Analysis of Algorithms - Scrimmage I (Solutions)

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Please attempt as many problems as you can in class. The scrimmage will not be graded, i.e. there are no points. The solutions are posted at:
`http://www.csee.wvu.edu/~ksmani/courses/fa02/cs320/cs320.html`

1. Prove using mathematical induction:

   (a) $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n\cdot(n+1)}=\frac{n}{n+1}$.
   <u>Proof</u>: *Base case $P(1)$:*

$$\begin{aligned} LHS &= \frac{1}{1\cdot 2} \\ &= \frac{1}{2} \\ RHS &= \frac{1}{1+1} \\ &= \frac{1}{2} \end{aligned}$$

Thus, $LHS = RHS$ and $P(1)$ is true.
Let us assume that $P(k)$ is true, i.e.

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k\cdot(k+1)}=\frac{k}{k+1}$$

We need to show that $P(k+1)$ is true.

$$\begin{aligned} LHS &= \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{(k+1)\cdot(k+2)} \\ &= \frac{k}{k+1}+\frac{1}{(k+1)\cdot(k+2)} \ \ (\textit{using the inductive hypothesis}) \\ &= \frac{1}{k+1}\cdot(k+\frac{1}{k+2}) \\ &= \frac{1}{k+1}\cdot(\frac{k^2+2k+1}{k+2}) \\ &= \frac{1}{k+1}\cdot(\frac{(k+1)^2}{k+2}) \\ &= \frac{k+1}{k+2} \\ &= RHS \end{aligned}$$

Thus, we have shown that $P(k) \rightarrow P(k+1)$; applying the principle of mathematical induction, we conclude that the conjecture is true. □

(b) $7^n - 2^n$ is divisible by 5

<u>Proof</u>: *Base case $P(1)$: Observe that $LHS = 7^1 - 2^1 = 7 - 2 = 5$, which is divisible by 5; therefore $P(1)$ is true.*

Let us assume that $P(k)$ is true for some integer k, i.e. $7^k - 2^k$ is divisible by 5 for some integer k. This means $7^k - 2^k = 5 \cdot m$ for some integer m. Now consider $P(k+1)$.

$$\begin{aligned} LHS &= 7^{k+1} - 2^{k+1} \\ &= 7 \cdot 7^k - 2 \cdot 2^k \\ &= 7 \cdot (5m + 2^k) - 2 \cdot 2^k \ \ (\textit{using the inductive hypothesis}) \\ &= 35m + 7 \cdot 2^k - 2 \cdot 2^k \\ &= 35m + 5 \cdot 2^k \\ &= 5 \cdot (7m + 2^k) \\ &= 5q \ \ for \ \ some \ \ q \end{aligned}$$

Thus, the LHS is divisble by 5 and $P(k+1)$ is true. We have shown that $P(k) \rightarrow P(k+1)$; applying the principle of mathematical induction, we conclude that the conjecture is true.

□

(c) Show that $13^n - 6^n$ is divisible by 7.

<u>Proof</u>: *Base Case: When $n = 1$, $13^n - 6^n = 13 - 6 = 7$ is divisible by 7; thus $P(1)$ is true.*

Let us assume that $P(k)$ is true, i.e. $13^k - 6^k$ is divisible by 7, for some $k > 1$. Thus, $13^k - 6^k = 7 \cdot m$, for some integer m. Now consider $P(k+1)$.

$$\begin{aligned} 13^{k+1} - 6^{k+1} &= 13 \cdot 13^k - 6 \cdot 6^k \\ &= 13 \cdot [6^k + 7m] - 6 \cdot 6^k \\ &= 13 \cdot 7m + 13 \cdot 6^k - 6 \cdot 6^k \\ &= 13 \cdot 7m + 6^k \cdot [13 - 6] \\ &= 13 \cdot 7m + 6^k \cdot 7 \\ &= 7 \cdot [13m + 6^k] \\ &= 7 \cdot q \ \ for \ \ some \ \ integer \ \ q \end{aligned}$$

Thus, applying the principle of mathematical induction, we conclude that the conjecture is true.

