Advanced Analysis of Algorithms - Scrimmage I

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Please attempt as many problems as you can in class. The scrimmage will not be graded, i.e. there are no points. The solutions are posted at:
`http://www.csee.wvu.edu/~ksmani/courses/fa03/algos/algos.html`

1 Summation

1. Assume that $0 < |x| < 1$. Derive closed forms for the following sums:
   (a) $\sum_{k=0}^{\infty} x^k$.
   (b) $\sum_{k=0}^{\infty} k \cdot x^k$.
   (c) $\sum_{k=0}^{\infty} k^2 \cdot x^k$.
2. Show that $\sum_{k=1}^{n} \frac{1}{2 \cdot k - 1} = \ln(\sqrt{n}) + O(1)$.
3. Show that $\sum_{k=0}^{\infty} \frac{k-1}{2^k} = 0$.

2 Counting

Let $C(n, k)$, $k \leq n$, denote the number of ways of selecting k objects from a set of n objects.

1. Show that $C(n, k) = C(n-1, k) + C(n-1, k-1)$.
2. Show that $C(n, k) = \frac{n}{k} C(n-1, k-1)$.
3. Prove that $\sum_{i=1}^{n} i = C(n+1, 2)$.

3 Probability

Let X and Y random variables defined on a sample space S.

1. Show that $\mathbf{E}[a] = a$, if a is a constant.
2. If X and Y are non-negative, show that $\mathbf{E}[\max(X, Y)] = \mathbf{E}[X] + \mathbf{E}[Y]$.
3. Prove that $\mathbf{Var}[aX] = a^2 \mathbf{Var}[X]$, a constant.
4. Assume that X can take on only two values, viz. 0 and 1. Show that $\mathbf{Var}[X] = \mathbf{E}[X] \cdot \mathbf{E}[1 - X]$.
5. Let X be non-negative. Show that $\mathbf{Pr}\{X \geq t\} \leq \frac{\mathbf{E}[X]}{t}$, for $t > 0$. (*Markov's inequality.*)
6. Let μ and σ denote the expectation and variance of X respectively. Prove that $\mathbf{Pr}\{|X - \mu| \geq t\sigma\} \leq \frac{1}{t^2}$. (*Chebyshev's inequality.*)