Thus,*

$$
\begin{aligned}
\sum_{i=1}^{n} \log i &= \log 1 + \log 2 + \ldots + \log n \\
&\geq \log(\frac{n}{2}+1) + \log(\frac{n}{2}+2) \ldots + \log n \\
&\geq \log(\frac{n}{2}) + \log(\frac{n}{2}) + \ldots + \log(\frac{n}{2}) \\
&= \frac{n}{2} \log(\frac{n}{2}) \\
&= \frac{n}{2} \log n - \frac{n}{2} \\
&\geq \frac{1}{10} n \log n \quad (for \quad n \quad \geq \quad 4)
\end{aligned}
$$

   Note:

$$
\begin{aligned}
5n \log n - 5n &\geq n \log n \\
5n \log n - n \log n &\geq 5n \\
4n \log n &\geq 5n \\
\log n &\geq \frac{5}{4} \\
n &\geq 2^{\frac{5}{4}}
\end{aligned}
$$

   We can thus choose $n \geq 4$, since $2^2 = 4$.
   Then by definition of 'Ω', $\sum_{i=1}^{n} \log i = \Omega(n \log n)$. □

5. Let T be a proper binary tree of height h having n nodes. What is the minimum value for n as a function of h? Justify your answer.

   The minimum value for n as a function of h is: $n = 2h + 1$

   Observe that the minimum value for n in a proper binary tree of height h occurs when the tree is unbalanced. In this case, each internal node has exactly one child that is an external node, except for the internal node at level $h - 1$, which has 2 children which are external nodes. This means that each level of the tree contains exactly two nodes, except for level 0 which contains only 1 node (the root). It follows that a tree of height h would have $2h + 1$ nodes.