□

(d) Show that

$$(cos\theta + i \ \sin\theta)^n = \ cos \ n\theta + i \ \sin \ n\theta \ \ \forall n \geq 1$$

<u>Proof</u>: *Base case P(1):*

$$\begin{aligned} LHS &= (\cos\theta + i \ \sin\theta)^1 \\ &= \cos\theta + i \sin\theta \\ &= RHS \end{aligned}$$

Thus, $P(1)$ is true. Let us assume that $P(k)$ is true for some $k > 1$, i.e.

$$(cos\theta + i\ \sin\theta)^k = \cos k\theta + i \sin k\theta \tag{1}$$

Let us show that $P(k+1)$ is true.

$$\begin{aligned}
(cos\theta + i\ \sin\theta)^{k+1} &= (cos\theta + i\sin\theta)\cdot(cos\theta + i\sin\theta)^k \\
&= (\cos\theta + i\sin\theta)\cdot(cosk\theta + i\sin k\theta)\ \ using\ \ the\ \ inductive\ \ hypothesis \\
&= \cos\theta\cdot\cos k\theta + i\cos\theta\cdot\sin k\theta + i\sin\theta\cdot\cos k\theta + i^2\sin\theta\sin k\theta \\
&= \cos\theta\cdot\cos k\theta - \sin\theta\sin k\theta + i\cdot(\cos\theta\cdot\sin k\theta + \sin\theta\cdot\cos k\theta) \\
&= \cos(1+k)\theta + i\cdot sin(1+k)\theta \\
&= \cos(k+1)\theta + i\cdot sin(k+1)\theta \\
&= RHS
\end{aligned}$$

We apply the principle of mathematical induction and conclude that the conjecture is true. □

2. Compare $f(n)$ and $g(n)$ using asymptotic notation; you may either describe $f(n)$ in terms of $g(n)$ (for instance, $f(n) = O(g(n))$) or $g(n)$ in terms of $f(n)$ (for instance, $g(n) = \omega(f(n))$). Make sure that your description is as precise as possible.

   (a) $f(n) = n\log^5 n$, $g(n) = n^2$
   Observe that

$$\begin{aligned}
\lim_{n\to\infty}\frac{f(n)}{g(n)} &= \lim_{n\to\infty}\frac{n\log^5 n}{n^2} \\
&= \lim_{n\to\infty}\frac{\log^5 n}{n} \\
&= \lim_{n\to\infty}\frac{(5\log^4 n)\frac{1}{n}}{1}\ \ (by\ applying\ L'Hospital's\ rule!) \\
&= \lim_{n\to\infty}\frac{5\log^4 n}{n} \\
&= \lim_{n\to\infty}\frac{5\cdot4\cdot3\cdot2\cdot1}{n}\ \ (by\ repeated\ application\ of\ L'Hospital's\ rule!) \\
&= 0
\end{aligned}$$

   It follows that $f(n) = o(g(n))$.

   (b) $f(n) = n\log_4 n$, $g(n) = n\log_{35} n$.
   Note that $f(n)$ can be written as $n\log_2 n\cdot(\frac{1}{\log_2 4})$, by using the Logarithm rules. Likewise, $g(n)$ can be written as: $n\log_2 n\cdot(\frac{1}{\log_2 35})$. It is clear that $f(n)$ and $g(n)$ differ only by a constant in their rates of growth and hence $f(n) = \Theta(g(n))$.

   (c) $f(n) = \log^3 n$, $g(n) = n^{\frac{1}{3}}$.
   By repeated application of L'Hospital's rule, we see that

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = 0$$

   Thus, $f(n) = o(g(n))$.

   (d) $f(n) = 2^n$, $g(n) = 2^{n+1}$
   Observe that $f(n) = \frac{1}{2}g(n)$ and $g(n) = 2\cdot f(n)$, i.e. the two functions differ at most by a constant in their rates of growth. It follows that $f(n) = \Theta(g(n))$.