# Advanced Analysis of Algorithms - Scrimmage I

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## 1 Summation

1. Assume that $0 < |x| < 1$. Derive closed forms for the following sums:

   (a) $\sum_{k=0}^{\infty} x^k$.
   **Solution:** Observe that: $\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \ldots = S$. Multiplying $S$ by $x$ we get:

   $$\begin{aligned} x \cdot S &= x(1 + x + x^2 + x^3 + \ldots) \\ &= x + x^2 + x^3 + x^4 + \ldots \end{aligned}$$

   Observe that $S - x \cdot S = S(1-x) = 1$ and by solving for $S$, we get $S = \frac{1}{1-x}$. Thus, $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$. □

   (b) $\sum_{k=0}^{\infty} k \cdot x^k$.
   **Solution:** Using the result of problem $1(a)$, we have: $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$. Differentiating both sides of the above equation with respect to $x$ and making use of the fact that the derivative of a sum is equal to the sum of derivatives, we get:

   $$\begin{aligned} (\sum_{k=0}^{\infty} x^k)' &= (\frac{1}{1-x})' \\ \sum_{k=0}^{\infty} k \cdot x^{k-1} &= \frac{0 \cdot (1-x) - (1) \cdot (-1)}{(1-x)^2} \\ \sum_{k=0}^{\infty} k \cdot x^{k-1} &= \frac{0+1}{(1-x)^2} \\ \sum_{k=0}^{\infty} k \cdot x^{k-1} &= \frac{1}{(1-x)^2} \end{aligned}$$

   Multiplying both sides of the above equation by $x$, we get $\sum_{k=0}^{\infty} k \cdot x^k = \frac{x}{(1-x)^2}$. □

   (c) $\sum_{k=0}^{\infty} k^2 \cdot x^k$.
   **Solution:** Using the result of problem $1(b)$, we have: $\sum_{k=0}^{\infty} k \cdot x^k = \frac{x}{(1-x)^2}$. Differentiating both sides of the above equation with respect to $x$ and making use of the fact that the derivative of a sum is equal to the sum of derivatives, we get:

   $$\begin{aligned} (\sum_{k=0}^{\infty} k \cdot x^k)' &= (\frac{x}{(1-x)^2})' \\ \sum_{k=0}^{\infty} k^2 \cdot x^{k-1} &= \frac{1 \cdot (1-x)^2 - x \cdot (2 \cdot (1-x) \cdot (-1))}{(1-x)^4} \end{aligned}$$

$$\begin{aligned}
\sum_{k=0}^{\infty} k^2 \cdot x^{k-1} &= \frac{(1-x)^2 + 2\cdot x \cdot (1-x)}{(1-x)^4} \\
\sum_{k=0}^{\infty} k^2 \cdot x^{k-1} &= \frac{(1-x)[1-x+2\cdot x]}{(1-x)^4} \\
\sum_{k=0}^{\infty} k^2 \cdot x^{k-1} &= \frac{1+x}{(1-x)^3}
\end{aligned}$$

Multiplying both sides of the above equation by $x$, we get: $\sum_{k=0}^{\infty} k^2 \cdot x^k = \frac{x(1+x)}{(1-x)^3}$. □

2. Show that $\sum_{k=1}^{n} \frac{1}{2\cdot k-1} = \ln(\sqrt{n}) + O(1)$.
**Solution:** Observe that $\sum_{k=1}^{n} \frac{1}{2\cdot k-1} = 1 + \frac{1}{3} + \frac{1}{5} + \ldots$ and that $\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$. By subtracting $\frac{1}{2} \cdot \sum_{k=1}^{n} \frac{1}{k}$ from $\sum_{k=1}^{n} \frac{1}{k}$ we get:

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{k} - \frac{1}{2} \cdot \sum_{k=1}^{n} \frac{1}{k} &= 1 + \frac{1}{3} + \frac{1}{5} + \ldots \\
&= \sum_{k=1}^{n} \frac{1}{2\cdot k - 1}
\end{aligned}$$

Using the fact that $H_n = \sum_{i=1}^{n} \frac{1}{i} = ln(n) + O(1)$, we have:

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{2\cdot k-1} &= H_n - \frac{1}{2} \cdot H_n \\
&= ln(n) + O(1) - \frac{1}{2}(ln(n) + O(1)) \\
&= \frac{1}{2} \cdot ln(n) + O(1) \\
&= ln(\sqrt{n}) + O(1)
\end{aligned}$$

Thus, $\sum_{k=1}^{n} \frac{1}{2\cdot k-1} = ln(\sqrt{n}) + O(1)$. □

3. Show that $\sum_{k=0}^{\infty} \frac{k-1}{2^k} = 0$.
**Solution:** Observe that we can rewrite the sum as,

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{k-1}{2^k} &= \sum_{k=0}^{\infty} (\frac{k}{2^k} - \frac{1}{2^k}) \\
&= \sum_{k=0}^{\infty} \frac{k}{2^k} - \sum_{k=0}^{\infty} \frac{1}{2^k}
\end{aligned}$$

