Automata Theory - Homework I (Solutions)

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1 Problems

1. Let A and B be propositions. Argue that the following two statements are tautologies:

   (a) $A \rightarrow A$,

   (b) $[A \wedge (A \rightarrow B)] \rightarrow B$

   Solution: We use truth-tables to establish the above tautologies.

A	$A \rightarrow A$
T	**T**
F	**T**

(a)

A	B	$A \rightarrow B$	$A \wedge (A \rightarrow B)$	$[A \wedge (A \rightarrow B)] \rightarrow B$
T	**T**	**T**	**T**	**T**
T	**F**	**F**	**F**	**T**
F	**T**	**T**	**F**	**T**
F	**F**	**T**	**F**	**T**

(b)

□

2. Explain the difference between the *converse* of a theorem and its *contra-positive*.

   Solution: A typical theorem has the form: *If* A_1, *then* A_2, where A_1 and A_2 are boolean propositions. The contra-positive of a theorem is a restatement of the above theorem as: *If not* A_2, *then not* A_1.

   The converse of the above theorem, on the other hand is: *If* A_2, *then* A_1.

   Observe that the contra-positive of a theorem is always true. However, depending upon the theorem in question, its converse may or may not be true.

   □

3. Use mathematical induction to show that $7^n - 2^n$ is divisible by 5, for all $n \geq 0$.

   Solution: Let $S(n)$ denote the proposition that $7^n - 2^n$ is divisible by 5, for all $n \geq 0$.

   Observe that $S(0)$ is the statement that $7^0 - 2^0 = 1 - 1 = 0$ is divisible by 5. Since every number other than zero divides zero, it follows that 5 divides 0 and $S(0)$ is true.

Assume that $S(k)$ is true, for some $k > 0$, i.e., assume that $7^k - 2^k$ is divisible by 5. Accordingly, we can write,

$$7^k - 2^k = 5 \cdot m, \tag{1}$$

where m is some integer. (Note that this is the meaining of divisibility in the first place!)

Now we have to prove that $S(k+1)$ is true.

Observe that,

$$\begin{aligned} 7^{k+1} - 2^{k+1} &= 7 \cdot 7^k - 2 \cdot 2^k \\ &= 7 \cdot (2^k + 5 \cdot m) - 2 \cdot 2^k \ \ (from \ \ (1) \\ &= 7 \cdot (2^k) + 7 \cdot (5 \cdot m) - 2 \cdot 2^k \\ &= (7-2) \cdot 2^k + 5 \cdot (7 \cdot m) \\ &= 5 \cdot 2^k + 5 \cdot (7 \cdot m) \\ &= 5 \cdot q, \end{aligned}$$

where $q = 2^k + 5 \cdot (7 \cdot m)$ is an integer, since k and m are integers. It follows that $7^{k+1} - 2^{k+1}$ is divisible by 5, i.e., $S(k+1)$ is true. □

4. Let S and T denote two sets, which are subsets of a set U. Let S' and T' denote the complements of S and T in U respectively. Prove the following set equivalence:

$$(S \cup T)' = S' \cap T'.$$

Solution: Let E denote the set $(S \cup T)'$ and let F denote the set $S' \cap T'$.

Observe that,

$$\begin{aligned} x \in E &\rightarrow x \in (S \cup T)' \\ &\rightarrow x \notin (S \cup T) \\ &\rightarrow x \notin S \text{ and } x \notin T \\ &\rightarrow x \in S' \text{ and } x \in T' \\ &\rightarrow x \in S' \cap T' \\ &\rightarrow x \in F \end{aligned}$$

Likewise, observe that,

$$\begin{aligned} x \in F &\rightarrow x \in S' \cap T' \\ &\rightarrow x \in S' \text{ and } x \in T' \\ &\rightarrow x \in S' \text{ and } x \in T' \\ &\rightarrow x \notin S \text{ and } x \notin T \\ &\rightarrow x \notin S \cup T \\ &\rightarrow x \notin (S \cup T) \\ &\rightarrow x \in (S \cup T)' \\ &\rightarrow x \in E \end{aligned}$$

It therefore follows that the set equivalence holds. □

5. Let $\Sigma = \{0, 1\}$ denote an alphabet. Enumerate five elements of the following languages:

   (a) Even binary numbers,

   (b) The number of zeros is not equal to the number of ones in a binary string.

Solution:

(a) $L = \{0,\ 10,\ 100,\ 1000,\ 1010\}$

(b) $L = \{1,\ 100,\ 101,\ 110,\ 010\}$.

□