Notice that $\sum_{k=0}^{\infty} \frac{k}{2^k} = \sum_{k=0}^{\infty} k \cdot (\frac{1}{2})^k$. Now, we can use the result obtained from problem $1(b)$ in order to get

$$\begin{aligned}
\sum_{k=0}^{\infty} k \cdot (\frac{1}{2})^k &= \frac{\frac{1}{2}}{(1-\frac{1}{2})^2} \\
&= \frac{\frac{1}{2}}{(\frac{1}{2})^2} \\
&= \frac{\frac{1}{2}}{\frac{1}{4}}
\end{aligned}$$

$$\begin{aligned} &= \frac{4}{2} \\ &= 2 \end{aligned}$$

Next, notice that $\sum_{k=0}^{\infty} \frac{1}{2^k} = \sum_{k=0}^{\infty} (\frac{1}{2})^k$, which is an infinite decreasing geometric series. Thus, $\sum_{k=0}^{\infty} (\frac{1}{2})^k = \frac{1}{1-\frac{1}{2}} = 2$.
Now, by substituting back into our equation, we get

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{k}{2^k} - \sum_{k=0}^{\infty} \frac{1}{2^k} &= 2 - 2 \\ &= 0 \end{aligned}$$

□

# 2 Counting

Let $C(n, k)$, $k \leq n$, denote the number of ways of selecting $k$ objects from a set of $n$ objects.

1. Show that $C(n, k) = C(n-1, k) + C(n-1, k-1)$.
   **Solution:** Observe that

$$\begin{aligned} C(n-1,k) + C(n-1,k-1) &= \frac{(n-1)!}{k! \cdot (n-1-k)!} + \frac{(n-1)!}{(k-1)! \cdot (n-k)!} \\ &= \frac{(n-1)!}{(k-1)! \cdot k \cdot (n-1-k)!} + \frac{(n-1)!}{(k-1)! \cdot (n-1-k)! \cdot (n-k)} \\ &= \frac{(n-1)! \cdot (n-k) + (n-1)! \cdot k}{(k-1)! \cdot k \cdot (n-1-k)! \cdot (n-k)} \\ &= \frac{(n-1)! \cdot n}{(k-1)! \cdot k \cdot (n-1-k)! \cdot (n-k)} \\ &= \frac{n!}{k! \cdot (n-k)!} \end{aligned}$$

   Thus, $C(n, k) = C(n-1, k) + C(n-1, k-1) = \frac{n!}{k! \cdot (n-k)!}$. □

2. Show that $C(n, k) = \frac{n}{k} C(n-1, k-1)$.
   **Solution:** Observe that

$$\begin{aligned} \frac{n}{k} \cdot C(n-1,k-1) &= \frac{n}{k} \cdot \frac{(n-1)!}{(k-1)! \cdot (n-k)!} \\ &= \frac{n \cdot (n-1)!}{k \cdot (k-1)! \cdot (n-k)!} \\ &= \frac{n!}{k! \cdot (n-k)!} \end{aligned}$$

   Thus, $C(n, k) = \frac{n}{k} \cdot C(n-1, k-1)$. □

3. Prove that $\sum_{i=1}^{n} i = C(n+1, 2)$.
   **Proof:** Observe that $\sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots = \frac{n \cdot (n+1)}{2}$. We also have that

$$\begin{aligned} C(n+1,2) &= \frac{(n+1)!}{2! \cdot (n-1)!} \\ &= \frac{(n+1)!}{2 \cdot (n-1)!} \end{aligned}$$

$$
\begin{aligned}
&= \frac{(n+1)\cdot n\cdot(n-1)!}{2\cdot(n-1)!} \\
&= \frac{(n+1)\cdot n}{2}
\end{aligned}
$$

Thus, $\sum_{i=1}^{n} i = C(n+1,2)$. □

# 3 Probability

Let $X$ and $Y$ random variables defined on a sample space $S$.

1. Show that $\mathbf{E}[a] = a$, if $a$ is a constant.
**Solution:** Let $g(x) = a$ for all $x$. Then

$$
\begin{aligned}
\mathbf{E}[a] &= \mathbf{E}[g(x)] \\
&= \sum_{x\cdot p(x)>0} g(x)\cdot p(x) \\
&= \sum_{x\cdot p(x)>0} a\cdot p(x) \\
&= a\cdot \sum_{x\cdot p(x)>0} p(x) \\
&= a\cdot 1 \\
&= a
\end{aligned}
$$

Thus, the expectation of a constant is the constant. □

2. If $X$ and $Y$ are non-negative, show that $\mathbf{E}[\max(X,Y)] = \mathbf{E}[X] + \mathbf{E}[Y]$.
**Solution:** Since $X$ and $Y$ are positive, we have $\max(X,Y) \leq X+Y$. This gives us $\mathbf{E}[\max(X,Y)] = \mathbf{E}[X+Y]$ and by linearity of expectation, we then have $\mathbf{E}[X+Y] = \mathbf{E}[X]+\mathbf{E}[Y]$. Thus, $\mathbf{E}[\max(X,Y)] = \mathbf{E}[X]+\mathbf{E}[Y]$. □

3. Prove that $\mathbf{Var}[aX] = a^2\mathbf{Var}[X]$, if $a$ is a constant.
**Proof:** Let $Y = aX$. Then $\mathbf{Var}[aX] = \mathbf{Var}[Y]$. Observe that

$$
\begin{aligned}
\mathbf{Var}[Y] &= \mathbf{E}[Y^2] - (\mathbf{E}[Y])^2 \;\; \textit{by definition of variance} \\
&= \mathbf{E}[a^2\cdot X^2] - (\mathbf{E}[a\cdot X])^2 \\
&= a^2\cdot \mathbf{E}[X^2] - a^2(\mathbf{E}[X])^2
\end{aligned}
$$

By definition, $a^2\cdot\mathbf{E}[X^2]-a^2\cdot(\mathbf{E}[X])^2 = a^2\cdot(\mathbf{E}[X^2]-(\mathbf{E}[X])^2) = a^2\mathbf{Var}[X]$. Therefore, $\mathbf{Var}[aX] = a^2\mathbf{Var}[X]$. □

4. Assume that $X$ can take on only two values, viz. 0 and 1. Show that $\mathbf{Var}[X] = \mathbf{E}[X]\cdot\mathbf{E}[1-X]$.
**Solution:** Let $\mathbf{Pr}\{X=0\} = 1-p$ and let $\mathbf{Pr}\{X=1\} = p$. Then, by defintion, $\mathbf{E}[X] = 0\cdot(1-p)+1\cdot p = p$ and $\mathbf{E}[X^2] = 0^2\cdot(1-p)+1^2\cdot p = p$, thus $\mathbf{E}[X^2] = \mathbf{E}[X]$. Observe that, by definition, $\mathbf{Var}[X] = \mathbf{E}[X^2]-(\mathbf{E}[X])^2$. Since, $\mathbf{E}[X^2] = \mathbf{E}[X]$, we have

$$
\begin{aligned}
\mathbf{Var}[X] &= \mathbf{E}[X] - (\mathbf{E}[X])^2 \\
&= \mathbf{E}[X]\cdot(1-\mathbf{E}[X]) \\
&= \mathbf{E}[X]\cdot\mathbf{E}[1-X] \;\; \textit{by linearity of expectation}
\end{aligned}
$$

Thus, $\mathbf{Var}[X] = \mathbf{E}[X]\cdot\mathbf{E}[1-X]$. □

5. Let $X$ be non-negative. Show that $\mathbf{Pr}\{X \geq t\} \leq \frac{\mathbf{E}[X]}{t}$, for $t > 0$. (*Markov's inequality.*)
**Proof:** Let $E[X] = \sum_{x \cdot p(x) > 0} x \cdot p(x)$. By splitting the summation, we obtain $\sum_{x \cdot p(x) > 0} x \cdot p(x) = \sum_{x \geq t} x \cdot p(x) + \sum_{x < t} x \cdot p(x)$. Using the technique of bounding terms, we then obtain

$$\begin{aligned} \sum_{x \geq t} x \cdot p(x) + \sum_{x < t} x \cdot p(x) &\geq \sum_{x \geq t} x \cdot p(x) \\ &\geq \sum_{x \geq t} t \cdot p(x) \end{aligned}$$

Since, $\sum_{x \geq t} t \cdot p(x) = t \cdot \sum_{x \geq t} p(x) = t \cdot \mathbf{Pr}\{x \geq t\}$, we have that $\mathbf{Pr}\{X \geq t\} \leq \frac{\mathbf{E}[X]}{t}$. $\square$

6. Let $\mu$ and $\sigma$ denote the expectation and variance of $X$ respectively. Prove that $\mathbf{Pr}\{|X - \mu| \geq t\sigma\} \leq \frac{1}{t^2}$. (*Chebshev's inequality.*)
**Proof:** Observe that $(X - \mu)^2$ is a non-negative random variable, with expectation $\sigma$ (as per the definition of variance), i.e., $\mathbf{E}[(X - \mu)^2] = \sigma^2$.

Therefore, we can use Markov's Inequality, to get $\mathbf{Pr}\{(X - \mu)^2 \geq \sigma^2 \cdot t^2\} \leq \frac{\mathbf{E}[(X-\mu)^2]}{\sigma^2 \cdot t^2} = \frac{\sigma^2}{\sigma^2 \cdot t^2} = \frac{1}{t^2}$. Observe that since $(X - \mu)^2$ is non-negative, $(X - \mu)^2 \geq t^2\sigma^2 \Leftrightarrow |X - \mu| \geq t \cdot \sigma$. Therefore, $\mathbf{Pr}\{|X - \mu| \geq t \cdot \sigma\} \leq \frac{1}{t^2}$. $\square$